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Principles of Engineering Mechanics

Volume 2
Dynamics—The Analysis of Motion

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Principles of Engineering Mechanics

Volume 2

Dynamics—The Analysis of Motion

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Springer

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To my wife and best friend
NADINE CHUMLEY BEATTY,
to our children
LAURA, ANN and SCOTT,
and to our grandchildren
BOONE, BRITTANY, HUNTER,
TANNER and TREVOR

Preface

This volume, which begins with Chapter 5, is a vector treatment of the principles of mechanics written primarily for advanced undergraduate and first year graduate students specializing in engineering. A substantial part of the material, however, exclusive of obvious advanced topics, has been tailored to provide a first course for sophomore or junior level undergraduates having two years of university mathematics through differential equations, which may be taken concurrently. The book also serves as a sound, self-study review for advanced students specializing in mechanics, engineering science, engineering physics, and applied mathematics. It is prerequisite, of course, that the reader be familiar with vector and elementary matrix methods, the primitive concepts of mechanics, and the basic kinematical equations developed in Volume 1, which comprises the first four chapters of this two volume series. In addition, it is recommended that advanced students be familiar with the elements of tensor algebra and finite rigid body rotations treated there in Chapter 3.

The arrangement of the subject into the separate parts kinematics and dynamics has always seemed to me the best didactic approach, and it still does. Unfortunately, the luxury of a slow-paced two semester course on these subjects is seldom afforded. Therefore, in a concentrated, fast-paced single semester course, especially for a diverse group of undergraduates, a level and variety of topics may be chosen for presentation to address student needs and to satisfy curriculum objectives. But a great deal of the subject matter also can be assigned first to reading and later discussed briefly in lectures. Assigned reading of my discourse on the foundation principles of classical mechanics, for example, might offer an opportunity for an interesting dialogue among students and the instructor. At the bottom line, however, the classroom usage of this material must be assessed by the teacher in keeping with the class level, the course objectives, the curriculum schedule, and the desired emphasis.

Naturally, the presentation is influenced by my personal interests and background in mechanical engineering, engineering science, and mechanics. Consequently, my approach definitely is somewhat more sophisticated and mathematical than is commonly found in traditional textbooks on engineering mechanics.

In keeping with this approach, the prerequisite mathematics, principally that of the seventeenth and eighteenth centuries, is used without apology. Nevertheless, aware that many readers may not have mastered these prerequisite materials, I have exercised care to reinforce the essential tools both directly within the text and indirectly in the illustrations, exercises and problems selected for study. Elements of the theory of linear differential equations of the second order, for example, are presented; hyperbolic functions are introduced and illustrated graphically; the analytic geometry of conic sections is reviewed; and advanced students will find a thoughtful discussion on the subject of elliptic functions and integrals applied in several examples. These and other topics are illustrated in many physical applications throughout the book; and several exercises and problems expand on related details. Instances where material may be omitted without loss of continuity are clearly indicated.

This presentation of dynamics focuses steadily on illustrating the predictive value of the methods and principles of mechanics in describing and explaining the motion and physical behavior of particles and rigid bodies under various kinds of forces and torques. The emphasis throughout this book is principally analytical, not computational, though many numerical examples and problems are provided to expose the relevant aspects of an example or to illustrate certain theoretical details. Examples have been selected for their interest and instructive value and to help the student achieve understanding of the various concepts, principles, and analytical methods. In some instances, experimental results, factual situations, and applications that confirm analytical predictions are described. I hope the historical remarks and several relevant short stories add interest and reality to the subject. Numerous assignment problems, ranging from easy, straightforward extensions or reinforcements of the subject matter to more difficult problems that challenge the creative skills of better students, are provided at the end of each chapter. To assist the student's study of dynamics, answers to the odd-numbered problems are provided at the back of the book.

It is axiomatic that physical intuition or insight cannot be taught. On the other hand, competence in mathematical and physical reasoning may be developed so that these special human qualities may be intelligently cultivated through study of physical applications that mirror the world around us and through practice of the rational process of reasoning from first principles. With these attributes in mind, one objective of this volume is to help the engineering student develop confidence in transforming problems into appropriate mathematical language that may be manipulated to derive substantive and useful physical conclusions or specific numerical results. I intend that this treatment should present a penetrating and interesting treatise on the elements of classical mechanics and their applications to engineering problems; therefore, this text is designed to deepen and broaden the student's understanding and to develop his or her mastery of the fundamentals. However, to reap a harvest from the seeds sown here, it is important that the student work through many of the examples and problems provided for study. When teaching this course over the years, I have usually assigned six problems per week, two per class period, roughly ninety per semester. If the assignment problems chosen range from easy to difficult, I don't think it is wise to assign more. The mere understanding that one may apply theoretical concepts and formulas to solve a particular problem is not equivalent to possession of the knowledge and skills required to produce its solution. These talents grow only from experience in dealing repeatedly with these matters. My view of the importance of solving many problems is expressed further at the beginning of the problem set for Chapter 5, the first chapter of this volume. The attitude emphasized there is echoed throughout this text. It is my hope that these books on kinematics and dynamics may provide engineering students and others with solid mathematical and mechanical foundations for future advanced study of topics in mechanical design analysis, advanced kinematics of mechanisms and analytical dynamics, mechanical vibrations and controls, and areas of continuum mechanics of solids and fluids.

The Contents of This Volume

Volume 2: Dynamics, concerns the analysis of motion based on classical foundation principles of mechanics due to Newton and Euler, later cast in a generalized energy based formulation by Lagrange. A fresh development of a classical subject like this is seldom seen. I believe, however, that the reader will find within these pages many fresh developments, beginning with my discussion of the fundamental classical principles of mechanics and concluding with the presentation of the elegant analytical formulation of the Lagrange equations of motion. And in between, the reader will find here a refreshingly different, consistent, logical, and gradual development and careful application of all of the classical principles of mechanics for a particle, a system of particles, a rigid body, actually any number of particles and rigid bodies, subjected to various kinds of forces and torques. Some unusual, hopefully interesting things to look for are sketched below.

Chapter 5: The Foundation Principles of Classical Mechanics, presents a detailed study and fresh discussion of Newton's laws of motion, Newton's theory of gravitation, the role of inertial reference frames, and Newton's second law of motion relative to the moving Earth frame. Further, two fascinating applications of Coulomb's laws of friction having engineering significance are presented. The chapter concludes with an advanced topic borrowed from continuum mechanics, here applied to deduce the general law of mutual interaction between two particles. This interesting topic has not previously appeared in any textbook.

The kinematical equations characterizing the velocity and acceleration of a particle in terms of rectangular, intrinsic, cylindrical, and spherical coordinates are applied in Chapter 6: Dynamics of a Particle, in the formulation of the equations of motion in a variety of special applications. Besides gravitational and frictional forces, additional electromagnetic, elastic spring, and drag force laws are introduced and illustrated in some atypical examples that include the mechanics of an ink jet printer and Millikan's famous oil drop experiment. Of course, familiar elementary problems, sometimes having a novel twist, appear here as well. The elements of projectile motion with examples that include drag force effects and the fanciful exploits of Percy Panther and Arnold Aardvark surely will attract attention. Several additional distinctive examples include a nonlinear oscillator, the finite nonlinear radial oscillation of a particle of an incompressible rubber tube, and problems of motion relative to the moving Earth frame that include Reich's experiment on the deflection of falling pellets and Foucault's experiment demonstrating the rotation of the Earth. The projectile deflection problem is solved; and, with this in mind, the historic tragic battle of the Falkland Islands is described.

The first integral principles of mechanics for a particle—impulse-momentum, torque-impulse, work-energy, and the conservation laws of momentum, moment of momentum, and energy—are developed in Chapter 7: Momentum, Work, and Energy. Along with several typical applications, some advanced topics treated there include an introduction to elliptic functions and integrals in the problem of the finite amplitude oscillations of a pendulum. The isochronous cycloidal pendulum and Huygens's clock, and orbital motion and Kepler's laws, are other interesting examples.

The classical principles of mechanics and first integrals for a particle are extended to study in Chapter 8: Dynamics of a System of Particles. The importance of the center of mass of the system, first mentioned in Chapter 5, is now evident. The two body problem correcting Kepler's third law is a distinctive illustration. In preparation for later work on rigid bodies, the moment of momentum principle with respect to a moving reference point, first derived in Chapter 6 for a particle, and the reduction to its simplest form, is carefully described. Several examples of simple systems of particles are demonstrated.

The elements of tensor algebra, including the Cartesian tensor transformation law, introduced in Volume 1, Chapter 3, are exploited in Chapter 9: The Moment of Inertia Tensor. Here the inertia tensor and its general properties are studied and applied to complex structured (composite) bodies. The center of mass concept for a simple rigid body, first studied in Chapter 5, is extended to composite bodies as well. The tensorial form of the parallel axis theorem is proved, and its interpretation is illustrated in examples. The principal (extremal) values of the inertia tensor and their directions are derived by the method of Lagrange multipliers; and their easy geometrical interpretation in terms of Cauchy's momental ellipsoid is described. These topics play a key role in the general theory of rigid body motion. Some advanced analytical topics on the principal values of general symmetric tensors and the multiplicity of principal values and corresponding principal axes are explored at the end.

Chapter 10: Dynamics of a Rigid Body, the most difficult of the subjects treated in this book, focuses on Euler's generalization of Newton's principles of mechanics, now extended to characterize the equations of motion for all bodies—solids, fluids, gasses—but herein restricted principally to the motions of rigid solid bodies. Euler's laws are set down separately and discussed in detail, somewhat parallel to my earlier discussion of Newton's principles; but there are sharp differences. First, the principle of determinism includes both forces and torques, and the third law of mutual interaction of both forces and torques follows as a theorem from Euler's first and second laws. The remaining developments, however, are fairly standard leading to Euler's nonlinear vector differential equation of motion for a rigid body in terms of its moment of inertia tensor. The key role of the principal body axes in the simplification of Euler's scalar equations is demonstrated. The usual general first integral principles on impulse-momentum, torque-impulse, conservation of momentum and moment of momentum, the work-energy and energy conservation principles for a rigid body are obtained. It is worth noting, however, that two forms of Euler's second law of motion with respect to a moving reference point, seldom discussed in other works, are derived. Their reduction to the simplest common case is thoroughly discussed, and several typical sorts of applications are studied. Some advanced examples describe the effect of the Earth's rotation in controlling the motion of a gyrocompass, the stability of the torque-free rotation of a spinning rigid body, the sliding, slipping and rolling motion of a billiard ball, and the motion of a general rigid lineal body in a Stokes medium.

A careful, thorough introduction to Lagrange's equations of motion for holonomic dynamical systems is provided in Chapter 11: Introduction to Advanced Dynamics. The general form of Lagrange's equations, an energy based formulation of the equations of motion, is first motivated in their derivation for a holonomic system consisting of a single particle. The principle of virtual work relates the generalized forces in these equations to the physical forces that act on the particle. The work-energy principle, Lagrange's equations for conservative scleronomic systems, and their first integral principle of conservation of energy follow. Precisely all of the same equations are shown to hold for any system of discrete material points having any number of degrees of freedom. For conservative systems, recognition of so-called ignorable coordinates in the Lagrangian energy function leads immediately to easy first integrals of Lagrange's equations. Generalized momenta are introduced and Lagrange's equations for each generalized coordinate are recast in a form having the appearance of Newton's second law in which the Lagrange forces consist of the generalized forces and certain pseudoforces of the sort encountered differently in earlier studies. Generalized first integral principles of conservation of momenta and the generalized impulse-momentum principle follow. All of the results are illustrated in several examples together with the physical interpretation of the terms. Lagrange's equations for both conservative and nonconservative general dynamical systems are then deduced from Hamilton's variational principle of least action. As a consequence, Lagrange's equations are immediately applicable to study the motion of rigid bodies; in fact, any dynamical system consisting of any number of particles and rigid bodies. It is only necessary to define the Lagrangian energy function, however complex, and to identify the corresponding generalized forces. Accordingly, the Lagrange equations are applied to formulate the equations of motion for several such physical systems. The theory of small vibrations for nondegenerate, multidegree of freedom systems is derived and demonstrated in an example. Finally, Lagrange's equations are modified to account for the effect of Stokes type damping in a general formulation of the Rayleigh dissipation function, and the result is illustrated in some concluding examples that include an application to the theory of small vibrations with damping. This relatively brief introduction to advanced dynamics presents the reader with tools adequate to pursue more advanced, abstract studies of analytical dynamics.

In every chapter throughout this work the reader will find a great many historical remarks, annotations, and references on the several topics treated here. I hope these little stories, digressions, and annotations will add life and bring to the subject a reality not seen in most books of this kind. Countless examples and problems are provided as aids to understanding and visualizing the details of the theory; and, in most instances, a physical interpretation is presented or perhaps some unusual effect is identified. This sort of discovery and prediction derived from the principles of mechanics and leading to results that most often are not intuitively evident, drive the purpose of the analysis. It is important that the student cultivate the habit of attempting to interpret the physical phenomena or unusual effects that emerge from problem analysis. Ample opportunity for the reader to do so is provided within the problems provided at the end of each chapter. So far as I am aware, the treatment of problems provided herein usually is quite different and more thorough than found in other comparable sources known to me. Nevertheless, by consulting the listed references or their own favorite books, both the teacher and the student should be able to supplement the many examples and problems to meet their special needs.

In writing this book, I have appealed over many years to numerous sources. While I do not necessarily subscribe to their approach to mechanics, several of the references on classical and engineering mechanics that I found particularly helpful are cited at the end of each chapter, usually with annotations to describe the substance of the work and to identify specific topics that may be consulted for collateral study. The presentation has also benefited greatly from the very many historical papers and books referenced there. I hope the acknowledgments are complete. It is impossible, of course, to be precise in citing my specific use of every source, and I apologize if I may have overlooked or forgotten a particular reference.

Acknowledgment

I owe much to Mr. Joe Haas at the University of Kentucky during the 1980s, for his good humor, untiring patience, and painstaking care in the tedious task of adding typeset lettering to my numerous ink drawings. I came to appreciate this even more when years later I did this myself. I am also grateful to Ms. Marjorie Bisbee and Mr. James Pester at the University of Nebraska-Lincoln in the 1990s, and I thank Mr. Haas as well, for applying their artistic talents and computer skills

to draw from my pencil sketches some of the illustrations that appear here. I hope they are not offended by my modifications and addition of shading to all of the illustrations.

The students that took my courses in dynamics at the University of Delaware, the University of Kentucky, and the University of Nebraska-Lincoln deserve my special thanks for the opportunity to learn from them as well. It was especially exciting to find in grading some assignment papers, an essential but usually dull and tiresome task, that from time to time a student would discover an interesting solution method that I had not thought of. In this regard, I thank my former graduate student at Nebraska, Professor Alex Elías-Zúñiga at the Instituto Technológico y de Estudios Superiores de Monterrey, Mexico; he read an early draft of the manuscript in the 1990s and worked through countless problems in confirmation of my solutions. As a consequence of such student efforts over many years, for which I am most grateful, virtually all of the problems in this book have been tested and their solutions confirmed. Any errors that may yet appear in the answers at the back of this volume are mine.

The joy of work with graduate students in areas of finite elasticity, related funded research interests, and classroom, professional, and administrative obligations as Chairman of the Department of Engineering Mechanics at Nebraska, constantly took precedence over my efforts to complete this volume started so long ago. When at last the manuscript was finished in the spring of 2004, I sought the criticism and comment of several respected colleagues who kindly accepted the burden to read all or a portion of the manuscript. Their thoughtful and constructive critiques led to many improvements that served to enhance the quality of the book. I am grateful for their input and for their unsolicited supportive complimentary comments. Such errors as may remain, however, are my sole responsibility. It is a pleasure to thank Professor Eveline Baesu, University of Nebraska-Lincoln, and Professor David Steigmann, University of California, Berkeley; they provided helpful remarks and useful suggestions on various parts of the manuscript. I am indebted to Professor Emeritus Oscar W. Dillon, University of Kentucky; his constructive critical comments, helpful discussions, and thoughtful recommendations and remarks covering a great many aspects of the manuscript have been invaluable. Finally, I am indeed most grateful to Professor Emeritus Michael A. Hayes, University College, Dublin, for his thoughtful, painstaking review of both the manuscript and the proof; he exposed in both a great many typographical slips and provided numerous beneficial suggestions and invaluable comments from which the final version of this book has greatly benefited throughout.

Millard F. Beatty, Jr.
Lexington, Kentucky

Contents

Note to the reader: This book, which begins with Chapter 5, is the second of a series of two volumes on the *Principles of Engineering Mechanics*. Chapters 1 through 4 and Appendices A and B appear in *Volume 1: Kinematics—The Geometry of Motion*, which includes the Answers to Selected Problems for that volume and a separate Index on its contents. Volume 2 includes Chapters 5 through 11, Appendices C and D, Answers to Selected Problems for this volume, and a separate Index on its contents.

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Volume 2

Dynamics

The Analysis of Motion

Mechanics is the paradise of the mathematical sciences, because here we come to the fruits of mathematics.

Leonardo da Vinci
The Notebooks

5

The Foundation Principles of Classical Mechanics

> I would mention the experience that it is exceedingly difficult to expound to thoughtful hearers the very introduction to mechanics without being occasionally embarrassed, without feeling tempted now and again to apologize, without wishing to get as quickly as possible over the rudiments and on to examples which speak for themselves. I fancy that Newton himself must have felt this embarrassment....
>
> Heinrich Hertz
> *The Principles of Mechanics*

5.1. Introduction

Dynamics is the theory of motion and the forces and torques that produce it. This theory integrates our earlier studies of kinematics, the geometry of motion, with certain fundamental laws of nature that relate force, torque, and motion. In this chapter the primitive concepts of mass and force introduced in Chapter 1 are related to motion through some basic principles commonly known as *Newton's laws.* Sir Isaac Newton (1642–1727) in his *Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy), often referred to as simply the *Principia,* published in 1687, formalized and extended earlier achievements of others by creating an axiomatic structure for the foundation principles of mechanics. By the organization of problems around his fundamental laws, Newton successfully demonstrated the application of his theory to the study of problems of mechanics of the solar system. He thus began the idea that the motions of bodies may be deduced from a few simple principles.

The formulation and application of Newton's laws entail the use of analytical methods of differential equations. Surprisingly, however, Newton never recorded or applied his laws in any general mathematical form; and historians (e.g. Truesdell) have found no evidence to suggest that he was able to set up differential equations for the mechanical systems he investigated. Others (e.g. Bixby) suggest that for the benefit of scholars, in those times well-versed in geometry, Newton's arguments were laboriously worked out by geometrical methods, rather than in terms of his emerging new calculus, so that mathematicians and scientists would be able to understand his new ideas on the motions of bodies. In fact, it was not until 1750 that Newton's laws for material points were first formulated more generally by Leonhard Euler (1707–1783) as differential equations relating force, torque, and motion for all bodies, including deformable bodies. Thus, it was not Newton; it was Euler who demonstrated countless times how to set up mechanical problems as definite mathematical problems formulated from basic, first principles. Therefore, it is not uncommon nowadays that the basic laws of mechanics are often referred to as *Euler's laws*. The classical, mathematical principles of mechanics created by Newton and Euler thus establish the fundamental laws governing the motions of all bodies. They provide the foundation for our study of dynamics—*the analysis of motion*.

The simplest kind of dynamical problem is to find the force needed to produce a specified motion of a particle. The converse problem of finding the motion arising from the application of known forces of various kinds is more difficult. This problem requires the solution of differential equations. Our earlier practice with simple integration methods applied in kinematics, therefore, will prove useful in the study of problems of this kind.

To formulate these types of problems, we need to know how to specify mathematically the nature of various kinds of forces that act between pairs of bodies. These forces are of two general kinds, *contact force* and *body force*. The weight of a body is a familiar example of a body force that arises from the mutual action between pairs of separated bodies in accordance with *Newton's law of universal gravitational attraction*. This basic body force law is studied in this chapter. Of course, two bodies may also interact by contact, i.e. by mutual touching. Everyone knows, for example, that when two blocks are pressed together, a force tangent to their common surfaces must be applied in order to slide one block on the other. But once the sliding has begun, the force needed to sustain the motion is somewhat smaller than that required to initiate it. The fundamental laws that characterize these familiar experiences are studied here too. These principles, called *Coulomb's laws,* relate the normal and the tangential components of the contact force that acts between two bodies to oppose their relative sliding motion. Other kinds of viscous, elastic, electromagnetic, and time varying forces are introduced in the next chapter. In addition, we are going to find that certain pseudoforces act on bodies having motion relative to an accelerating, rotating reference frame.

The effect of the motion of the frame of reference on the form of Newton's second law of motion is investigated. It turns out that our moving Earth frame is not the reference frame with respect to which Newton's laws hold. Therefore, we must learn how the governing laws are to be modified so that they may be applied to problems in any moving frame, including our Earth frame. In addition to aiding our understanding of the extent of the error that may be expected when the motion of the reference frame is neglected, the theory will also reveal in later applications some interesting and subtle physical phenomena that arise from the Earth's rotation. The idea that laws governing the internal forces between the parts of a system should be independent of any external reference frame used to describe them is expressed in the *principle of frame indifference* studied here in the context of the mutual force that acts between two particles and depends on only their spatial positions. Application of this idea leads to the most general form of the law of mutual action between two particles as a function of only their distance of separation.

Our main objective in this chapter is to study the foundation principles of classical mechanics. The *Newton–Euler laws of mechanics* are here formulated in a manner that parallels that introduced by Newton and generalized by Euler. The content, utility, and the predictive value of these rules in relation to special force laws, like those that govern gravitation and sliding friction for example, are explored in their application to physical theory and problems, and in some cases by comparison of their theoretical predictions with experimental observations. A few introductory illustrations of these qualities are investigated here; and many additional examples and practice problems and solution techniques for particle dynamics are presented in Chapter 6. Some other useful principles of momentum, work, and energy that derive from the primary Newton–Euler law for a particle or center of mass object are presented in Chapter 7. The structure used in these beginning chapters is extended in Chapter 8 to the motion of a system of particles. The moment of inertia tensor is introduced in Chapter 9; and then Euler's grand generalization of Newton's principles of mechanics are formulated for a rigid body in Chapter 10. Our study ends in Chapter 11 with an introduction to the methods of advanced dynamics. The formulation of Lagrange's equations and Hamilton's principle for analytical mechanics are explored there. This is the point where books on advanced dynamics usually begin. Construction of a foundation for these future studies begins with one particle. First, it is recommended that the reader review the primitive terms and concepts introduced on pages 3–7 in Chapter 1.

5.2. Mass, Momentum, and the Center of Mass

The mass of a particle, a system of particles, and a rigid body, and the corresponding principle of conservation of mass for each of these is introduced. The momentum of a particle, and the momentum and the center of mass of a system of particles and of a rigid body are defined. The latter ideas are then applied to learn how the momentum of a system of particles and of a rigid body are related to the momentum of their respective centers of mass. These preliminary concepts and results on the center of mass are important to our future study of the classical principles of mechanics.

5.2.1. Mass and Momentum of a Particle

We recall from Chapter 1 that the mass $m(P, t)$ of a particle P is a positive scalar measure of its material content. The physical dimension of mass is denoted by $[M]$. It is a postulate of Newton's mechanics that *the mass of a particle is invariant in time, that is,*

$$m(P, t) = m(P) \tag{5.1}$$

for all times t. This axiom is called *the principle of conservation of mass.* It emphasizes that the mass of a particle is an invariant measure of its material content alone. Of course, the mass of another particle may be different.

The *momentum* $\mathbf{p}(P, t)$ of a particle P in a reference frame Φ is a vector-valued function of time t defined by the product of the mass $m(P)$ of the particle and its velocity $\mathbf{v}(P, t)$ in Φ:

$$\mathbf{p}(P, t) \equiv m(P)\mathbf{v}(P, t). \tag{5.2}$$

Sometimes the momentum is called the *linear momentum* to distinguish it from the moment of momentum introduced later on. It is seen from (5.2) that the momentum vector has the physical dimensions $[\mathbf{p}] = [MV] = [MLT^{-1}]$. Specific measure units are reviewed in the Appendix following the References at the end of this chapter and in the Problems.

5.2.2. Mass, Momentum, and Center of Mass of a System of Particles

We recall from Chapter 1 that a body $\beta = \{P_k\}$ consisting of n discrete particles P_k having mass $m_k = m(P_k)$, $k = 1, 2, \ldots, n$, is called a *system of particles.* It is clear that mass is an additive scalar measure on β. *Hence, the mass $m(\beta)$ of the system of particles is defined by the sum of the masses m_k of the particles P_k of β:*

$$m(\beta) \equiv \sum_{k=1}^{n} m_k. \tag{5.3}$$

The principle of conservation of mass (5.1) requires that the mass of the system is constant: $dm(\beta)/dt = 0$. Clearly, in a system of particles the mass may vary from one particle to another; and the mass of another system may be different.

5.2.2.1. Momentum of a System of Particles

By (5.2), each particle P_k has a momentum $\mathbf{p}_k \equiv \mathbf{p}(P_k, t) = m_k\mathbf{v}_k$ for which $\mathbf{v}_k \equiv \mathbf{v}(P_k, t)$ denotes the velocity of P_k in Φ. Therefore, *the momentum of the*

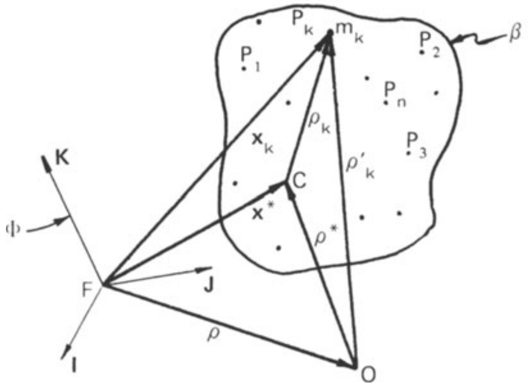


Figure 5.1. Schema for the center of mass properties of a system of particles.

system β *in frame* Φ is defined by

$$\mathbf{p}(\beta, t) \equiv \sum_{k=1}^{n} \mathbf{p}_k = \sum_{k=1}^{n} m_k \mathbf{v}_k. \tag{5.4}$$

5.2.2.2. Center of Mass of a System of Particles

The center of mass of a system of particles is an important concept that enables us to reduce the momentum (5.4) of the system to the momentum of a single, fictitious particle—a neat trick that proves most useful in future work. With this objective in mind, consider a system of particles shown in Fig. 5.1 in frame $\Phi = \{F; \mathbf{I}_j\}$, a set comprising an origin point F and an orthonormal vector basis $\mathbf{I}_j$, as defined in Chapter 1. Let $\mathbf{x}_k \equiv \mathbf{x}(P_k, t)$ denote at time t the position vector of a particle P_k whose mass is m_k. The *center of mass of a system of particles* $\beta = \{P_k\}$ is defined as the point in Φ whose position vector $\mathbf{x}^* \equiv \mathbf{x}^*(\beta, t)$ is determined by

$$m(\beta)\mathbf{x}^* = \sum_{k=1}^{n} m_k \mathbf{x}_k, \tag{5.5}$$

wherein we recall (5.3) for the mass $m(\beta)$ of the system. In this sense, the weighted-average motion of the particles of the system is described by the motion $\mathbf{x}^*(\beta, t)$ of a single, fictitious particle of mass $m(\beta)$, the mass of the system. Some properties of the center of mass are discussed next.

We first note that *the center of mass need not be a place occupied by a particle of* β, *but it may be*. Consider for example a system $\beta = \{P_1, P_2\}$ of two particles of equal mass $m_1 = m_2 = m$, one at the origin $\mathbf{x}_1 = \mathbf{0}$ and the other at an arbitrary place $\mathbf{x}_2 = \mathbf{d}$ in Φ at an instant t. Then by (5.3), we have $m(\beta) = 2m$; and (5.5) provides $2m\mathbf{x}^* = \sum_{k=1}^{2} m_k \mathbf{x}_k = m\mathbf{d}$. Hence, the center of mass of this system at the instant t is at the place $\mathbf{x}^* = \mathbf{d}/2$ in Φ—a place that is not occupied by either particle of β. On the other hand, consider a system of three particles of equal mass m; one at $\mathbf{x}_1 = \mathbf{0}$, one at $\mathbf{x}_2 = \mathbf{d}/2$, and the other at $\mathbf{x}_3 = \mathbf{d}$ in Φ at time t. In this case, (5.5) shows that the center of mass of the system at the instant t is at the place $\mathbf{x}^* = \mathbf{d}/2$ occupied by the particle P_2.

We show next that *the center of mass is a unique point whose definition is independent of the reference origin in* Φ. First consider the reference origin. Identify another reference point O at $\boldsymbol{\rho}$ from F in Φ in Fig. 5.1. Introduce $\mathbf{x}_k = \boldsymbol{\rho} + \boldsymbol{\rho}_k'$ and $\mathbf{x}^* = \boldsymbol{\rho} + \boldsymbol{\rho}^*$, where $\boldsymbol{\rho}_k'$ and $\boldsymbol{\rho}^*$ are the respective position vectors of the particle P_k and of the center of mass C from O. Then (5.5), with the aid of (5.3), becomes

$$m(\beta)(\boldsymbol{\rho} + \boldsymbol{\rho}^*) = \sum_{k=1}^{n} m_k \boldsymbol{\rho} + \sum_{k=1}^{n} m_k \boldsymbol{\rho}_k' = m(\beta)\boldsymbol{\rho} + \sum_{k=1}^{n} m_k \boldsymbol{\rho}_k'.$$

It thus follows that for an arbitrary point O,

$$m(\beta)\boldsymbol{\rho}^* = \sum_{k=1}^{n} m_k \boldsymbol{\rho}_k'$$

has the same form as (5.5). *Therefore, the definition* (5.5) *for the center of mass is independent of the choice of the reference origin in* Φ.

Now let us choose O at the center of mass C so that $\boldsymbol{\rho}^* = \boldsymbol{\rho}_k' - \boldsymbol{\rho}_k = \mathbf{0}$ in Fig. 5.1. Then relative to the center of mass, we have

$$\sum_{k=1}^{n} m_k \boldsymbol{\rho}_k = \mathbf{0}, \tag{5.6}$$

wherein $\boldsymbol{\rho}_k$ is the position vector of the particle P_k from C at time t. Clearly, (5.6) simply states that the position vector of the center of mass from itself is the zero vector.

It is now easy to prove that the center of mass is the only point with respect to which (5.6) holds for a system of particles. Indeed, suppose there exists another point C', say, at the place $\mathbf{r}$ from C such that (5.6) holds. Then $\sum_{k=1}^{n} m_k \mathbf{r}_k = \mathbf{0}$, where $\mathbf{r}_k$ is the position vector of P_k from C'. However, substitution of $\boldsymbol{\rho}_k = \mathbf{r} + \mathbf{r}_k$ into (5.6) shows that $\mathbf{r} = \mathbf{0}$; that is, the points C and C' coincide. *Therefore, at each instant, the center of mass of a given system of particles is the unique point for which* (5.6) *holds*. Plainly, if the system is altered in any way, so is its center of mass.

5.2.2.3. Momentum of the Center of Mass of a System of Particles

We now derive an important result relating the momentum of a system of particles to the momentum of its center of mass. Of course, the system of particles is generally in motion in Φ with momentum (5.4), in which $\mathbf{v}_k \equiv \dot{\mathbf{x}}_k$. We recall (5.5) and define $\mathbf{v}^* \equiv \dot{\mathbf{x}}^*(\beta, t)$, the velocity of the center of mass. Then, differentiation of (5.5) with respect to time in Φ, the mass of the system being conserved, and use of (5.4), yields the important result

$$\mathbf{p}^* \equiv m(\beta)\mathbf{v}^* = \sum_{k=1}^{n} m_k \mathbf{v}_k = \mathbf{p}(\beta, t). \tag{5.7}$$

The vector $\mathbf{p}^*$ defined by the first equation in (5.7) is the momentum of an imaginary particle of mass $m(\beta)$ that moves with the velocity $\mathbf{v}^*$ of the center of mass. This particle is named the *center of mass particle* (or *object*); and $\mathbf{p}^*$ is called briefly the *momentum of the center of mass*. The result (5.7) thus shows that *the momentum of a system of particles is equal to the momentum of its center of mass*: $\mathbf{p}(\beta, t) = \mathbf{p}^*(\beta, t)$.

Further, differentiation of (5.6) yields

$$\sum_{k=1}^{n} m_k \dot{\boldsymbol{\rho}}_k = \mathbf{0}. \tag{5.8}$$

Hence, the momentum of a system of particles relative to its center of mass particle is always zero.

Example 5.1. A system $\beta = \{P_1, P_2, P_3\}$ consists of three particles with mass $m_1 = m$, $m_2 = 2m$, $m_3 = 3m$ and having the respective constant velocities $\mathbf{v}_1 = v(6, -7, 0)$, $\mathbf{v}_2 = v(0, 2, -3)$, $\mathbf{v}_3 = v(2, -1, -2)$ in frame $\Phi = \{F; \mathbf{I}_k\}$. Determine the momentum of the system in Φ, find the velocity of each particle relative to the center of mass C, and thus confirm (5.8).

Solution. First recall (5.4) for the momentum of the system. The momentum of each particle is determined by (5.2); and from the assigned data, we obtain

$$\mathbf{p}_1 = m_1\mathbf{v}_1 = mv(6, -7, 0), \qquad \mathbf{p}_2 = m_2\mathbf{v}_2 = 2mv(0, 2, -3),$$
$$\mathbf{p}_3 = m_3\mathbf{v}_3 = 3mv(2, -1, -2). \tag{5.9a}$$

Then, by (5.4), the momentum of the system in Φ is given by

$$\mathbf{p}(\beta, t) = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 6mv(2\mathbf{I} - \mathbf{J} - 2\mathbf{K}). \tag{5.9b}$$

The velocity of the particle P_k relative to C is given by $\dot{\boldsymbol{\rho}}_k = \mathbf{v}_k - \mathbf{v}^*$, in which the velocity of C may be found from (5.7). Hence, with (5.3), the momentum of C is $\mathbf{p}^* = 6m\mathbf{v}^* = \mathbf{p}(\beta, t)$; and use of (5.9b) yields $\mathbf{v}^* = v\,(2\mathbf{I} - \mathbf{J} - 2\mathbf{K})$. Therefore,

$$\dot{\boldsymbol{\rho}}_1 = \mathbf{v}_1 - \mathbf{v}^* = v(4, -6, 2), \qquad \dot{\boldsymbol{\rho}}_2 = \mathbf{v}_2 - \mathbf{v}^* = v(-2, 3, -1),$$
$$\dot{\boldsymbol{\rho}}_3 = \mathbf{v}_3 - \mathbf{v}^* = \mathbf{0}, \tag{5.9c}$$

identify the velocity of each particle relative to C in Φ; and hence

$$\sum_{k=1}^{3} m_k \dot{\rho}_k = m_1\dot{\rho}_1 + m_2\dot{\rho}_2 + m_3\dot{\rho}_3 = mv(4, -6, 2) + 2mv(-2, 3, -1) + \mathbf{0} = \mathbf{0}, \tag{5.9d}$$

in agreement with the general result (5.8). □

5.2.3. Mass, Momentum, and Center of Mass of a Rigid Body

Let us consider a rigid body $\mathscr{B}$, and let $dm(P)$ denote an additive parcel (or element) of mass at the material point P. Then *the mass of the body* is defined by

$$m(\mathscr{B}) \equiv \int_{\mathscr{B}} dm(P) = \int_{\mathscr{B}} \rho(P)\,dV(P), \tag{5.10}$$

wherein $dV(P)$ is the elemental material volume of $\mathscr{B}$ at P, and $\rho(P) \equiv dm(P)/dV(P)$, the ratio of the element of mass at P to its element of volume at P, that is, the mass per unit volume of $\mathscr{B}$, is called the *mass density*. The subscript $\mathscr{B}$ on the integral sign, here and throughout this volume, means that the integration, with appropriate limits, is over the bounded region defined by the body $\mathscr{B}$. Neither the mass density nor the material volume of a rigid body can change with time, so the principle of balance of mass is satisfied: $dm(\mathscr{B})/dt = 0$. In general, however, the density may vary from one material point to another. A rigid body $\mathscr{B}$ is said to be *homogeneous* whenever its mass density is constant throughout $\mathscr{B}$. Thus, by (5.10), the mass of a homogeneous rigid body is simply the product of the mass density and the material volume of $\mathscr{B}$, namely, $m(\mathscr{B}) = \rho V(\mathscr{B})$, where $V(\mathscr{B}) = \int_{\mathscr{B}} dV(P)$.

5.2.3.1. Momentum of a Body

The momentum of a body element of mass $dm(P)$ at P in Φ is $dm(P)\mathbf{v}(P, t)$. Hence, the *momentum* $\mathbf{p}(\mathscr{B}, t)$ *of a body* in a reference frame Φ is defined by

$$\mathbf{p}(\mathscr{B}, t) \equiv \int_{\mathscr{B}} \mathbf{v}(P, t)\,dm(P). \tag{5.11}$$

In general, both the velocity and mass distributions must be known to effect the integration of (5.11). Consider, for example, a rigid body $\mathscr{B}$ having a uniform motion in the frame Φ. In this case, the velocity of every particle of $\mathscr{B}$ is a constant vector $\mathbf{v}(P, t) = \mathbf{v}$, so equations (5.11) and (5.10) yield $\mathbf{p}(\mathscr{B}, t) = \mathbf{v}\int_{\mathscr{B}} dm\,(P) = m(\mathscr{B})\mathbf{v}$. Hence, the momentum of a rigid body $\mathscr{B}$ having a uniform motion is the same as that of a single particle of mass $m(\mathscr{B})$ moving with the constant velocity $\mathbf{v}$. We shall see next that this imaginary particle is the center of mass of the body.

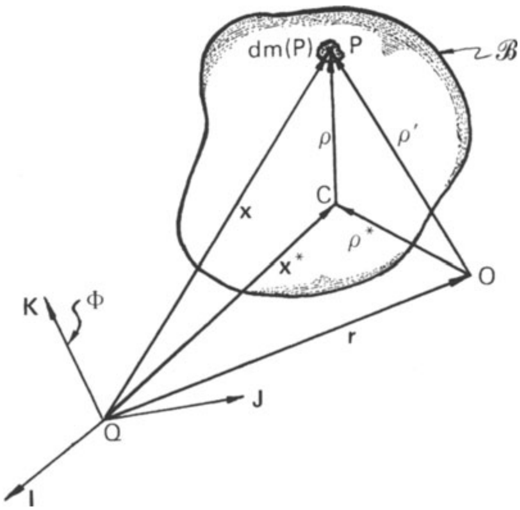


Figure 5.2. Schema for the center of mass properties of a body.

5.2.3.2. Center of Mass of a Body

We shall soon discover that the dynamics of a rigid body involves the motion of its center of mass, an important concept by which the momentum (5.11) of a body may be replaced by the momentum of a single, imaginary particle situated at its center of mass. With this in mind, let $\mathbf{x}(P, t)$ denote at time t the position vector of the material parcel $dm(P)$ of a body $\mathscr{B}$ in a spatial frame $\Phi = \{Q; \mathbf{I}_k\}$ shown in Fig. 5.2. The *center of mass of the body* $\mathscr{B}$ is the unique point in Φ whose position vector $\mathbf{x}^* \equiv \mathbf{x}^*(\mathscr{B}, t)$ at time t is determined by

$$m(\mathscr{B})\mathbf{x}^*(\mathscr{B}, t) = \int_{\mathscr{B}} \mathbf{x}\,(P, t)\, dm\,(P)\,, \tag{5.12}$$

in which we recall (5.10). In this sense, the weighted-average motion of the particles of the body is described by the motion $\mathbf{x}^*(\mathscr{B}, t)$ of a single, fictitious particle of mass $m(\mathscr{B})$, called *the center of mass particle*. Some properties of the center of mass are described next.

It is easy to prove that *the definition* (5.12) *is independent of the choice of reference origin Q in* Φ. Therefore, relative to the center of mass point itself, (5.12) becomes

$$\int_{\mathscr{B}} \boldsymbol{\rho}(P, t) dm(P) = \mathbf{0}, \tag{5.13}$$

where $\boldsymbol{\rho}(P, t)$ is the position vector from the center of mass C to the parcel $dm(P)$ at P in frame Φ, as shown in Fig. 5.2. Thus, by an argument similar to that used for a system of particles, it follows that *at each instant t the center of mass is the unique point with respect to which* (5.13) *holds*. Indeed, its unique location in a rigid body is determined relative to a body reference frame $\varphi = \{O; \mathbf{i}_k\}$ with respect to which the position vectors in (5.12) and (5.13) are independent of time. Therefore, the center of mass of a rigid body is a unique point determined by the geometry and material content of that body alone—it always occupies the same place in the body reference frame relative to which (5.13) holds. The center of mass moves with the body, and, of course, its position vector with respect to different spatial reference frames will naturally vary.

Exercise 5.1. (a) Show that the definition (5.12) for the center of mass of a body is independent of the choice of reference origin. (b) Prove that the center of mass is the *unique* point for which (5.13) holds. □

The center of mass of a homogeneous body often may be easily identified. For a homogeneous body, the constant mass density may be eliminated from (5.12) to obtain at time t the familiar *formula for the geometrical centroid of $\mathscr{B}$*:

$$V(\mathscr{B})\mathbf{x}^*(\mathscr{B}, t) = \int_{\mathscr{B}} \mathbf{x}(P, t)\, dV(P), \tag{5.14}$$

wherein $V(\mathscr{B})$ is the material volume of $\mathscr{B}$. *Thus, the mass center of a homogeneous body coincides with its centroid.* Of course, very often, the centroid is easy to identify.

In general, the center of mass need not be a place occupied by a particle of $\mathscr{B}$. It is clear, for example, that the center of mass of a homogeneous, circular cylindrical tube is at the geometrical center on its axis—plainly a place that is not occupied by a particle of the tube. On the other hand, the center of mass of a similar solid cylinder has the same location. These assertions are evident from symmetry considerations. Nevertheless, it is instructive to review integration methods typically involved in the use of (5.12) or (5.14), because similar techniques are used for both homogeneous and nonhomogeneous bodies for which symmetry may not be so evident.

Example 5.2. (i) Compute the location of the center of mass of the homogeneous, cylindrical tube described in Fig. 5.3. (ii) Find the center of mass when the density varies linearly from the constant value ρ_o at $z = 0$ to $2\rho_o$ at $z = \ell$.

Solution of (i). The circular tube shown in Fig. 5.3 has an inner radius r_i, outer radius r_o, and length ℓ. Because the material is homogeneous, the center of mass is at the centroid determined by (5.14) in which

$$V(\mathscr{B}) = \pi\ell\left(r_o^2 - r_i^2\right) \tag{5.15a}$$

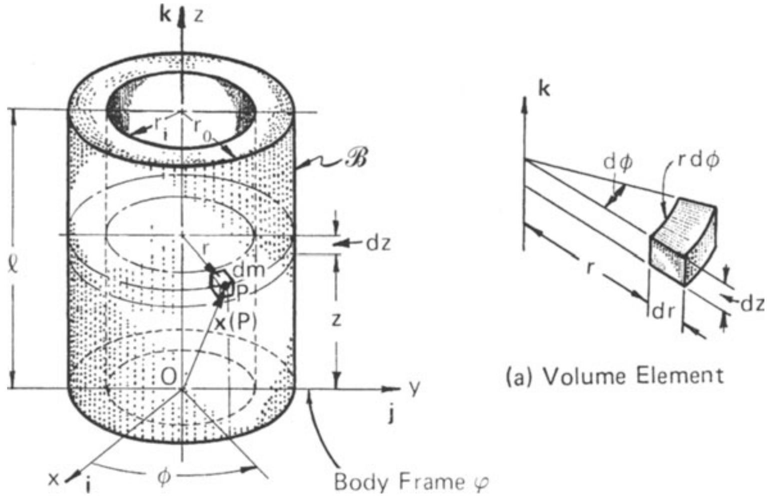


Figure 5.3. Geometry for determination of the center of mass of a tube.

is the material volume of the tube. It is natural to introduce cylindrical coordinates in the imbedded frame $\varphi = \{O; \mathbf{i}_k\}$, whose origin is at the base of the tube. Then the position vector $\mathbf{x}(P, t) \equiv \mathbf{x}(P)$ of a particle P of $\mathscr{B}$ and the elemental volume at P in Fig. 5.3 may be expressed as $\mathbf{x}(P) = r(\cos\phi\mathbf{i} + \sin\phi\mathbf{j}) + z\mathbf{k}$ and $dV(P) = rdrd\phi dz$. Hence, with (5.14), the center of mass location $\mathbf{x}^*(\mathscr{B}, t) \equiv \mathbf{x}^*(\mathscr{B})$ in φ is given by

$$V(\mathscr{B})\mathbf{x}^*(\mathscr{B}) = \int_0^{2\pi}\int_0^{\ell}\int_{r_i}^{r_o} (r\cos\phi\mathbf{i} + r\sin\phi\mathbf{j} + z\mathbf{k}) r dr dz d\phi. \quad (5.15b)$$

The first two integrals in the angle ϕ vanish. Therefore, as anticipated from the symmetry, the center of mass lies on the axis of the tube. Integration of the remaining term in (5.15b) and use of (5.15a) yields $\mathbf{x}^*(\mathscr{B}) = \ell/2\mathbf{k}$, that is, the center of mass is at the center of the void. We notice also that $\mathbf{x}^*$ is independent of the radii of the tube, so the location of the center of mass in φ is the same for all radii. In particular, for a solid cylinder for which $r_i = 0$, $\mathbf{x}^*(\mathscr{B}) = \ell/2\mathbf{k}$ holds as well. Of course, whatever reference point may be used, the center of mass of the rigid tube remains at the same central position; and as the tube moves in space, its center of mass retains its central location in the moving, imbedded frame.

In problems of this kind it is often easier to simplify the integration in (5.15b) by use of the method of slices. The application of this method to the previous homogeneous problem is left as a review exercise for the reader to show that $\ell\mathbf{x}^*(\mathscr{B}) = \int_0^\ell z\mathbf{k}dz$, which yields $\mathbf{x}^*(\mathscr{B}) = \ell/2\mathbf{k}$, as before. We next apply this method to solve the variable density problem.

Solution of (ii). We are given that the mass density of the tube varies linearly from ρ_o at $z = 0$ to $2\rho_o$ at $z = \ell$, and hence $\rho = \rho_o\,(1 + z/\ell)$. Because ρ varies only along the tube's length, the simultaneous geometrical and mass distribution symmetries about the tube's axis imply that the center of mass is on the axis. Therefore, $x^* = y^* = 0$ and only the z^* component need be found. Hence, (5.12) yields

$$m(\mathscr{B})z^* = \int_{\mathscr{B}} z\,dm. \tag{5.15c}$$

The method of slices shows that for the annular ring in Fig. 5.3 the volume element $dV = A\,dz$, where $A = \pi\left(r_o^2 - r_i^2\right)$ is the constant area of the ring. The mass is then found by (5.10):

$$m(\mathscr{B}) = \rho_o A \int_0^{\ell} \left(1 + \frac{z}{\ell}\right) dz = \frac{3}{2} A\rho_o \ell,$$

and the right-hand side of (5.15c) becomes

$$\int_{\mathscr{B}} z\,dm = \rho_o A \int_0^{\ell} z\left(1 + \frac{z}{\ell}\right) dz = \frac{5}{6}\rho_o A \ell^2.$$

Therefore, by (5.15c), the center of mass is on the axis of the tube at $z^* = 5\ell/9$ from its base at O. Clearly, the center of mass is not the centroid, which is located at $z^* = \ell/2$ in accordance with (5.14). □

5.2.3.3. Momentum of the Center of Mass of a Rigid Body

We shall now derive an important result relating the momentum of a rigid body to the momentum of its center of mass. The body is generally in motion in Φ with momentum defined by (5.11), in which $\mathbf{v}(P, t) \equiv \dot{\mathbf{x}}(P, t)$. The motion of the center of mass is defined by (5.12), and hence $\mathbf{v}^*(\mathscr{B}, t) \equiv \dot{\mathbf{x}}^*(\mathscr{B}, t)$ defines the velocity of the center of mass. Thus, differentiation of (5.12) with respect to time and use of (5.11) for a rigid body yields the important result

$$\mathbf{p}^*(\mathscr{B}, t) \equiv m(\mathscr{B})\mathbf{v}^*(\mathscr{B}, t) = \int_{\mathscr{B}} \mathbf{v}(P, t)\,dm(P) = \mathbf{p}(\mathscr{B}, t). \tag{5.16}$$

The vector $\mathbf{p}^*(\mathscr{B}, t)$ defined by the first equation in (5.16) is the momentum of a fictitious particle of mass $m(\mathscr{B})$ that moves with the velocity $\mathbf{v}^*(\mathscr{B}, t)$ of the center of mass. Our imaginary particle is sometimes called the *center of mass particle* (or *object*). Hence, $\mathbf{p}^*$ is called briefly the *momentum of the center of mass*. The result (5.16) thus shows that *the momentum of a rigid body is equal to the momentum of its center of mass*: $\mathbf{p}(\mathscr{B}, t) = \mathbf{p}^*(\mathscr{B}, t)$.

Moreover, differentiation of (5.13) in the spatial frame Φ yields

$$\int_{\mathscr{B}} \dot{\boldsymbol{\rho}}(P, t)\,dm\,(P) = \mathbf{0}. \tag{5.17}$$

Hence, the momentum of a body relative to its center of mass in Φ is always zero.

The definitions (5.10) and (5.11) may be readily extended to a deformable body whose volume and density may vary with time, and for which similar center of mass properties hold at each instant. In this case, however, greater care must be exercised in differentiation of the integrals in (5.12) and (5.13) because the region $\mathscr{B}$ of the integration over the deforming body varies with time; and the location of the center of mass will vary with the deformation. Of course, the body region $\mathscr{B}$ for a rigid body is the same for all time. Deformable bodies are not studied in this text.

5.3. Moment of a Vector About a Point

The moment of a vector about a point occurs frequently in future work. This operation is first defined in general terms; and the transformation rule that describes the effect of a change of the reference point follows. The familiar idea of the moment of a force about a point is then reviewed; and the moment of momentum vector is introduced in the next section.

We start with the general idea. Let $\mathbf{x}_Q(P)$ be the position vector of a point P from a point Q, and let $\mathbf{u}(P)$ denote a vector quantity at P in Fig. 5.4. *The moment about Q of the vector $\mathbf{u}(P)$ is a vector entity $\boldsymbol{\mu}_Q(P)$ defined by the rule*

$$\boldsymbol{\mu}_Q(P) \equiv \mathbf{x}_Q(P) \times \mathbf{u}(P). \tag{5.18}$$

This vector is perpendicular to both $\mathbf{x}_Q(P)$ and $\mathbf{u}(P)$. It is represented in Fig. 5.4 as a vector line with an arrow turning about it in the right-hand sense of (5.18).

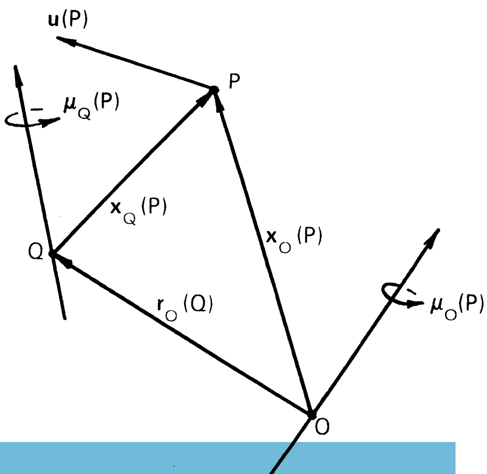


Figure 5.4. Schema for the moment of a vector about a point.

5.3.1. Reference Point Transformation Rule

The vector $\boldsymbol{\mu}_Q(P)$ depends on the choice of Q. The moment of the same vector $\mathbf{u}(P)$ about another reference point O in Fig. 5.4 is given by

$$\boldsymbol{\mu}_O(P) = \mathbf{x}_O(P) \times \mathbf{u}(P),$$

where $\mathbf{x}_O(P)$ is the position vector of P from O. It is seen in Fig. 5.4 that $\mathbf{x}_O(P) = \mathbf{r}_{OQ} + \mathbf{x}_Q(P)$, in which $\mathbf{r}_{OQ} \equiv \mathbf{r}_O(Q)$ is the position vector of Q from O. Hence, substitution of this relation into the previous equation and use of (5.18) yields *the transformation rule relating the moments of the same vector* $\mathbf{u}(P)$ *about the points* O *and* Q:

$$\boldsymbol{\mu}_O(P) = \boldsymbol{\mu}_Q(P) + \mathbf{r}_{OQ} \times \mathbf{u}(P). \tag{5.19}$$

It is seen that $\boldsymbol{\mu}_O(P) = \boldsymbol{\mu}_Q(P)$ when and only when the nonzero vector $\mathbf{r}_{OQ}$ is parallel to $\mathbf{u}(P)$.

5.3.2. Moment of a Force About a Point

We recall the familiar idea of the moment of a force about a point. In Fig. 5.4, let $\mathbf{u}(P) \equiv \mathbf{F}(P)$ denote a force acting on a particle P whose position vector from point Q is $\mathbf{x}_Q(P)$, and write $\boldsymbol{\mu}_Q(P) \equiv \mathbf{M}_Q(P)$. Then, by (5.18), *the moment about* Q *of the force* $\mathbf{F}(P)$ *is the vector* $\mathbf{M}_Q(P)$ *defined by the rule*

$$\mathbf{M}_Q(P) \equiv \mathbf{x}_Q(P) \times \mathbf{F}(P). \tag{5.20}$$

The moment vector is a measure of the turning or twisting effect of the force about the reference point. Hence, the moment of a force is also called the *torque*; its physical dimensions are $[\mathbf{M}_Q] = [FL]$.

If $\mathbf{a}$ is a vector from Q to *any* point A on the action line of $\mathbf{F}(P)$, the vector defined by $\mathbf{r} \equiv \mathbf{x}_Q(P) - \mathbf{a}$ is parallel to $\mathbf{F}(P)$. It thus follows from (5.20) that $\mathbf{M}_Q(P) = \mathbf{a} \times \mathbf{F}(P)$ holds for any point A on the action line of the force acting on P. *Therefore, the moment of the force* $\mathbf{F}(P)$ *about* Q *is independent of the actual point of application of the force along its line of action*; *and hence only the component of* $\mathbf{x}_Q(P)$ *that is perpendicular to* $\mathbf{F}(P)$ *determines the torque of* $\mathbf{F}(P)$ *about* Q. Thus, in abbreviated notation, the magnitude $|\mathbf{M}_Q| = |\mathbf{x}_Q|\,|\mathbf{F}|$ $\sin < \mathbf{x}_Q, \mathbf{F} >$ of the moment vector $\mathbf{M}_Q$ is equal to the product of the magnitude of the force $F \equiv |\mathbf{F}|$ and the perpendicular distance $d \equiv |\mathbf{x}_Q| \sin < \mathbf{x}_Q, \mathbf{F} >$ from Q to the action line of $\mathbf{F}$, where $< \mathbf{x}_Q, \mathbf{F} >$ denotes the smaller angle between $\mathbf{x}_Q$ and $\mathbf{F}$, as usual; that is, $|\mathbf{M}_Q| = Fd$, a familiar elementary rule.

The definition (5.20) may be applied to each particle P_k of a system of particles. In this case, the total, or resultant, moment about a point Q of the several forces $\mathbf{F}_k = \mathbf{F}(P_k)$ that act on a system of n particles $\beta = \{P_k\}$ is defined by the sum of the moments about Q of all of the forces $\mathbf{F}_k$ that act on β:

$$\mathbf{M}_Q(\beta) \equiv \sum_{k=1}^{n} \mathbf{M}_Q(P_k) = \sum_{k=1}^{n} \mathbf{x}_{Qk} \times \mathbf{F}_k, \tag{5.21}$$

where $\mathbf{x}_{Qk} \equiv \mathbf{x}_Q(P_k)$ is the position vector of particle P_k from Q; and the total, or resultant, force is defined by $\mathbf{F}(\beta) \equiv \sum_{k=1}^{n} \mathbf{F}_k$.

The same rule may be applied to determine the total moment about a point Q of all the concentrated and distributed forces that act on a rigid body $\mathscr{B}$. For the elemental force distribution $d\mathbf{F}_d(P)$ acting on a material parcel at P, for example, the total torque about a point Q of the distributed force is defined by

$$\mathbf{M}_Q(\mathscr{B}) \equiv \int_{\mathscr{B}} \mathbf{x}_Q(P) \times d\mathbf{F}_d(P), \tag{5.22}$$

where $\mathbf{x}_Q(P)$ is the position vector from Q to the parcel at P. A formula similar to (5.21) holds for n concentrated forces $\mathbf{F}_k(\mathscr{B})$ acting on $\mathscr{B}$.

Now consider the point transformation rule. Clearly, the turning effect of a force about another reference point at O in Fig. 5.4 will be different from that about Q. The transformation rule (5.19) shows that *the moment of the same force about the reference point O is related to its moment* (5.20) *about the point Q by the rule*

$$\mathbf{M}_O(P) = \mathbf{M}_Q(P) + \mathbf{r}_{OQ} \times \mathbf{F}(P). \tag{5.23}$$

We recall that $\mathbf{r}_{OQ}$ is the position vector of point Q from O; and hence $\mathbf{r}_{OQ} \times \mathbf{F}(P)$ is the moment about O of the total force as though it were placed at Q.

The same point transformation rule applies to (5.21) and (5.22); thus,

$$\mathbf{M}_O(\mathscr{B}) = \mathbf{M}_Q(\mathscr{B}) + \mathbf{r}_{OQ} \times \mathbf{F}(\mathscr{B}), \tag{5.24}$$

where the total force acting on $\mathscr{B}$, namely, $\mathbf{F}(\mathscr{B}) = \mathbf{F}_d(\mathscr{B}) + \mathbf{F}_c(\mathscr{B})$, is the sum of the total distributed force $\mathbf{F}_d(\mathscr{B}) \equiv \int_{\mathscr{B}} d\mathbf{F}_d(P)$ and the total of all concentrated forces $\mathbf{F}_c(\mathscr{B}) \equiv \sum_{k=1}^{n} \mathbf{F}_k(\mathscr{B})$. Also, $\mathbf{M}_O(\mathscr{B})$ and $\mathbf{M}_Q(\mathscr{B})$ are the total moments about points O and Q of all of these forces. Therefore, by (5.24), *the total moment of force about a point O is equal to the total moment of force about point Q plus the moment about O of the total force placed at Q.*

The rule (5.24) relates the moments of the force about any two points. In particular, if O is the center of mass at C, then $\mathbf{r}_{OQ} = -\mathbf{r}_{QC} = -\mathbf{x}_Q^*(\mathscr{B})$ and (5.24) is written

$$\mathbf{M}_Q(\mathscr{B}) = \mathbf{M}_Q^*(\mathscr{B}) + \mathbf{M}_C(\mathscr{B}), \tag{5.25}$$

in which $\mathbf{M}_C(\mathscr{B})$ is the total moment of force about the center of mass and $\mathbf{M}_Q^*(\mathscr{B})$ is the moment about Q of the total force placed at the center of mass:

$$\mathbf{M}_Q^*(\mathscr{B}) = \mathbf{x}_Q^*(\mathscr{B}) \times \mathbf{F}(\mathscr{B}). \tag{5.26}$$

Thus, in physical terms (5.25) shows that *the total moment of force about any point Q is equal to the moment about Q of the total force placed at the center of mass plus the total moment of force about the center of mass.*

5.3.3. Equipollent Force Systems

Now consider two systems of forces and torques. These systems are said to be *equipollent* if and only if they have the same total force and the same total torque about the same point. That is, a system A with total force $\mathbf{F}^A$ and total torque $\mathbf{M}_Q^A$ about a point Q is equipollent to a system B with total force $\mathbf{F}^B$ and total torque $\mathbf{M}_Q^B$ about the same point Q when and only when

$$\mathbf{F}^A = \mathbf{F}^B \quad \text{and} \quad \mathbf{M}_Q^A = \mathbf{M}_Q^B. \tag{5.27}$$

It follows from the point transformation rule (5.23) or (5.24) that if two force systems are equipollent with respect to a point Q, they are equipollent with respect to any other point O.

We know from (5.20) that the moment about Q of a *single* force is perpendicular to the force and to the position vector from Q to its point of application. In general, however, this is not true for a *system* of forces—the total torque $\mathbf{M}_Q$ about a point Q of a system of forces generally is not perpendicular to the total force acting on the system. Here we focus on the special case when the system of forces A is such that $\mathbf{M}_Q^A \cdot \mathbf{F}^A = 0$; then, by (5.27) the same holds for the equipollent system B. Consider, for example, a distributed system of forces $\mathbf{F}^B(\mathscr{B}) = \mathbf{F}_d(\mathscr{B})$ with a total torque $\mathbf{M}_Q^B(\mathscr{B})$ equal to (5.22) such that $\mathbf{M}_Q^B(\mathscr{B}) \cdot \mathbf{F}^B(\mathscr{B}) = 0$. Then, this system is equipollent to a single force $\mathbf{F}^A(\mathscr{B}) = \mathbf{P}$ located at distance from Q such that

$$\mathbf{P} = \int_{\mathscr{B}} d\mathbf{F}_d(P) = \mathbf{F}_d(\mathscr{B}), \tag{5.28}$$

$$\mathbf{M}_Q^A(\mathscr{B}) \equiv \bar{\mathbf{x}}_Q \times \mathbf{P} = \int_{\mathscr{B}} \mathbf{x}_Q(P) \times d\mathbf{F}_d(P) = \mathbf{M}_Q^B(\mathscr{B}), \tag{5.29}$$

where the locus of the unknown vector $\bar{\mathbf{x}}_Q$ from Q traces the line of action of $\mathbf{P}$. Of course, $\bar{\mathbf{x}}_Q$ is necessarily perpendicular to $\mathbf{M}_Q^B(\mathscr{B})$. Now, bearing in mind that only the component of $\bar{\mathbf{x}}_Q$ perpendicular to the line of action of the force $\mathbf{P}$ influences the torque about Q, the relation (5.29) determines the place $\bar{\mathbf{x}}_Q^*$, say, on the line from Q perpendicular to $\mathbf{P}$, called the *center of force with respect to Q*, through which the force $\mathbf{P}$ must act to produce the same total torque about Q. Of course, the center of force with respect to another moment center at O, say, though also on the line of action of $\mathbf{P}$, will be different. Notice that (5.29) may be written as $\bar{\mathbf{x}}_Q \times \mathbf{F}_d(\mathscr{B}) = \int_{\mathscr{B}} \mathbf{x}_Q(P) \times d\mathbf{F}_d(P)$. Specifically, for any system of planar forces or for any system of parallel forces, the total moment of the forces about an arbitrary point is plainly perpendicular to the total force; therefore, in accordance with (5.28) and (5.29), each of these systems may be reduced to a single force

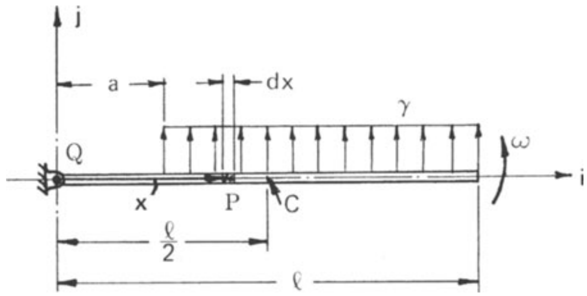


Figure 5.5. A homogeneous, thin rigid rod under a uniformly distributed load.

acting at its center of force. Clearly, for a system of discrete forces, the procedure is similar. (See Problem 5.35.) For further discussion on the reduction of force systems for the general case see the referenced texts on statics.

Example 5.3. A homogeneous, thin rigid rod of length ℓ is supported at one end by a smooth hinge at Q and is subjected to a load of magnitude γ per unit length distributed uniformly over the region $[a, \ell]$ shown in Fig. 5.5. (i) Find the force system with respect to Q that is equipollent to the distributed load. (ii) Determine the moment of the distributed load about the center of mass of the rod at C.

Solution of (i). The total force $\mathbf{F}^A = \mathbf{P}$ equipollent to the distributed load $\mathbf{F}^B = \mathbf{F}_d(\mathscr{B})$ for which $d\mathbf{F}_d(P) = \gamma dx\mathbf{j}$ is given by (5.28). Thus,

$$\mathbf{P} = \int_a^\ell \gamma dx\mathbf{j} = \gamma(\ell - a)\mathbf{j}. \tag{5.30a}$$

The total moment of the distribution about the hinge point Q is given by (5.22) in which $\mathbf{x}_Q(P) = x\mathbf{i} + y\mathbf{j}$;

$$\mathbf{M}_Q^B(\mathscr{B}) = \int_a^\ell x\mathbf{i} \times \gamma dx\mathbf{j} = \frac{\gamma}{2}(\ell^2 - a^2)\mathbf{k}. \tag{5.30b}$$

Of course, for the system B only the component $x\mathbf{i}$ of $\mathbf{x}_Q(P)$ that is perpendicular to the distribution contributes to the torque about Q.

Notice that this is a system of parallel forces, and $\mathbf{M}_Q^B(\mathscr{B})$ is perpendicular to $\mathbf{P}$. Thus, with $\bar{\mathbf{x}}_Q = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$ and (5.30a), we may write $\mathbf{M}_Q^A = \bar{\mathbf{x}}_Q \times \mathbf{P} = \bar{x}\gamma(\ell - a)\mathbf{k}$. Here we see that for the system A only the component $\bar{x}\mathbf{i}$ of $\bar{\mathbf{x}}_Q$ that is perpendicular to $\mathbf{P}$ contributes to the torque about Q. Thus, with (5.30b), (5.29) yields $\bar{x} = \frac{1}{2}(\ell + a)$; that is, with respect to Q, the center of force $\bar{\mathbf{x}}_Q^*$ for $\mathbf{P}$ is at

$$\bar{\mathbf{x}}_Q^* = \tfrac{1}{2}(\ell + a)\mathbf{i} = \left[a + \tfrac{1}{2}(\ell - a)\right]\mathbf{i}. \tag{5.30c}$$

The line of action of $\mathbf{P}$ is traced by $\bar{\mathbf{x}}_Q = \bar{\mathbf{x}}_Q^* + \bar{y}\mathbf{j}$ for all values of $\bar{y}$. Equation (5.30c) shows that the center of force for the uniformly distributed load is at the geometrical center of the loaded portion of the rod in Fig. 5.5. The force system consisting of the single force $\mathbf{P}$ acting at the center of force $\bar{\mathbf{x}}_Q^*$ in (5.30c) is equipollent to the assigned uniformly distributed force system; it consists of the same total force (5.30a) and produces the same total moment about Q in (5.30b).

Solution of (ii). The moment of the same distribution about point C may be found from the transformation rule (5.25). In accordance with (5.26), consider the load $\mathbf{P}$ placed at the center of mass of the homogeneous rod at $\mathbf{x}_Q^*(\mathscr{B}) = \frac{1}{2}\ell\mathbf{i}$, and recall (5.30a) to determine $\mathbf{M}_Q^*(\mathscr{B}) = \mathbf{x}_Q^* \times \mathbf{P} = \frac{1}{2}\gamma\ell(\ell - a)\mathbf{k}$. Then by (5.25) and (5.30b), we find

$$\mathbf{M}_C^B(\mathscr{B}) = \mathbf{M}_Q^B(\mathscr{B}) - \mathbf{M}_Q^*(\mathscr{B}) = \gamma\frac{a}{2}(\ell - a)\mathbf{k}. \tag{5.30d}$$

The same result may be obtained by our noting that the equipollent system consists of the single force (5.30a) acting at $\bar{\mathbf{x}}_Q^*$ in (5.30c). Hence, its moment about C at $\mathbf{x}_Q^* = \frac{1}{2}\ell\mathbf{i}$ is given by $(\bar{\mathbf{x}}_Q^* - \mathbf{x}_Q^*) \times \mathbf{P} = \frac{1}{2}\gamma a(\ell - a)\mathbf{k}$, which is the same as (5.30d). □

Finally, notice that if $\mathbf{F}(\mathscr{B}) \equiv \mathbf{0}$, then (5.24) shows that $\mathbf{M}_O(\mathscr{B}) = \mathbf{M}_Q(\mathscr{B})$ and hence the resultant moment is independent of the choice of reference point. In this case, the force system is called a *couple.* A force system consisting of a noncollinear pair of equal and oppositely directed forces is a familiar example. If both $\mathbf{F}(\mathscr{B}) \equiv \mathbf{0}$ and $\mathbf{M}_O(\mathscr{B}) \equiv \mathbf{0}$, then $\mathbf{M}_Q(\mathscr{B}) \equiv \mathbf{0}$ as well. In this case the resultant moment with respect to any reference point vanishes, and the force system is said to be *equipollent to zero*. It is an exercise for the reader to show that any force system can be reduced to a single force acting at an arbitrary point together with a couple. A torque $\mathbf{M}_Q$ induced by essentially twisting a body about an axis at a point Q is called a *concentrated couple*. Tightening a screw in a wooden body by twisting the screw about its axis is a physical example that may be modeled as a concentrated couple acting on the wooden body. We may think of a concentrated couple at Q as a pair of equal and opposite, noncollinear forces of very large intensity and having a very small moment arm, the perpendicular distance between the force pair, at Q.

None of the foregoing results for a body require that it be rigid. Moreover, although explicit dependence on time t is not indicated, it is clear that all of the foregoing vector entities also may vary with time. Another useful application of the moment of a vector about a point follows.

5.4. Moment of Momentum

Here we introduce an important vector quantity called the moment of momentum. The moment of momentum of a particle, a system of particles, and a body are defined in turn.

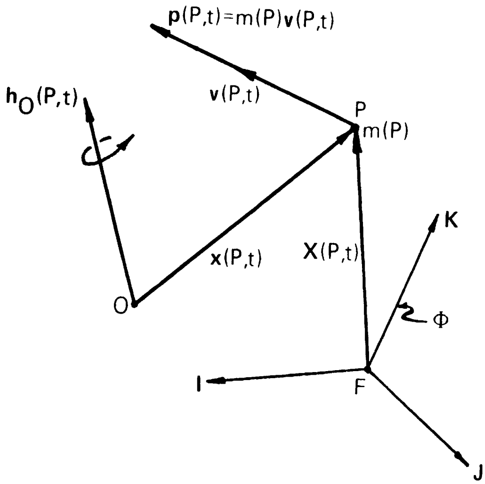


Figure 5.6. Schema for the moment about a point O of the momentum of a particle P relative to frame Φ.

5.4.1. Moment of Momentum of a Particle

Let $\mathbf{x}_O(P,t) = \mathbf{x}(P,t)$ denote the position vector of a particle P from an arbitrary spatial point O in a reference frame $\Phi = \{F; \mathbf{I}_k\}$ shown in Fig. 5.6. The velocity of P relative to Φ is given by $\mathbf{v}(P,t) = \dot{\mathbf{X}}(P,t)$, where $\mathbf{X}(P,t)$ is the position vector of P from F, as usual; and the momentum of P is defined by (5.2). In accordance with (5.18), *the moment about point O of the momentum of P relative to Φ, denoted by $\mathbf{h}_O(P,t)$, is a vector-valued function of time defined by*

$$\mathbf{h}_O(P,t) \equiv \mathbf{x}_O(P,t) \times \mathbf{p}(P,t) = \mathbf{x}(P,t) \times m(P)\,\mathbf{v}(P,t). \tag{5.31}$$

Notice that two reference points are involved in this definition, the origin F of frame Φ and the spatial point O. The moment about reference points O and Q of the same momentum vector $\mathbf{p}(P,t)$ are related by $\mathbf{h}_O(P,t) = \mathbf{h}_Q(P,t) + \mathbf{r}_{OQ} \times \mathbf{p}(P,t)$ in accordance with the transformation rule (5.19).

The moment of momentum is also known as the *angular momentum*, a term frequently used in other texts. It follows from (5.31) that moment of momentum has the physical dimensions $[\mathbf{h}_O] \equiv [H] = \left[ML^2T^{-1}\right]$.

5.4.2. Moment of Momentum of a System of Particles

Each particle of a system $\beta = \{P_k\}$ of n particles has a moment of momentum about point O given by (5.31), so that $\mathbf{h}_{Ok} \equiv \mathbf{h}_O(P_k,t) = \mathbf{x}_{Ok} \times \mathbf{p}_k$, where $\mathbf{x}_{Ok} \equiv \mathbf{x}_O(P_k,t)$ is the position vector of P_k from O, and $\mathbf{p}_k = m_k\mathbf{v}_k = m_k\dot{\mathbf{X}}_k$ is its momentum relative to Φ. *Relative to a frame $\Phi = \{F; \mathbf{I}_k\}$, the moment of* *momentum* $\mathbf{h}_O(\beta, t)$ *of a system of particles about a point* O *in* Φ *is defined by*

$$\mathbf{h}_O(\beta, t) \equiv \sum_{k=1}^{n} \mathbf{h}_{Ok} = \sum_{k=1}^{n} \mathbf{x}_{Ok} \times \mathbf{p}_k = \sum_{k=1}^{n} \mathbf{x}_{Ok} \times m_k \mathbf{v}_k. \tag{5.32}$$

Example 5.4. At an instant of interest t_0, the three particles described in Example 5.1, page 9, are situated at $\mathbf{x}_{O1} = (0, 0, -1)$, $\mathbf{x}_{O2} = (-3, -2, 2)$, and $\mathbf{x}_{O3} = (6, -2, -4)$ from a point O located at $\mathbf{B} = (2, -1, 3)$ from F in frame $\Phi = \{F; \mathbf{I}_k\}$. Compute the moment of momentum of the system about O at t_0.

Solution. The moment about O of the momentum of a particle is determined by (5.31). Thus, for the system of three particles with momenta (5.9a), we find

$$\mathbf{h}_{O1} = \mathbf{x}_{O1} \times \mathbf{p}_1 = -\mathbf{K} \times mv\,(6\mathbf{I} - 7\mathbf{J}) = mv\,(-7\mathbf{I} - 6\mathbf{J}),$$

$$\mathbf{h}_{O2} = \mathbf{x}_{O2} \times \mathbf{p}_2 = 2mv \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -3 & -2 & 2 \\ 0 & 2 & -3 \end{vmatrix} = 2mv\,(2\mathbf{I} - 9\mathbf{J} - 6\mathbf{K}),$$

$$\mathbf{h}_{O3} = \mathbf{x}_{O3} \times \mathbf{p}_3 = 3mv \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 6 & -2 & -4 \\ 2 & -1 & -2 \end{vmatrix} = 6mv\,(2\mathbf{J} - \mathbf{K}).$$

Then, by (5.32), the moment of momentum of the system about point O in Φ is

$$\mathbf{h}_O(\beta, t_0) = \mathbf{h}_{O1} + \mathbf{h}_{O2} + \mathbf{h}_{O3} = -3mv\,(\mathbf{I} + 4\mathbf{J} + 6\mathbf{K}).$$

□

Exercise 5.2. What is the moment of momentum about F in Φ for the system of particles described above? Derive the reference point transformation rule for the moment of momentum of a system of particles. □

5.4.3. Moment of Momentum of a Body

Consider a body $\mathscr{B}$ in Fig. 5.7 and recall that the momentum in Φ of a parcel of mass $dm(P)$ of $\mathscr{B}$ at P is defined by $\mathbf{v}(P, t)\,dm(P)$. Thus, *for a body* $\mathscr{B}$ *the moment of momentum about a point* O *in* $\Phi = \{F; \mathbf{I}_k\}$ is defined by

$$\mathbf{h}_O(\mathscr{B}, t) \equiv \int_{\mathscr{B}} \mathbf{x}_O\,(P, t) \times \mathbf{v}\,(P, t)\,dm\,(P). \tag{5.33}$$

Herein $\mathbf{x}_O\,(P, t) = \mathbf{x}\,(P, t)$ is the position vector of the material point P from the point O in Φ and $\mathbf{v}(P, t) = \dot{\mathbf{X}}(P, t)$ is its velocity relative to Φ. While in this book we shall be concerned only with bodies that are rigid, the definitions (5.11) for the momentum and (5.33) for the moment of momentum of a body hold more generally for all deformable solid and fluid bodies.

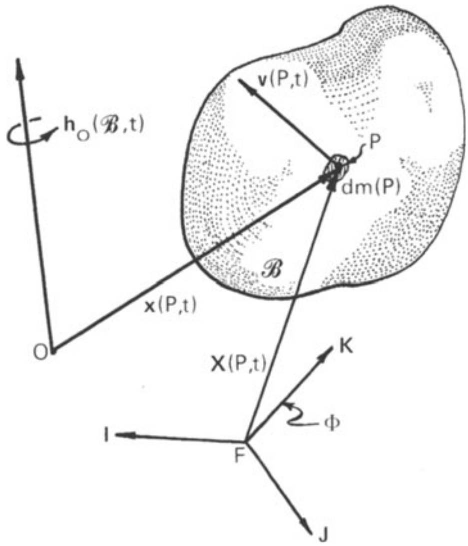


Figure 5.7. Schema for the Moment about a point O of the momentum of a body $\mathscr{B}$ relative to frame Φ.

Example 5.5. Find the moment about O in Φ of the momentum of a body having a constant translational acceleration relative to Φ.

Solution. Since $\mathbf{a}(P,t) = \mathbf{a}^*(\mathscr{B})$ is a constant vector for all particles of $\mathscr{B}$, the translational velocity $\mathbf{v}(P,t) = \mathbf{v}^*(\mathscr{B},t) = \mathbf{a}^*t + \mathbf{v}_0^*(\mathscr{B})$ is also the same for all particles of $\mathscr{B}$, where $\mathbf{v}(P,0) = \mathbf{v}_0^*(\mathscr{B})$ is the translational velocity of the center of mass of $\mathscr{B}$ initially. Hence, (5.33) may be written as

$$\mathbf{h}_O(\mathscr{B},t) = \int_{\mathscr{B}} \mathbf{x}_O(P,t)\, dm(P) \times \mathbf{v}^*(\mathscr{B},t).$$

Recalling (5.12) and (5.16), we obtain

$$\mathbf{h}_O(\mathscr{B},t) = \mathbf{x}_O^*(\mathscr{B},t) \times m(\mathscr{B})\,\mathbf{v}^*(\mathscr{B},t) = \mathbf{x}_O^*(\mathscr{B},t) \times \mathbf{p}^*(\mathscr{B},t),$$

in which $\mathbf{x}_O^*(\mathscr{B},t)$ is the position vector of the center of mass from O. This equation has the same form as (5.31) for a single particle. Thus, with respect to an arbitrary point O, the moment of momentum of a body having a uniform translational acceleration is equal to the moment of momentum of its center of mass. □

The forgoing concepts on the mass, momentum, and moment of momentum of a particle, a system of particles, and a body have been assembled here for future convenience and to emphasize their parallel definitions and structure. These ideas, including the notion of the center of mass of a system and a body, will also be helpful in our introduction and discussion of the basic laws of mechanics to be studied next. Their main thrust, however, will appear later as the theory unfolds leading eventually to the analysis of the motion of a system of particles and of a rigid body.

5.5. Newton's Laws of Motion

The structure of classical dynamics rests upon three foundation axioms introduced by Sir Isaac Newton in 1687. These are known as Newton's laws of motion. In their original form, however, Newton's principles are inadequate for the study of the motion of a rigid or a deformable body. These applications require a brilliant generalization introduced by Leonhard Euler in 1750 and thereafter. Here we follow the course of classical developments and begin with an introduction to the foundation principles of mechanics for a particle.* Principles for systems and continua are discussed briefly below and in greater detail in later chapters. In the meanwhile, we shall see in the following two chapters that our subject is rich with interesting and useful results that derive from the following principles of classical mechanics.

1. **The first law of motion:** *In every material universe, the motion of a particle in a preferential reference frame Φ is determined by the action of forces whose total vanishes for all times when and only when the velocity of the particle is constant in Φ. That is, a particle initially at rest or in uniform motion in the preferential frame Φ continues in that state unless compelled by forces to change it.*
2. **The second law of motion:** *There exists a material universe, called the world, wherein the total force* $\mathbf{F}(P, t)$ *exerted on a particle* P *in the preferential frame* Φ *is equal to the time rate of change of the momentum of* P *in* Φ:
$$\mathbf{F}(P, t) = \frac{d\mathbf{p}(P, t)}{dt} = \frac{d}{dt}\left[m(P)\,\mathbf{v}(P, t)\right]. \tag{5.34}$$
3. **The law of mutual action:** *To every action force* $\mathbf{A}$ *there corresponds an equal and oppositely directed reaction force* $\mathbf{R}$. *That is, the mutual actions of two particles, one on the other, are oppositely directed vectors*: $\mathbf{R} = -\mathbf{A}$.

These foundation principles characterize a material universe that is intended to model the physical world, the real world in which we live. Indeed, a large body of practical experience and the test of many experiments have shown that these

* In the statement of his laws, Newton uses the term "body" or "bodies". The least of these, however, is a single particle; and we shall see later on that for a body of finite size the laws may be stated in terms of its center of mass particle. Moreover, we recall that Newton's theory focuses principally on its applications to the motions of celestial bodies whose dimensions are small compared with their enormous distances of separation, so heavenly bodies are usually modeled as particles.

laws model very well mechanical phenomena in the real world. Therefore, they are employed universally with confidence in their predictive value. On the other hand, there may exist other material universes where these rules do not hold, or they hold only approximately. We shall say more about this later on. Let us look more closely at their content.

5.5.1. The Material Universe and Forces

In analytical terms, the material universe is the set $\mathscr{U} = \{O_k\}$ whose elements O_k are material objects; and a body $\mathscr{B}$ is a subset of $\mathscr{U}$, the least of which consists of a single particle P. Forces can exist only in the presence of pairs of bodies. A force acts on a body $\mathscr{B}$ only when there exists another body $\hat{\mathscr{B}}$ separate from $\mathscr{B}$ which is the source of the action. Moreover, the action of a force in one direction is not the same as its action in another direction. Thus, force is a vector-valued entity defined on pairs of separate bodies in $\mathscr{U}$.

The forces of interaction between pairs of material objects are classified as *contact forces* and *body forces.* Contact force arises from the mutual action of material objects that touch one another. Body force arises from the mutual action between a pair of separated objects, and for this reason body force is often called *action at a distance.* Gravitational, electrical, and magnetic forces are familiar examples of body forces. However, forces are not always what they seem to be. Artificial gravity, for example, can be created by the whirling motion of a human centrifuge used to train astronauts. This apparent gravity is felt by the astronaut as a contact force when pressed hard into the seat by the centrifuge motion; and everyone has witnessed the apparent increase and decrease in gravity while riding up and down, respectively, in a fast moving elevator. A similar feeling of artificial gravity would be experienced in an elevator in outer space moving "upward" with a constant acceleration. And we all know that astronauts experience "weightlessness" (actually the absence of contact force in a perpetual free fall within the spacecraft), because the gravitational force that continues to act on them is very nearly balanced by a certain pseudo-force that arises from the orbital motion of the rapidly moving spacecraft and its passengers.

Interaction between material objects in $\mathscr{U}$ may be internal or external to a subset $\mathscr{S}$ of $\mathscr{U}$. This is diagrammed in Fig. 5.8. A force exerted on part $\mathscr{P}$ (a subset) of a body $\mathscr{S} \subset \mathscr{U}$ by another disjoint part $\hat{\mathscr{P}}$ of the same body is called an *internal force.* The force exerted on a part $\mathscr{P}$ of a body $\mathscr{S} \subset \mathscr{U}$ due to another body $\hat{\mathscr{S}} \subset \mathscr{U}$ that is not contained in $\mathscr{S}$ is called an *external force.* The collection of forces that act on a body is assumed additive. We remember that a part $\mathscr{P}$ of a body is itself a body. Hence, the total force exerted on a body $\mathscr{P}$ in $\mathscr{S}$ is defined as the vector sum of all internal and external forces that act on $\mathscr{P}$. Since the first two laws apply only to a body $\mathscr{S}^*$ consisting of a single particle (see Fig. 5.8.), it follows that the total force in these laws is necessarily the total external force that acts on that particle. Whatever may be the physical nature

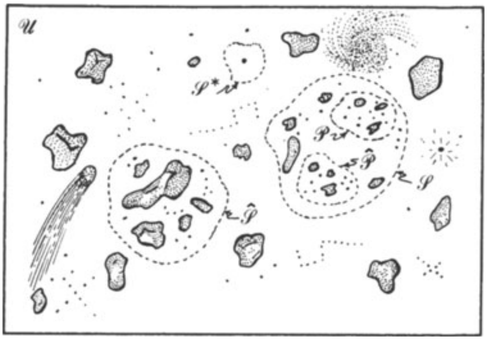


Figure 5.8. The material universe and its interacting parts.

of a force, its physical dimensions are defined on the basis of (5.34); namely, $[\mathbf{F}] \equiv [F] = [MVT^{-1}] = [MLT^{-2}]$. (See also the preface to the Problems for this chapter.) The three foundation laws are next discussed in turn.

5.5.2. The First Law of Motion

It is important to observe that Newton's laws hold only with respect to a certain preferential frame Φ. This special frame is called a *Newtonian* or *inertial reference frame*. The properties of the inertial frame will be studied later. For the time being, let us accept the idea that there exists in the universe an inertial frame that may serve as the preferred frame of Newton's laws, and continue.

The first law of motion postulates the existence of at least one preferred frame Φ and specifies that any disturbance of a particle P which is at rest or in uniform motion relative to this frame can occur only in response to force, while an arbitrary uniform motion or stationary state of P in Φ requires no force at all. So, explicitly, if $\mathbf{F}(P, t)$ denotes the total force acting on a particle P in any material universe whatever, the motion $\mathbf{x}(P, t)$ of P relative to Φ is determined by a certain functional relation (i.e., an equation in which the variable itself is a function or a set of functions) $\mathbf{x}(P, t) = \chi\,(\mathbf{F}(P, t))$, more commonly expressed in the standard form

$$\mathbf{F}(P, t) = \mathscr{F}(\mathbf{x}(P, t)). \tag{5.35}$$

Moreover, whatever its form, this general functional equation must satisfy the specified necessary and sufficient condition for a uniform motion in Φ, namely,

$$\mathbf{x}(P, t) = \mathbf{x}_0(P) + \mathbf{v}_0(P)t \Leftrightarrow \mathbf{F}(P, t) = \mathbf{0} \text{ for all } t, \tag{5.36}$$

wherein $\mathbf{x}_0$ and $\mathbf{v}_0$ are constant vectors. A rest state corresponds to the trivial case $\mathbf{v}_0 = \mathbf{0}$. Accordingly, the first law states that the unique solution of the equation $\mathbf{F}(P, t) = \mathscr{F}(\mathbf{x}(P, t)) = \mathbf{0}$ valid for all t in Φ is the uniform motion in (5.36). Or, conversely, if the motion is uniform in Φ, then $\mathbf{F}(P, t) = \mathscr{F}(\mathbf{x}_0 + \mathbf{v}_0 t) = \mathbf{0}$ for all t.

Alternatively, since a motion is uniform in Φ when and only when the acceleration in Φ is zero for all times t, (5.36) may be written as

$$\mathbf{F}(P, t) = \mathbf{0} \text{ for all } t \Leftrightarrow \mathbf{a}(P, t) = \mathbf{0} \text{ for all } t. \tag{5.37}$$

Because there is no inherent difference between a uniform motion and a state of rest, by definition, a stationary or uniform state of motion in the preferred frame Φ is called an *equilibrium state* in Φ. Thus, in accordance with (5.36) and (5.37), the first law specifies that *in every material universe, a condition necessary and sufficient for equilibrium of a particle P in an inertial reference frame is that the total force acting on P shall vanish for all times*:

$$\text{Equilibrium} \Leftrightarrow \mathbf{F}(P, t) = \mathbf{0} \Leftrightarrow \mathbf{a}(P, t) = \mathbf{0}. \tag{5.38}$$

Thus, Newton's first law postulates the general rule of determinism (5.35) and it specifies, by (5.36) or (5.38), a *universal principle of equilibrium* for a particle. It provides the foundation for the important special branch of dynamics called *statics*—the study of forces on bodies *at rest* in Φ.

The principle of equilibrium is the same in every material universe—it is a universal rule. However, when the motion is not uniform, the form of the functional equation (5.35) will depend upon the nature of the material universe it describes. In this respect, the first law is intentionally vague. The second law, on the other hand, is specific about the form of (5.35).

5.5.3. The Second Law of Motion

The second law of motion identifies a special material universe, called the *world,* for which the definite relation (5.34) between force and motion is introduced to describe the mechanical nature of things in the world. Of course, the abstract world of the second law is our analytical model of the real material universe, the real world where we live. However, the rule (5.34) must respect the conditions set in (5.36) or (5.37). Clearly, $\mathbf{F}(P, t) = \mathbf{0}$ for all t holds in (5.34) when and only when the momentum $\mathbf{p}(P, t) = m(P)\mathbf{v}(P, t) = \mathbf{p}_0(P)$ is a constant vector. *Hence, the motion is uniform if and only if the mass $m(P)$ is constant* (which it is).

On the other hand, imagine a different material universe in which (5.34) holds but now the mass varies with the particle speed. The second law would still support the conditions of the first law in this other material universe. In classical mechanics, however, the mass of a given body is an invariant, fixed property of the body—it is independent of the position, velocity, temperature, or any other influence acting on the body, so long as no part of the body disappears; that is, the mass of the body, or any part of the body, does not change in time. The principle of conservation of mass (5.1) invokes this condition for every motion of a particle. In consequence, from the rule (5.34), we obtain the basic formula popularly known as **Newton's**

equation of motion:

$$\mathbf{F}(P, t) = m(P)\mathbf{a}(P, t), \tag{5.39}$$

in which $\mathbf{a}(P, t)$ is the acceleration of P in the inertial frame.

The condition (5.37) imposed by the first law for every material universe strongly suggests that the simplest law of motion for the world is one for which $\mathbf{F} \propto \mathbf{a}$, so that $\mathbf{F} = \mathbf{0}$ implies a uniform motion in the inertial frame Φ, and conversely. This means we should have $\mathbf{F} = k\mathbf{a}$, where k is some constant characteristic of the particle. And what more appropriate constant might we select than the invariant mass of the object? Indeed, this is just the way it turned out in (5.39).

Thus, according to the first law, there may exist infinitely many material universes, or worlds, all having the same law of equilibrium but each characterized by a special equation of motion of its own, conceivably quite different from (5.34). The second law, however, provides a simple mathematical model to study the nature of most, though not all, physical phenomena in our world. Let us briefly look at its extension to a system of particles and to a continuum.

5.5.3.1. The Second Law for a System of Particles

The total force acting on a system of particles is defined as the sum of the forces that act on all of its particles. Let $\mathbf{F}_k = \mathbf{F}(P_k, t)$ denote the total force acting on the particle P_k of a system $\beta = \{P_k\}$ of n particles. Then, with (5.34) and (5.4), we *derive* **Newton's second law for a system of particles:** *The total force acting on a system of particles is equal to the time rate of change of the momentum of the system in the inertial frame, i.e.,*

$$\mathbf{F}(\beta, t) = \sum_{k=1}^{n} \mathbf{F}_k = \sum_{k=1}^{n} \frac{d\mathbf{p}_k}{dt} = \frac{d\mathbf{p}(\beta, t)}{dt}. \tag{5.40}$$

With the aid of (5.7) and the fact that mass is conserved, (5.40) may be cast in the same form as the basic equation of motion (5.39) for a single particle:

$$\mathbf{F}(\beta, t) = \frac{d\mathbf{p}^*(\beta, t)}{dt} = m(\beta)\mathbf{a}^*(\beta, t), \tag{5.41}$$

where $\mathbf{a}^*(\beta, t) = \dot{\mathbf{v}}^*(\beta, t)$ is the acceleration of the center of mass of the system. In words, *the total force acting on a system of particles is equal to the time rate of change of the momentum of its center of mass in the inertial frame Φ, and hence is equal to the product of the mass of the system and the acceleration of its center of mass in Φ*. The second law (5.41) for a system of particles thus aids the determination of the motion of the fictitious center of mass particle and external forces that control or constrain the motion of the system. In addition to (5.41), for a system of particles the auxiliary relations (5.5) through (5.8) are often needed in applications, as are the separate equations of motion of the particles. The equations of motion for a system of particles are discussed further in Chapter 8. Some further remarks on the equilibrium and interaction between the particles of the system follow shortly.

5.5.3.2. *Introduction to Euler's Laws for a Continuum*

We may visualize that as the number of particles of a system grows indefinitely, the system becomes a continuum $\mathscr{B}$ with momentum (5.11). In this case, the rule (5.34) is *replaced* by a more general principle known as **Euler's first law of motion:** *The total (external) force* $\mathbf{F}(\mathscr{B}, t)$ *acting on a body is equal to the time rate of change of its momentum in the preferred frame, i.e.,*

$$\mathbf{F}(\mathscr{B}, t) = \frac{d\mathbf{p}(\mathscr{B}, t)}{dt} = \frac{d}{dt}\int_{\mathscr{B}} \mathbf{v}(P, t)dm(P). \tag{5.42}$$

It is an amazing fact that this relation also may be written in the form of Newton's basic equation (5.39). We recall (5.16) and note that because the mass is conserved, Euler's first law (5.42) becomes

$$\mathbf{F}(\mathscr{B}, t) = \frac{d\mathbf{p}^*(\mathscr{B}, t)}{dt} = m(\mathscr{B})\mathbf{a}^*(\mathscr{B}, t). \tag{5.43}$$

Therefore, the total force acting on a body is equal to the time rate of change of the momentum of its center of mass, and hence is equal to product of the mass of the body and the acceleration $\mathbf{a}^*(\mathscr{B}, t)$ *of its center of mass in the inertial frame.* Euler's first law for a body thus relates the applied force to the motion of the center of mass.

Euler's second law has no counterpart among Newton's laws of motion. Euler's second principle relates the rotational part of the body's motion to the applied torque—the total moment of the applied forces about a fixed point in the inertial frame; and it also involves the moment of momentum (5.33) for a body. Thus, to study the general motion of a rigid body, besides (5.43), we shall need **Euler's second law of motion:** *With respect to a fixed point O in the inertial frame Φ, the total torque* $\mathbf{M}_O(\mathscr{B}, t)$ *that acts on a body is equal to the time rate of change in Φ of the total moment of momentum of the body about O:*

$$\mathbf{M}_O(\mathscr{B}, t) = \dot{\mathbf{h}}_O(\mathscr{B}, t) = \frac{d}{dt}\int_{\mathscr{B}} \mathbf{x}_O(P, t) \times \mathbf{v}(P, t)dm(P). \tag{5.44}$$

Euler's basic laws (5.42) and (5.44) are postulated for *all* bodies, including deformable solid and fluid bodies. Their application in this book, however, is restricted to rigid bodies. In this case, the velocity $\mathbf{v}(P, t)$ of an arbitrary body particle P may be expressed in terms of the angular velocity vector. This fact suggests that (5.44) relates the body's angular velocity and angular acceleration to the total applied torque about a fixed point in the inertial frame. We thus envision that Euler's second law is useful in determination of the rotational motion of the rigid body.

It follows from (5.43) and (5.44) that *equilibrium of a rigid body* requires two conditions necessary and sufficient in order that *every* particle of the body initially at rest or in uniform motion in the inertial frame shall continue in that state. With the initial conditions in mind, equilibrium requires that both the total force *and* the total torque acting on the rigid body about a fixed point must vanish for all time, i.e. the system of forces must be equipollent to zero:

$$\text{Equilibrium} \Leftrightarrow \mathbf{F}(\mathscr{B}, t) = \mathbf{0} \quad \text{and} \quad \mathbf{M}_O(\mathscr{B}, t) = \mathbf{0} \text{ for all } t. \tag{5.45}$$

This rule and Euler's laws are discussed further in Chapter 10.

The principle (5.43) that the mass center moves like a particle having mass equal to the mass of the body and acted upon by a force equal to the total force acting on the body means that the motion of the center of mass of a body often may be found by the methods of particle dynamics. Therefore, in our future study of the dynamics of a particle, it should be clear that it is correct to model a body of finite size by its center of mass particle. In general, however, because the equations of motion (5.43) and (5.44) for a body may be coupled, we cannot suppose that a problem of rigid body motion may be split into simple separate parts—a problem of particle dynamics and one of rotation of the body about an axis. In problems where rotational effects are absent, however, Euler's first law for a rigid body, or equivalently, Newton's second law for a particle, may be used to determine the motion of the center of mass particle and related unknown forces that drive or constrain that motion. The effects due to torques that may act on the body are studied later. Further discussion of (5.40) through (5.44) is reserved for their own place later; but, as we continue, we shall need to consider continua and systems of particles in discussion of their mutual interactions.

5.5.4. The Law of Mutual Action

Newton's third law admits that particles may exert *mutual forces* on one another to induce motion in accordance with the previous laws; however, whatever the nature of the force, the reaction of one particle in response to the action of another must be of equal, but oppositely directed intensity. Of course, this does not mean that these two forces will cancel from the equations of motion (5.39) for the particles, for the forces of action and reaction do not act on the same particle.

On the other hand, when the two particles are treated as a system, the mutual forces have no influence in the equation of motion (5.40) for the system. To see this, let us consider a system $\beta = \{P_1, P_2\}$ in which the particles P_1 and P_2 exert mutual force on one another. Let $\mathbf{F}_{12} = \mathbf{F}(P_1, t)$ be the force exerted on particle P_1 by particle P_2, and $\mathbf{F}_{21} = \mathbf{F}(P_2, t)$ the force exerted on particle P_2 by particle P_1. Then the third law requires that $\mathbf{F}_{12} = -\mathbf{F}_{21}$. These mutual forces are internal forces, and hence the total internal force is $\mathbf{F}(P_1, t) + \mathbf{F}(P_2, t) = \mathbf{0}$. Therefore, such mutual pairs of internal forces do not contribute to the total force

$\mathbf{F}(\beta, t)$ in the equation of motion (5.40), or (5.41), for the system. On the other hand, if only one particle P_1, say, is considered, then the mutual force $\mathbf{F}_{12}$ acts on this new "system", and it does not vanish in the equation of motion (5.39) for P_1.

This example shows the importance of carefully distinguishing the system being considered. The system chosen for study in a particular situation is called a *free body.* A drawing that shows all of the forces acting on the free body is called a *free body diagram,* a device introduced to facilitate the solution of a problem. To construct a free body diagram for any system, we need only recall that there are two classes of forces: contact forces and body forces. Therefore, we may begin by asking the question—What bodies are touching our free body? We then show in the free body sketch the appropriately directed contact forces exerted on the free body by each contacting body. Next, we ask—What bodies exert forces at a distance that are acting on our free body? And we show these appropriately directed body forces in the free body diagram. This simple but important initial procedure in the analysis of problems is illustrated many times in the sequel. It is essential that the student learn how to do this.

It is also important to mention that although the total internal force acting on a system of two particles is always zero, this does not imply that the system is in equilibrium. The particles could be moving with proportional acceleration vectors directed along the same line, or perhaps moving on distinct parallel lines. Also, particles of a system need not have the same uniform motion to be in equilibrium. On the other hand, for a system of two particles that separately are in equilibrium, the equal and oppositely directed mutual forces must be balanced by external forces so that the total force acting on each particle treated as a separate system is zero. Hence, the vanishing of the total force that acts on a system of particles is a necessary but not a sufficient condition for equilibrium. Moreover, if it is not required or otherwise established that mutual forces act along the line joining the particles, the force $\mathbf{F}_{12}$ exerted on P_1 by P_2 will have a definite turning effect on P_1 in moving it around P_2 as center. Newton's law of universal gravitational attraction assumes this collinearity, whereas, as shown later, the collinearity of mutual forces actually may be proved on the basis of a general rule governing the nature of mutual internal force that depends only on the locations of the two particles.

To advance further, however, we shall need to identify various kinds of forces. We begin by introducing the mutual gravitational force between two material objects.

5.6. Newton's Law of Gravitation

One kind of body force between two bodies is the mutual force of gravitational attraction, a basic force of nature that everyone knows as *gravity.* The theory of

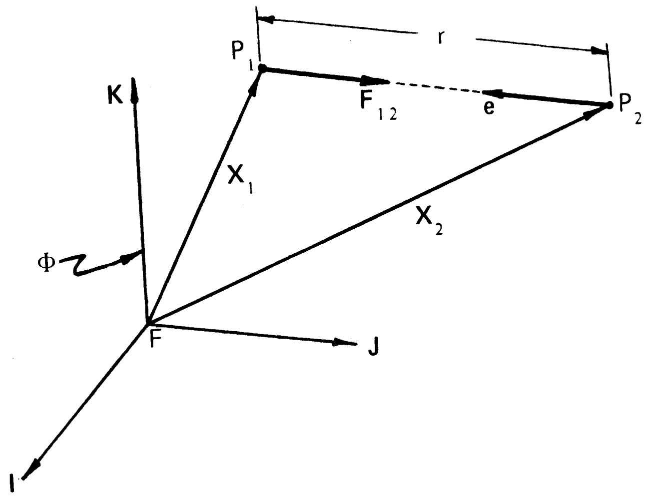


Figure 5.9. Schema for the mutual gravitational attraction of two particles.

gravitation invented by Newton to explain the motions of celestial bodies is studied here. The idea of a gravitational field created by the existence of matter is introduced to describe the gravitational field strength due to a particle, to a system of particles, and to a continuum; and the gravitational force exerted by these bodies on another particle, or body, is derived. We shall see that with regard to their gravitational attraction, bodies behave very much like particles, but not entirely. Our objective is to show that in all cases the gravitational force acting on a material object is equal to the product of its mass and the gravitational field strength it experiences. Afterwards, Newton's theory of gravitation is illustrated in a few examples. The gravitational attraction by an ideal planet is determined, and subsequently the definition of the weight of a body is introduced.

We begin with a pair of particles P_1, P_2 having mass m_1, m_2, respectively, and denote by $\mathbf{F}_{12}$ the force exerted on P_1 by P_2, as shown in Fig. 5.9. Let $\mathbf{e}$ be a unit vector directed from P_2, the source of the action, toward P_1; and write $r = |\mathbf{X}_2 - \mathbf{X}_1|$ for the distance between P_1 and P_2, wherein $\mathbf{X}_1$ and $\mathbf{X}_2$ are the respective distinct position vectors of P_1 and P_2 in any reference frame $\Phi = \{F; \mathbf{I}_k\}$. Clearly, only the relative position vector $\mathbf{r} \equiv r\mathbf{e}$ of P_1 from P_2 is important, so a reference frame is needed only for the solution of particular problems. These terms are used to state the following law of nature.

Newton's law of gravitation: *Between any two particles in the world, there exists a mutual gravitational force that is directly proportional to the product of their masses, inversely proportional to the square of their distance of separation, and directed in the sense of mutual attraction along their common line, i.e.,*

$$\mathbf{F}_{12} = -G\frac{m_1 m_2}{r^2}\mathbf{e} = -G\frac{m_1 m_2}{r^3}\mathbf{r}. \tag{5.46}$$

The positive constant G in (5.46) is named the *gravitational constant*, it is universal for all particles. Its physical dimensions consistent with (5.46) are $[G] = \left[FL^2/M^2\right] = \left[L^3/(MT^2)\right]$; its value will be given later. Of course, the roles of P_1 and P_2 are mutual and may be reversed. Hence, it is a consequence of the law itself that the mutual gravitational force exerted by P_1 on P_2 automatically respects the principle of mutual action, that is, $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

Newton's law describes the gravitational interaction between any two particles in the world; and it has the same form in every reference frame—it depends only on the relative positions of the particles and their invariant masses. It is conceivable, however, that there may exist other material worlds where the law of gravity is different, or where Newton's law may hold only approximately. In fact, in the real world it has been known for a long time that the observed orbit of the planet Mercury differs very slightly from the path determined from calculations based on Newton's law. Indeed, the combined gravitational influence of all the known planets has failed to account for the observed shift in Mercury's perihelion.

In 1915, however, Einstein proposed a theory of relativity by which he showed that for bodies that move with speeds that are small compared with the speed of light Newton's theory of gravitation is a first approximation to a more general theory of gravitational fields. Unlike Newton's theory, which introduces the idea of mysterious forces at a distance, Einstein's theory is based on a special geometry of space and time—a theory whose formulation far exceeds the scope of our studies here.

Of course, practically all deviations from Newton's law that are predicted by Einstein's theory are so small that even with precision instruments they are difficult to measure. The precessional motion of the elliptical orbit of Mercury is a model case for which measurements of the rotation of its major axis, about 43 arc-seconds each century, agree precisely with Einstein's prediction. The deflection of light by the gravitational field of a star, the influence of gravitational field strength on the frequency of emitted light, and an explanation of the expanding motion of galactic systems are other effects predicted by Einstein's theory of relativity and confirmed by observations. These delicate, fascinating phenomena cannot be explained by Newton's theory. There are, however, countless other phenomena in the world that are perfectly and more easily modeled by Newton's simpler theory of gravitation described by (5.46). The discovery of Neptune based on an incredibly tedious year long calculation in 1846 by Urbain Jean Joseph Le Verrier, for example, was an exceptional accomplishment of Newton's theory.

Irregularity in the orbit of the planet Uranus was also known for a long time.† Calculations by the astronomer Le Verrier of the path of a hypothetical planet, whose gravitational attraction in accordance with Newton's theory would produce the observed discrepancy in Uranus's orbit, predicted the position of a new body in the sky. And when eventually a telescope was focussed on this place, the new

† Historical details of the discovery of Neptune and the search for the putative planet Vulcan are provided in articles by J. D. Fernie cited in the References.

planet Neptune was discovered very close to its predicted position. The same trick was used by Le Verrier to try to account for the discrepancies in Mercury's orbit. But his hypothetical planet named Vulcan has never been found. Rather, it was Einstein's theory of gravitation in 1915 that eventually accounted for the orbital discrepancies of Mercury, and it predicted similar effects for other planets, including the Earth. These are impressive theoretical results. Nevertheless, it is fair to say that in general Newton's simpler law of gravitation provides an exceptionally good mathematical model for studying the nature of many, though certainly not all, gravitational phenomena in the world; and we may use it with confidence in its predictive value. The idea of a gravitational field based on Newton's theory is introduced next.

5.6.1. The Gravitational Field of a Particle

A *gravitational field* $\mathscr{G}$ is said to exist in all of space due to the mass m_o whenever a force of attraction is felt by another "test" particle placed anywhere in $\mathscr{G}$. Hence, m_o is named the *origin*, or *source*, of the gravitational field. The attractive force due to m_o, per unit mass of the test particle, is called the *strength* of the field $\mathscr{G}$. Let $\mathbf{g}(\mathbf{X})$ denote the field strength at $\mathbf{X}$. Then, in accordance with (5.46),

$$\mathbf{g}(\mathbf{X}) = -\frac{Gm_o}{r^2}\mathbf{e}, \tag{5.47}$$

where $\mathbf{e}$ is the unit vector directed from the source m_o to the field point $\mathbf{X}$ whose distance from m_o is r, as shown in Fig. 5.10. Since $\mathbf{g}$ is the gravitational force that a particle of unit mass will experience when placed at $\mathbf{X}$ in $\mathscr{G}$, the gravitational force $\mathbf{F}(P;\mathbf{X})$ exerted on a particle P of mass m at $\mathbf{X}$ is given by

$$\mathbf{F}(P;\mathbf{X}) = m(P)\mathbf{g}(\mathbf{X}). \tag{5.48}$$

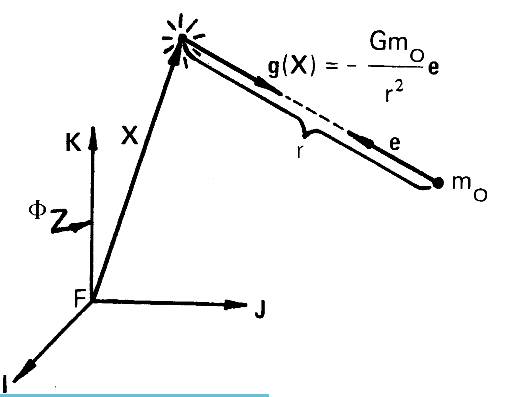


Figure 5.10. Gravitational field strength $\mathbf{g}(\mathbf{X})$ due to the mass point m_o.

Alternatively, with (5.47), $\mathbf{F}(P;\mathbf{X}) = -Gm_o m\mathbf{e}/r^2$, which is the same as (5.46). Observe again that the gravitational force is independent of the reference frame that may be used to identify the place $\mathbf{X}$.

5.6.2. The Gravitational Field of a System of Particles

The law of gravitation (5.46), hence also its alternate form (5.48), applies only to two particles. To find the gravitational force exerted on a particle P by a system of particles $\beta = \{P_k\}$, we use the fact that the field strength is a vector measure of force per unit mass. Since forces are vectorially additive, the separate field strengths of all particles of β must be vectorially additive. We suppose that the internal forces between the particles of β remain equal and opposite and in no way alter the individual field strengths $\mathbf{g}_k(\mathbf{X})$ due to the separate particles P_k of β. Then, with the aid of (5.48), the resultant gravitational force exerted on P by the totality of particles that comprise β is given by $\mathbf{F}(P;\mathbf{X}) = \sum_{k=1}^{n} \mathbf{F}_k(P;\mathbf{X}) = \sum_{k=1}^{n} m(P)\mathbf{g}_k(\mathbf{X})$, wherein $\mathbf{F}_k(P;\mathbf{X})$ is the gravitational force exerted by P_k on the particle P at $\mathbf{X}$. Thus, use of (5.47) for each source mass m_k in β yields the *resultant field strength* $\mathbf{g}(\mathbf{X})$ *for a system of n particles*:

$$\mathbf{g}(\mathbf{X}) \equiv \sum_{k=1}^{n} \mathbf{g}_k(\mathbf{X}) = -\sum_{k=1}^{n} \frac{Gm_k}{r_k^2}\mathbf{e}_k. \tag{5.49}$$

The interpretation of r_k and $\mathbf{e}_k$ is evident from Fig 5.11 in which the resultant field strength at $\mathbf{X}$ for a two particle system is illustrated. Hence, use of (5.49) yields the *resultant gravitational force on a particle P due to a system of particles*: $\mathbf{F}(P;\mathbf{X}) = m(P)\mathbf{g}(\mathbf{X})$, which has the same form as (5.48). Of course, the particle P exerts an equal but oppositely directed gravitational force on β. (See Problem 5.14.)

The direction of $\mathbf{F}(P;\mathbf{X})$ will depend on the direction of $\mathbf{g}(\mathbf{X})$, which is determined by the system β. In general, the resultant gravitational field strength (5.49), and hence the resultant gravitational force, does not pass through the center

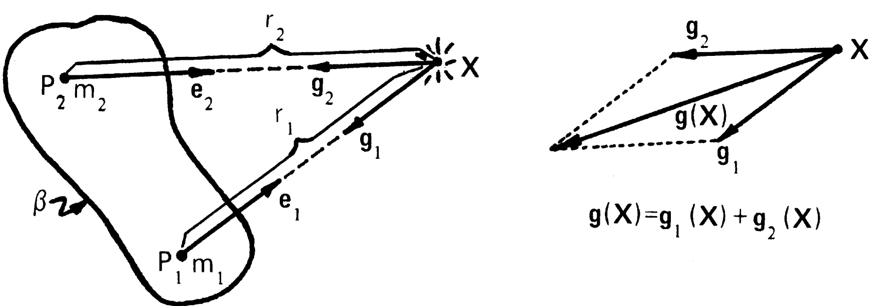


Figure 5.11. Resultant field strength $\mathbf{g} = \mathbf{g}_1 + \mathbf{g}_2$ of a system of particles $\beta = \{P_1, P_2\}$.

of mass of the field source β. Indeed, the field strength $\mathbf{g}^*(\mathbf{X})$ due to the center of mass particle at the place $\mathbf{r}^* = -r^*\mathbf{e}^*$ from $\mathbf{X}$ and having mass $m^* \equiv m(\beta)$ is given by (5.47). Writing $\mathbf{r}_k = -r_k\mathbf{e}_k$ (no sum) for the position vector of P_k from $\mathbf{X}$, as suggested in Fig. 5.11, and recalling (5.5) for the center of mass, we see by (5.47) and (5.49) that

$$\mathbf{g}^*(\mathbf{X}) = \frac{G}{r^{*3}} m^*\mathbf{r}^* = \sum_{k=1}^{n} \frac{Gm_k}{r^{*3}}\mathbf{r}_k \neq \sum_{k=1}^{n} \frac{Gm_k}{r_k^3}\mathbf{r}_k = \mathbf{g}(\mathbf{X}).$$

In general, therefore, $\mathbf{g}(\mathbf{X})$ is not parallel to $\mathbf{r}^*$, and hence the resultant gravitational force does not pass through the center of mass of β. Consequently, the gravitational force on P has a moment about the center of mass of the system. On the other hand, it may be seen that $\mathbf{g}(\mathbf{X}) = \mathbf{g}^*(\mathbf{X})$, very nearly, when the particle P is sufficiently far from the neighborhood of β so that the distance r_k of each particle P_k from $\mathbf{X}$ is equal, very nearly, to the distance r^* of the center of mass of β from $\mathbf{X}$. Precise demonstration of this statement based on the last relation above is left for the reader.

We have found that the formula for the resultant gravitational force on a particle due to a system of particles has the same form as the basic rule (5.48) for the gravitational force due to one particle. Derivation of a similar result for the gravitational interaction of two separate systems is left for the reader. The procedure and consequences are similar to those described below for two continuous bodies.

5.6.3. The Gravitational Field of a Body

The gravitational force due to a continuum acting on a particle may be found in a parallel manner. In this case, we generalize the particle theory by considering a gravitational field whose strength due to a parcel of mass dm_o of the body $\mathscr{B}_o$ is defined by $-(Gdm_o/r^2)\mathbf{e}$, where $\mathbf{e}$ is the unit vector directed from the source dm_o to the field point $\mathbf{X}$ shown in Fig. 5.12 at a distance r from dm_o. Then the *resultant field strength at* $\mathbf{X}$ *due to the body* $\mathscr{B}_o$ is defined by

$$\mathbf{g}(\mathbf{X}) \equiv -G\int_{\mathscr{B}_o} \frac{\mathbf{e}}{r^2(\mathbf{X})}\,dm_o. \tag{5.50}$$

Both $\mathbf{e}$ and r will vary in the integration over the source body $\mathscr{B}_o$, so they cannot be taken outside the integral. The *resultant gravitational force exerted by the body* $\mathscr{B}_o$ *on a particle* P *of mass* m *at* $\mathbf{X}$ is determined by $\mathbf{F}(P;\mathbf{X}) = m(P)\mathbf{g}(\mathbf{X})$, which has the same representation as the basic rule (5.48) for the attraction between two particles. Of course, the particle exerts an equal and oppositely directed gravitational force on the body.

The direction of the resultant force $\mathbf{F}(P;\mathbf{X})$ is the same as that of $\mathbf{g}(\mathbf{X})$, which is determined by the body $\mathscr{B}_o$. It may be seen that the resultant gravitational field strength (5.50), and hence the resultant gravitational force usually does not pass through the center of mass of $\mathscr{B}_o$. The proof is parallel to that for a system of particles. Hence, in general, the resultant gravitational force exerted on P by the

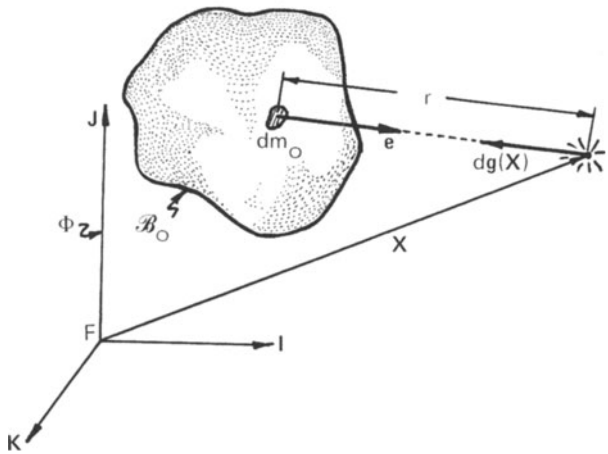


Figure 5.12. Elemental gravitational field strength $d\mathbf{g}(\mathbf{X})$ due to a parcel of mass dm_o of a body $\mathscr{B}_o$.

body is not the same as the gravitational force exerted on P by its center of mass particle. In consequence, the gravitational force on P exerts a torque about the center of mass of the body. Of course, when the particle P at $\mathbf{X}$ is sufficiently far from the neighborhood of $\mathscr{B}_o$ so that the distance of each of its particles from $\mathbf{X}$ is equal very nearly to the distance r^* of the center of mass of $\mathscr{B}_o$ from $\mathbf{X}$, the two field strengths are very nearly equal.

Finally, let us suppose that P is a material parcel $dm(P)$ of another body $\mathscr{B}$ with mass $m(\mathscr{B})$. Then use of the field strength (5.50) in integration over $\mathscr{B}$ determines the *resultant gravitational force* $\mathbf{F}(\mathscr{B})$ *exerted on* $\mathscr{B}$ *by* $\mathscr{B}_o$, namely,

$$\mathbf{F}(\mathscr{B}) = \int_{\mathscr{B}} \mathbf{g}(\mathbf{X})dm(P) \equiv m(\mathscr{B})\, \hat{\mathbf{g}}\, (\mathscr{B}). \tag{5.51}$$

The quantity $\hat{\mathbf{g}}\,(\mathscr{B})$ defined by (5.51) is named the *average,* or *mean field strength* due to $\mathscr{B}_o$. (See Problems 5.23 and 5.24.)

The gravitational force exerted by $\mathscr{B}$ on $\mathscr{B}_o$ is necessarily equal and oppositely directed to $\mathbf{F}(\mathscr{B})$; but the forces need not be collinear, nor pierce the center of mass of either body. Thus, with respect to an arbitrary reference point, in general the source body $\mathscr{B}_o$ will exert a *gravitational torque* on the body $\mathscr{B}$. If $\mathbf{x}_Q(P)$ is the position vector from a reference point Q to an element of mass $dm(P)$ of $\mathscr{B}$, the moment about Q of the gravitational force distribution exerted on $\mathscr{B}$ by the field source $\mathscr{B}_o$, in accordance with (5.22), is

$$\mathbf{M}_Q(\mathscr{B}) = \int_{\mathscr{B}} \mathbf{x}_Q(P) \times \mathbf{g}(\mathbf{X})dm(P). \tag{5.52}$$

This is illustrated in a subsequent Exercise 5.5, page 42, that includes discussion of the equipollent force and couple for the gravitational force system (5.51) and

(5.52). Of course, the gravitational torque may vanish and the mutual gravitational forces may pierce the centers of mass in special cases. This happens, for example, when $\mathscr{B}$ is sufficiently far from the source body $\mathscr{B}_o$.

Observations of the kind described above will be helpful in understanding the approximations assumed in our future studies of particle dynamics in which bodies of finite size occur in many of the problems. We have seen that with regard to the equation of motion, a body may be replaced by its corresponding center of mass object; and as regards the gravitational force acting on a body, there is presently only one rule that need concern us here. In sum, *regardless of the nature of the field source, the gravitational force* $\mathbf{F}(\mathscr{O})$ *acting on a material object* $\mathscr{O}$ *is equal to the product of its mass* $m(\mathscr{O})$ *and the total gravitational field strength* $\mathbf{g}(\mathscr{O})$ experienced by $\mathscr{O}$; *that is, in contracted notation,*

$$\mathbf{F}(\mathscr{O}) = m(\mathscr{O})\mathbf{g}(\mathscr{O}). \tag{5.53}$$

5.7. Some Applications of Newton's Theory of Gravitation

The application of Newton's theory of gravitation is illustrated next in two examples. The gravitational interaction between a wire ring and a particle, and between a wire ring and a thin rod are studied. It is confirmed that when a material object is sufficiently far from the field source, the gravitational interaction reduces to the fundamental law (5.46) for two particles. The gravitational torque exerted on a rod by a semicircular wire is then described in an exercise. We begin with the gravitational interaction between a solid body and a particle.

Example 5.6. *Interaction between a wire ring and a particle.* A homogeneous, thin circular wire $\mathscr{B}_o$ of radius R and mass m_o is shown in Fig. 5.13.

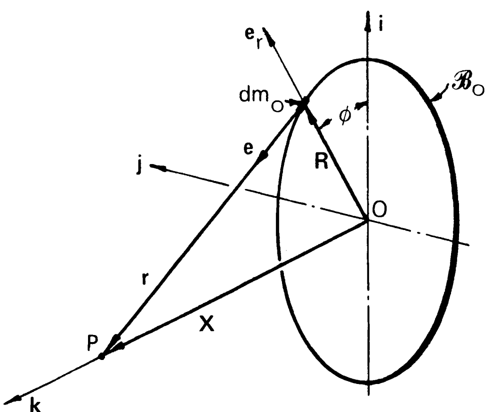


Figure 5.13. Geometry for the gravitational interaction between a wire ring and a particle.

Determine the gravitational field strength of the wire ring at a point P on the normal axis through its center O. Show that the resultant gravitational force exerted by $\mathscr{B}_o$ on a particle of mass m placed at P reduces to the gravitational force (5.46) between two particles when P is far enough from O such that $|\mathbf{X}| \gg R$.

Solution. The resultant field strength of the circular wire $\mathscr{B}_o$ at the place $\mathbf{X} = Z\mathbf{k}$ is determined by (5.50) in which the relative position vector $\mathbf{r}(\mathbf{X})$ of the point P from the parcel of mass dm_o of $\mathscr{B}_o$ is given by $\mathbf{r}(\mathbf{X}) = r\mathbf{e} = \mathbf{X} - \mathbf{R} = Z\mathbf{k} - R\mathbf{e}_r$ in terms of the cylindrical reference variables shown in Fig. 5.13. With $r^2 = Z^2 + R^2$, the integrand in (5.50) may be written as

$$\frac{\mathbf{e}}{r^2} = \frac{Z\mathbf{k} - R\mathbf{e}_r}{\left(Z^2 + R^2\right)^{\frac{3}{2}}}. \tag{5.54a}$$

Introducing $\sigma = m_o/2\pi R$, the mass per unit length of the homogeneous wire, and $ds = Rd\phi$, its elemental length, we have $dm_o = \sigma ds = \frac{1}{2\pi} m_o d\phi$. Then, with (5.54a) in (5.50), noting that both Z and R are fixed quantities, and setting the limits of integration over $\mathscr{B}_o$, we obtain the resultant field strength of the circular wire at $\mathbf{X}$:

$$\mathbf{g}(\mathbf{X}) = -\frac{Gm_o}{2\pi(Z^2 + R^2)^{\frac{3}{2}}} \left(Z\mathbf{k} \int_0^{2\pi} d\phi - R \int_0^{2\pi} \mathbf{e}_r d\phi \right), \tag{5.54b}$$

in which $\mathbf{e}_r = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}$. The last term vanishes; and the gravitational field strength at the place $\mathbf{X}$ due to the circular wire is thus given by

$$\mathbf{g}(\mathbf{X}) = -\frac{Gm_o Z}{\left(Z^2 + R^2\right)^{\frac{3}{2}}}\mathbf{k}. \tag{5.54c}$$

The field strength at the place P is directed toward the center of the ring.

A particle of mass m placed at $\mathbf{X}$ in the field (5.54c) experiences an attractive gravitational force given by (5.48) in accordance with the rule (5.53), namely,

$$\mathbf{F}(P;\mathbf{X}) = m(P)\mathbf{g}(\mathbf{X}) = -\frac{Gmm_o Z}{\left(Z^2 + R^2\right)^{\frac{3}{2}}}\mathbf{k}, \tag{5.54d}$$

directed through the center of mass of $\mathscr{B}_o$. Notice that if P is placed at the center O where $Z = 0$, the resultant, mutual gravitational force on P is zero.

Finally, suppose that P is far enough from O so that $R/Z \ll 1$, hence negligible. Then $r = Z$, $\mathbf{k} = \mathbf{e}$, approximately, and (5.54d) may be written as $\mathbf{F}(P;\mathbf{X}) = -Gmm_o\mathbf{e}/r^2$, which has the same form as Newton's law (5.46) for the gravitational force between two particles of mass m and m_o, respectively placed at P and O. □

We next study an application of (5.51) for the gravitational attraction between two solid bodies.

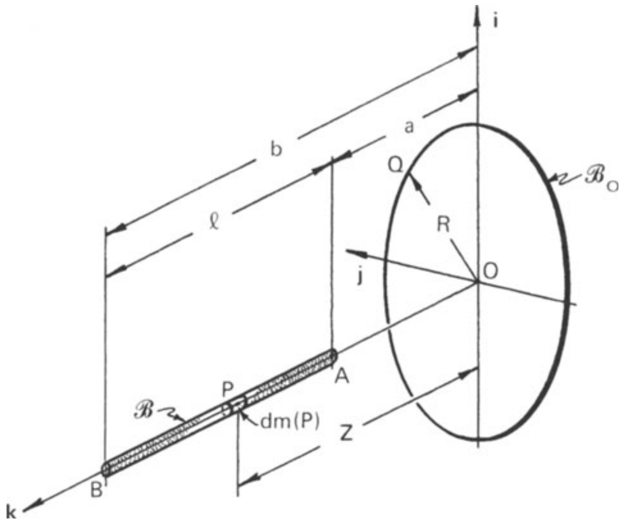


Figure 5.14. Gravitational interaction between a wire ring and a thin rod.

Example 5.7. *Interaction between a wire ring and a thin rod.* A homogeneous, thin rod $\mathscr{B}$ of length ℓ and mass $m(\mathscr{B})$ is placed along the normal axis of the wire ring described in the last example. What is the resultant gravitational force exerted by the rod on the ring? Find the mean field strength due to the ring.

Solution. Since the gravitational field strength of the wire ring is known by (5.54c), it is convenient to first find the resultant force that the ring exerts on the rod, and afterwards obtain the opposite force acting on the ring. The rod is placed along the central axis with its ends A and B at the respective distances a and b from the center O, as shown in the Fig. 5.14. For the homogeneous, thin rod, the parcel of mass at $\mathbf{X} = Z\mathbf{k}$ from O is $dm(P) = m(\mathscr{B})dZ/\ell$. Hence, the substitution into (5.51) of the gravitational field strength vector (5.54c) acting on $dm(P)$ determines the resultant gravitational force on the rod. Introducing the integration limits for the rod $\mathscr{B}$ and noting that $2Z\,dZ = d(Z^2 + R^2)$, we obtain

$$\mathbf{F}(\mathscr{B}) = -\frac{Gm_o m(\mathscr{B})}{2\ell}\mathbf{k}\int_a^b \frac{d\left(Z^2 + R^2\right)}{(Z^2 + R^2)^{\frac{3}{2}}}. \tag{5.55a}$$

This yields the resultant gravitational force on the rod $\mathscr{B}$ due to the wire ring $\mathscr{B}_o$:

$$\mathbf{F}(\mathscr{B}) = -\frac{Gm_o m(\mathscr{B})}{\ell}\mathbf{k}\left(\left(R^2 + a^2\right)^{-\frac{1}{2}} - \left(R^2 + b^2\right)^{-\frac{1}{2}}\right). \tag{5.55b}$$

The resultant gravitational force exerted on the ring is now given by $\mathbf{F}(\mathscr{B}_o) = -\mathbf{F}(\mathscr{B})$. This force pierces the mass centers of both homogeneous solids.

When the center of the rod is at O, $b = a = \ell/2$ and the mutual resultant gravitational force vanishes. When the rod is sufficiently far from the ring so that R/a and ℓ/a are both $\ll 1$, (5.55b) for the gravitational attraction between the two bodies reduces to (5.46) for two particles.

To determine the mean field strength due to the ring, first observe that

$$a_o \equiv \sqrt{R^2 + a^2}, \qquad b_o \equiv \sqrt{R^2 + b^2}, \tag{5.55c}$$

are the respective distances from any point Q on the ring to the end points A and B of the rod. Then, with (5.55b), the mean gravitational field strength due to the ring, in accordance with (5.51), is

$$\hat{\mathbf{g}}\,(\mathscr{B}) = \frac{\mathbf{F}(\mathscr{B})}{m(\mathscr{B})} = -\frac{Gm_o}{a_o b_o}\left(\frac{b_o - a_o}{\ell}\right)\mathbf{k}. \tag{5.55d}$$

□

The reader will find it informative to work through the following exercises. These review the previous examples in the solution of a similar problem for a semicircular wire. In addition, the gravitational torque effect is illustrated.

Exercise 5.3. *Interaction between a semicircular wire and a thin rod.* Suppose that the ring in the previous example is replaced by a semicircular wire of radius R in the upper half plane so that $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ (see Fig. 5.13), while the rod retains the configuration shown in Fig. 5.14. Recall the sequence of equations (5.54a) through (5.54d). Show that the semicircular wire produces on the normal axis through O at $\mathbf{X} = Z\mathbf{k}$ a resultant gravitational field strength given by

$$\mathbf{g}(\mathbf{X}) = \frac{Gm_o}{\pi\left(Z^2 + R^2\right)^{\frac{3}{2}}}\,(2R\mathbf{i} - \pi Z\mathbf{k})\,; \tag{5.56a}$$

and hence the resultant gravitational force exerted by the wire on the rod is

$$\mathbf{F}\,(\mathscr{B}) = \frac{Gm_o m}{\pi \ell a_o b_o}\left[\frac{2}{R}\,(ba_o - ab_o)\,\mathbf{i} - \pi\,(b_o - a_o)\,\mathbf{k}\right], \tag{5.56b}$$

where a_o and b_o are defined in (5.55c). □

Exercise 5.4. *Gravitational torque exerted by a semicircular wire on a thin rod.* It is seen in (5.56b) that the resultant gravitational force on the rod has a vertical component that has a moment about the center point O, for example. Therefore, the gravitational force distribution on the rod gives rise to a gravitational torque (5.52). Let Q be the reference point at O so that $\mathbf{x}_O = Z\mathbf{k}$ in (5.52). Show that the gravitational torque about O exerted on the rod by the semicircular wire is

$$\mathbf{M}_O(\mathscr{B}) = \frac{2RGm_o m}{\pi \ell a_o b_o}\,(b_o - a_o)\,\mathbf{j}. \tag{5.56c}$$

□

Exercise 5.5. The force system (5.56b) and (5.56c) is equipollent to a *gravitational force* $\mathbf{F}(\mathscr{B})$ at a certain point Q and a *gravitational couple* $\mathbf{C}(\mathscr{B}) \equiv \bar{\mathbf{x}}_O \times \mathbf{F}(\mathscr{B})$, where $\bar{\mathbf{x}}_O = (\bar{x}, \bar{y}, \bar{z})$ is a position vector from O to any point on the line of action of $\mathbf{F}(\mathscr{B})$. Hence, Q is an arbitrary point on this line; and $\mathbf{C}(\mathscr{B}) = \mathbf{M}_O(\mathscr{B})$ provides the equation of the line of action of the equipollent force. Find the equation of the line of action of the equipollent force for the gravitational force system exerted by the semicircular wire on the rod. Determine its intercepts $(\bar{x}_o, \bar{y}_o, \bar{z}_o)$ with the axes, and thus show that the line of action of the equipollent force acting on the rod pierces the center of mass of the homogeneous semicircular wire, but not that of the rod. Consequently, the rod exerts on the wire a gravitational force $\mathbf{F}(\mathscr{B}_o) = -\mathbf{F}(\mathscr{B})$ and a gravitational couple $\mathbf{C}(\mathscr{B}_o) = -\mathbf{C}(\mathscr{B})$ at its center of mass. □

5.8. Gravitational Attraction by an Ideal Planet

Though enormous in size compared to ordinary material things, heavenly bodies are separated by great distances, so the ratios d/D of their diameters d to their distances of separation D are small quantities. Consequently, as regards their gravitational interactions, the heavenly bodies typically are modeled as particles. Here we examine this hypothesis for an ideal planet and show that its gravitational field strength is the same as the field strength of a particle of equal mass placed at its center.

Every material object in the vicinity of the Earth experiences a gravitational attraction that arises principally from the attractive force exerted by all parts of the Earth on every part of the object. Of course, the dimensions of ordinary bodies are infinitesimal in comparison with the size of the Earth, so even when these bodies may be on or very near the Earth, it seems sensible in a first approximation to model the body in its relationship to the Earth as a particle or, more precisely, as a center of mass object of mass m. Since the mass of a planet like the Earth is so considerably greater than the mass of even the largest structures, like an aircraft, a ship, or a skyscraper, the mutual gravitational attractions of these bodies obviously are small in comparison with the total gravitational force due to the Earth. Indeed, in all of our experience we have suffered no apparent propulsion toward these objects, nor they toward one another. But when we have the misfortune to tumble from even the slightest height, it hurts! The effect would be the same if it happened on the Moon, but with much reduced intensity due to the Moon's smaller size and mass. (See Problems 5.25 and 5.26.) In any case, ignoring other bodies, we want to know—What is the gravitational force on a body due to the Earth?

To model the shape of a typical planet and its mass distribution, let us assume that (i) the planet is a sphere of radius A, and (ii) its mass density $\rho = \rho(R)$ varies only with the distance R from its center. The total gravitational force exerted by the sphere on an external material point P at a distance X from its center may be determined by use of (5.50) in (5.48), and cast in the spherical coordinates

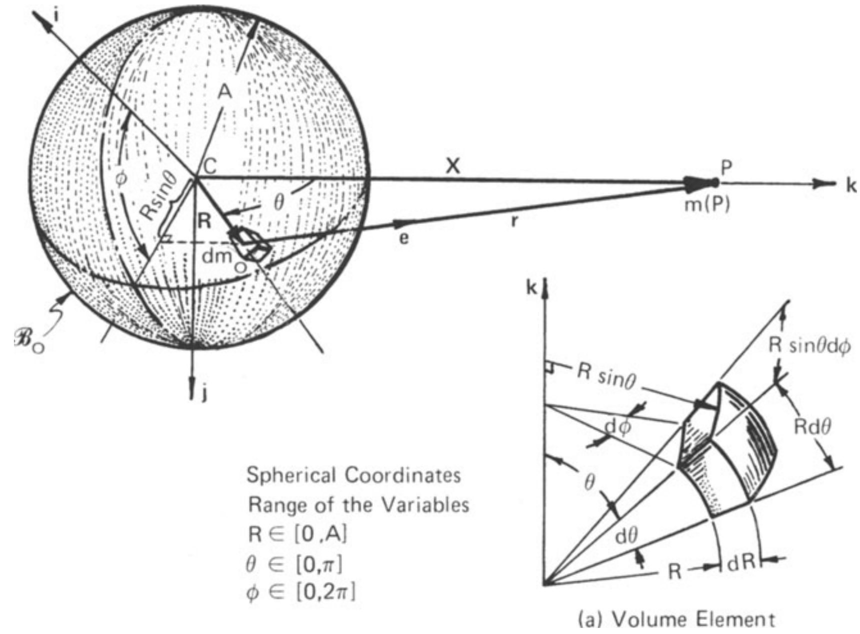


Figure 5.15. Geometry for the gravitational attraction of a particle P due to an ideal spherical planet.

(R, θ, ϕ) shown in Fig. 5.15. The material volume element is shown in Fig. 5.15a. Hence, the spherical element of mass is $dm_o = \rho(R)R^2 \sin\theta \; dRd\theta d\phi$. Also, $-\mathbf{e}/r^2 = (\mathbf{R} - \mathbf{X})/r^3$, wherein the unit source vector $\mathbf{e}$ is directed from the parcel dm_o at $\mathbf{R} = R\sin\theta(\cos\phi\mathbf{i} + \sin\phi\mathbf{j}) + R\cos\theta\mathbf{k}$ to the particle P at $\mathbf{X} = X\mathbf{k}$, and $r = (R^2 + X^2 - 2RX\cos\theta)^{\frac{1}{2}}$. Collecting these terms into (5.50) and setting the limits of integration over the sphere $\mathscr{B}_o$, we find the gravitational force (5.48) exerted by the ideal planet on a particle P of mass m at $\mathbf{X}$ is given by

$$\frac{\mathbf{F}(P;\mathbf{X})}{mG} = \int_0^A \int_0^\pi \int_0^{2\pi} \left(\frac{R\sin\theta(\cos\phi\mathbf{i} + \sin\phi\mathbf{j}) + (R\cos\theta - X)\mathbf{k}}{(R^2 + X^2 - 2RX\cos\theta)^{\frac{3}{2}}} \right) \times \rho(R)R^2 \sin\theta dRd\theta d\phi.$$

The integrations are not so formidable as may appear. In fact, integration with respect to ϕ yields $\int_0^{2\pi}(\cos\phi\mathbf{i} + \sin\phi\mathbf{j})d\phi = \mathbf{0}$ and $\int_0^{2\pi} d\phi = 2\pi$. Therefore, it follows, as one might expect from symmetry, that *the resultant gravitational force exerted by an ideal planet on a particle of mass m is directed toward the center of the sphere*:

$$\mathbf{F}(P; X) = -2\pi mG\mathbf{k} \int_0^A \int_0^\pi \frac{(X - R\cos\theta)\rho(R)R^2 \sin\theta dRd\theta}{\left(R^2 + X^2 - 2RX\cos\theta\right)^{\frac{3}{2}}}. \tag{5.57}$$

The reader may show directly that (5.57) may be obtained by use of symmetry about the $\mathbf{k}$-axis and by considering the attraction of a thin ring of radius $R\sin\theta$ and thickness dR at a central angle 2θ. So far the result (5.57) actually holds more generally for $\rho = \rho(R, \theta)$. To continue, however, we need $\rho = \rho(R)$.

Returning to (5.57) and integrating the functions in θ, being careful to observe that the particle at $\mathbf{X}$ lies *outside the sphere*, i.e. $X \geq R$, we eventually find the important result

$$\mathbf{F}(P;\mathbf{X}) \equiv -G\frac{m(P)m(\mathscr{B}_o)}{X^2}\mathbf{k} = m(P)\mathbf{g}(\mathbf{X}) \text{ for } X \geq R, \tag{5.58}$$

wherein $m(\mathscr{B}_o)$ is the mass of the sphere and $\mathbf{g}(\mathbf{X})$ is its field strength at $\mathbf{X}$:

$$m(\mathscr{B}_o) = \int_{\mathscr{B}_o} dm_o = \int_0^A 4\pi\rho(R)R^2dR, \qquad \mathbf{g}(\mathbf{X}) = -\frac{Gm(\mathscr{B}_o)}{X^2}\mathbf{k}. \tag{5.59}$$

The gravitational force (5.58) has precisely the same form as (5.46) for the gravitational attraction between two particles; and the gravitational field strength in (5.59) is the same as the field strength (5.47) of a particle of equal mass $m(\mathscr{B}_o)$ placed at the center of the sphere. *Therefore, as regards its gravitational attraction, a sphere of mass density $\rho(R)$ behaves like a particle having mass $m_o = m(\mathscr{B}_o)$, the mass of the sphere, and located at its center.* Thus, any planet that is essentially spherical and has an average density variation that depends only on the distance from its center will attract a particle of mass m with the central directed force (5.58) characteristic of a source particle located at its center. Plainly, our hypothetical planet does not represent accurately the true features of the Earth, nor any other real planet. This analysis provides only a simple first approximation of the field strength due to the Earth, or any similar body.

5.9. Gravitational Force on an Object Near an Ideal Planet

Let us consider the field strength in the vicinity of our ideal planet. The radius vector from its center to an object P in the neighborhood of its surface may be written as $\mathbf{X} = (A + \varepsilon)\mathbf{k}$, where ε, the normal distance of P from the surface, is very small compared with the planet's radius A. Then (5.58) may be written as

$$\mathbf{F}(P;\mathbf{A} + \varepsilon) = -\frac{Gmm_o\mathbf{k}}{A^2(1 + \varepsilon/A)^2}, \tag{5.60}$$

where $m_o \equiv m(\mathscr{B}_o)$ denotes the planet's mass. When $\varepsilon = 0$, we obtain the *gravitational force on P at the planet's surface*:

$$\mathbf{F}(P;\mathbf{A}) = m(P)\mathbf{g}(\mathbf{A}) \quad \text{with} \quad \mathbf{g}(\mathbf{A}) = -g\mathbf{k} \equiv -\frac{Gm_o}{A^2}\mathbf{k}. \tag{5.61}$$

The constant $g \equiv Gm_o/A^2$ is known as the *acceleration of gravity*; its value plainly depends upon the size and mass distribution of the planet. Although **g**, as its name implies, has the physical dimensions of acceleration, it is not a kinematical quantity; it is not the derivative of a velocity vector.

To determine the error committed by our neglecting the term ε/A in (5.60), the relation (5.61) and the binomial expansion of $(1+\varepsilon/A)^{-2}$ are used to obtain

$$\mathbf{F}(P;\mathbf{A}+\varepsilon) = \mathbf{F}(P;\mathbf{A})\left(1 - \frac{2\varepsilon}{A} + \frac{3\varepsilon^2}{A^2} - \cdots\right).$$

The first approximation $\varepsilon/A = 0$ yields (5.61). Therefore, the next term $2\varepsilon/A$ is a measure of the error committed when this term is ignored. For example, for an aircraft flying at an altitude of $\varepsilon = 10$ mile (16 km) above the Earth, whose average radius is 3960 mile (6373 km), $2\varepsilon/A = 0.005$, whereas for a spacecraft at an altitude of 100 mile (161 km), $2\varepsilon/A = 0.05$. In the first instance we commit an error of about 0.5% when using the estimate (5.61), in the second we err by nearly 5%. Thus, so long as the object P does not stray too far from the planet, to a close approximation, the gravitational force $\mathbf{F} = m\mathbf{g}$ is a constant vector given by (5.61). The extent to which this approximation may be useful depends on the particular application. In situations where gravitational variations with the altitude are important, the estimate (5.61) is not to be used. (See Problem 5.22.)

5.10. Weight of a Body and its Center of Gravity

The gravitational force exerted by a body $\mathscr{B}_1$ on another body $\mathscr{B}_2$ is called the *weight of $\mathscr{B}_2$ relative to $\mathscr{B}_1$*. The gravitational field strength of a body $\mathscr{B}_o$ is given by (5.50), and the gravitational force it exerts on an object $\mathscr{O}$ is described by (5.53). This is the weight of $\mathscr{O}$ relative to $\mathscr{B}_o$. Thus, specifically, the weight $\mathbf{W}(P;\mathbf{X})$ at $\mathbf{X}$ of a particle P of mass $m(P)$, relative to $\mathscr{B}_o$, is defined by

$$\mathbf{W}(P;\mathbf{X}) \equiv m(P)\mathbf{g}(\mathbf{X}). \tag{5.62}$$

The universal law of gravitation (5.46), hence (5.50), involves invariant quantities that are independent of the reference frame—it is the same for all observers. Therefore, the weight of an object is the same for all observers; but it varies with the relative gravitational source. The weight of a particle P near the Earth is estimated by the constant force (5.61). The weight of the same particle in the neighborhood of the Moon, say, is also estimated by (5.61), but its value differs from its weight relative to the Earth. (See Problem 5.25.) In both cases, however, the mass $m(P)$ is the same—mass is an invariant property of a body; its weight is not. Henceforward, unless stated otherwise, the *weight of a body* shall mean its weight relative to the Earth. Thus, by (5.61) and (5.62), the weight $\mathbf{W}$ of a body modeled as a particle of mass m is an attractive body force abbreviated by $\mathbf{W} = W\mathbf{n} = mg\mathbf{n} = m\mathbf{g}$, where $\mathbf{n}$ is a unit vector directed toward the center of the Earth. In accordance with (5.53), the weight of a system of particles and a continuum are regarded similarly.

5.10.1. The Local Acceleration of Gravity—An Estimate

It is known from experimental measurements that the gravitational constant has the value $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2 \cdot kg^{-2}} = 3.43 \times 10^{-8}\ \mathrm{lb \cdot ft^2 \cdot slug^{-2}}$. The estimated average mass density of the Earth is $\rho = 5520\ \mathrm{kg/m^3}$, and its average radius is $A = 6373$ km, very nearly. Hence, the constant acceleration of gravity in the vicinity of the Earth *estimated* by (5.61) is $g = 9.824\ \mathrm{m/sec^2} = 32.23\ \mathrm{ft/sec^2}$. These values are reviewed and refined later on. In most engineering applications, however, it is customary to use the estimate $g = 9.8\ \mathrm{m/sec^2} = 32.2\ \mathrm{ft/sec^2}$.

Since the gravitational constant G is so very small, even when two bodies may be very close to one another, the gravitational force between them, though measurable (as demonstrated in experiments to measure G), is insignificant unless the mass of at least one of the bodies, like the Earth, is enormous. Therefore, the mutual attractive forces of neighboring bodies other than the Earth are ignored, and hence the total attractive gravitational force on an object is its weight. (See Problem 5.26.)

5.10.2. Center of Gravity of a Body

So far as a particle may be concerned there is no ambiguity as to where the weight vector acts—it acts on the particle. But when the total weight of a system of particles or of a body is introduced, the place relative to their material points at which the total weight of these bodies may be supposed to act is not evident. The concept of the center of gravity is introduced to clarify this question. We shall discuss the center of gravity for a body and leave as an exercise the parallel development for a system of particles.

The weight of a material parcel of mass $dm(P)$ at a point P of a body $\mathscr{B}$ is $\mathbf{g}(P)dm(P)$, where $\mathbf{g}(P)$ is the gravitational field strength at P due to the Earth. In accordance with the first equation in (5.51), the *weight* of $\mathscr{B}$ is defined by

$$\mathbf{W}(\mathscr{B}) = \int_{\mathscr{B}} \mathbf{g}(P)dm(P). \tag{5.63}$$

If the gravitational field strength is uniform over $\mathscr{B}$ so that $\mathbf{g}(P) = \mathbf{g}$, a constant vector, the weight of $\mathscr{B}$ is simply the product of $\mathbf{g}$ and its mass $m(\mathscr{B})$: $\mathbf{W}(\mathscr{B}) = m(\mathscr{B})\mathbf{g}$.

Since the body over which the gravitational field acts is small compared with the Earth, the Earth's field, though directed approximately toward its center, may be modeled as a parallel field over the body region, so that $\mathbf{g}(P) = g(P)\mathbf{n}$, where

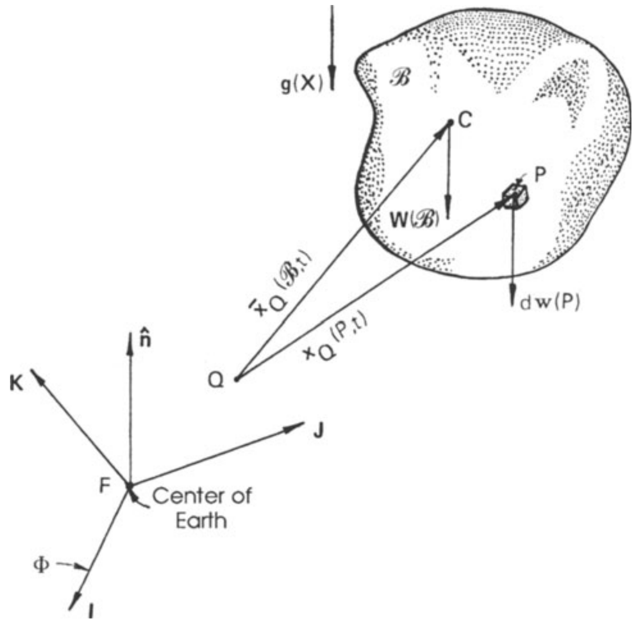


Figure 5.16. Schema for the equipollent moment condition in a parallel, variable gravity field.

$\mathbf{n}$ is a constant unit vector radially directed toward the Earth. Hence, (5.63) yields

$$W(\mathscr{B}) = \int_{\mathscr{B}} dw(P) = \int_{\mathscr{B}} g(P)dm(P), \tag{5.64}$$

in which $dw(P) \equiv g(P)dm(P)$ is the elemental weight of the parcel $dm(P)$.

By (5.63), the distribution of the weight of a body in a parallel, but variable gravitational field is equipollent to the single force $\mathbf{W}(\mathscr{B})$. In addition, for any assigned point Q in the Earth frame $\Phi = \{F; \mathbf{I}_k\}$ shown in Fig. 5.16, the moment $\mathbf{M}_Q$ about Q of the weight distribution $d\mathbf{w}(P) = \mathbf{n}dw(P)$ in the parallel Earth field is equipollent to the moment about Q of the total weight $\mathbf{W}(\mathscr{B}) = \mathbf{n}W(\mathscr{B})$ acting at a point C along its line of action in Φ. The unknown position vector of C from Q is denoted by $\bar{\mathbf{x}}_Q(\mathscr{B})$ in Fig 5.16. Thus, the equipollent moment condition (5.29) is

$$\mathbf{M}_Q = \bar{\mathbf{x}}_Q(\mathscr{B}) \times \mathbf{W}(\mathscr{B}) = \int_{\mathscr{B}} \mathbf{x}_Q(P) \times d\mathbf{w}(P), \tag{5.65}$$

wherein $\mathbf{x}_Q(P)$ is the position vector of a material parcel of weight $d\mathbf{w}(P)$ at P. With $d\mathbf{w} = dw\mathbf{n}$ and use of (5.64), (5.65) yields $W(\mathscr{B})\bar{\mathbf{x}}_Q \times \mathbf{n} = \int_{\mathscr{B}} \mathbf{x}_Q(P)dw(P) \times \mathbf{n}$. For simplicity, let us discard the subscript Q, and note that in general the position vectors may vary with time t, as suggested in Fig. 5.16. Then, with these adjustments, since Q may be chosen arbitrarily and $\mathbf{n}$ is a fixed direction, we may satisfy this equation by choosing the point at $\bar{\mathbf{x}}$ *defined by*

$$W(\mathscr{B})\bar{\mathbf{x}}(\mathscr{B}, t) = \int_{\mathscr{B}} \mathbf{x}(P, t)\, dw(P) \tag{5.66}$$

to provide the location from Q of the point C at which the weight of $\mathscr{B}$ acts to produce a moment about Q equal to that of its distribution. The point of the body $\mathscr{B}$ defined by $\bar{\mathbf{x}}(\mathscr{B}, t)$ in (5.66) is called the *center of gravity* of $\mathscr{B}$.

The location of the center of gravity will depend on the variable gravitational field strength $g(P)$ and the orientation of the body, which also might be nonhomogeneous. So, if the body is moved to a different configuration at another place in a variable gravity field, the center of gravity generally is not at the same place in the body frame; and hence the center of gravity generally is not a unique point in the body frame.

Example 5.8. A homogeneous cylinder $\mathscr{B}$ of height h and its base at the distance a from the Earth's center F in frame $\Phi = \{F; \mathbf{I}_k\}$ is shown in Fig. 5.17. Show that the center of gravity in a variable gravity field is not an invariant point in the body reference frame.

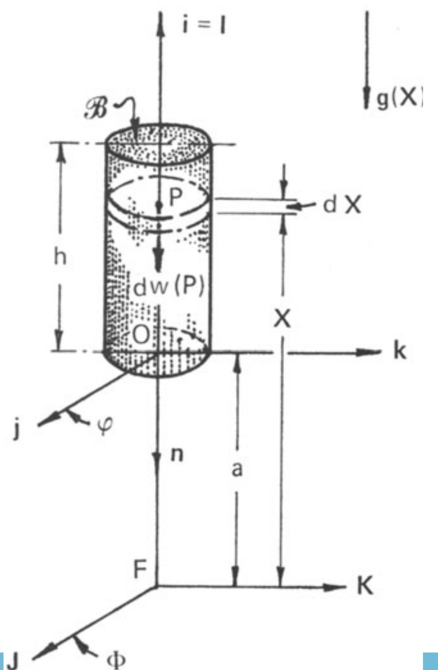


Figure 5.17. Schema for evaluation of the center of gravity of a uniform cylinder in a variable, parallel gravitational field.

Solution. The second equation in (5.59) gives the variable gravitational field strength $g(P) = MG/X^2$ at P due to the Earth. The Earth's mass is $M = m(\mathscr{B}_o)$ and X is the distance from F to a material parcel at P having weight $dw(P) = g(P)dm(P) = (MG/X^2)\sigma dX$, where $\sigma = m/h$ is the mass per unit length of $\mathscr{B}$. Integration in accordance with (5.64) shows that the weight of the cylinder in the given configuration will vary with the distance a from the Earth:

$$W(\mathscr{B}) = \frac{mMG}{h}\int_a^{a+h}\frac{dX}{X^2} = \frac{mMG}{a(a+h)}. \tag{5.67a}$$

The location $\bar{\mathbf{x}}(\mathscr{B}) = \bar{X}\mathbf{I} + \bar{Y}\mathbf{J} + \bar{Z}\mathbf{K}$ of the center of gravity from F is given by (5.66). With $\mathbf{x}(P) = X\mathbf{I} + Y\mathbf{J} + Z\mathbf{K}$ in Fig. 5.17, we find by symmetry about the $\mathbf{I}$-axis that $\bar{Y} = \bar{Z} = 0$ and

$$W(\mathscr{B})\bar{X} = \frac{MmG}{h}\int_a^{a+h}\frac{dX}{X} = \frac{MmG}{h}\ln\left(\frac{a+h}{a}\right). \tag{5.67b}$$

Using (5.67a) and introducing $\bar{x} \equiv \bar{X} - a$, we obtain the location $\bar{x}$ of the center of gravity in the body frame $\varphi = \{O; \mathbf{i}_k\}$ in Fig. 5.17:

$$\bar{x} = a\left[\frac{1 + h/a}{h/a}\ln\left(1 + \frac{h}{a}\right) - 1\right]. \tag{5.67c}$$

This result shows that the center of gravity in the body frame varies with a, the vertical distance of O from the center of the Earth. If the body is moved vertically to another place, the location $\bar{x}$ of the center of gravity in the body frame will change. Hence, in contrast with the invariant center of mass of the same body, the center of gravity generally is not a unique point in the body reference frame φ. The center of gravity is not an invariant property of the body.

On the other hand, the variable gravity effect on the position of the center of gravity of an ordinary body usually may be considered negligible. Because the body's height h is small compared to the radial distance a from the center of the Earth, we may ignore in the last formula all terms of order greater than the first in $h/a \ll 1$. We recall the series expansion $ln(1 + z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \cdots$ valid for $0 < z = h/a < 1$ and thus obtain the *unique, approximate location* $\bar{x} = h/2$ of the center of gravity in the body frame φ. □

In most practical cases of interest, the gravitational field throughout a body $\mathscr{B}$ that is small compared with the Earth may be approximated by a constant field of strength g throughout that body. Hence, the center of gravity of a body $\mathscr{B}$, even in a variable, parallel gravitational field, is the unique point in the body frame φ whose position vector $\bar{\mathbf{x}}$ from any assigned point Q in Φ is given by (5.66), very nearly. Therefore, so far as its weight is concerned, the body may be replaced by a particle of weight $\mathbf{W}(\mathscr{B})$ located at its center of gravity. Of course, the center of gravity particle need not be a material point of $\mathscr{B}$, but it may be. In a fixed configuration of the body, the definition (5.66) is independent of the choice of the reference point Q in Φ, and hence in a locally, constant gravity field, the center of gravity is the unique point C in the body frame relative to which

$$\int_{\mathscr{B}} \boldsymbol{\rho}(P,t)dw(P) = \mathbf{0}. \tag{5.68}$$

Here $\boldsymbol{\rho}(P,t)$ is the position vector from C to the parcel of weight $dw(P)$ at P. Equation (5.68) states that the moment of the weight distribution of the body about its center of gravity vanishes. The foregoing construction does not specify that the body be rigid. For a rigid body, however, $\boldsymbol{\rho}(P,t)$ is independent of time in a body frame.

We have learned that in general the center of gravity is not an invariant property of the body—it varies with the gravitational field strength in the region of space that the body currently occupies. However, because the field strength due to the Earth varies insignificantly over ordinary bodies, it is quite reasonable to replace the variable, parallel gravitational field by a locally uniform, parallel field. In this case, (5.66) reduces to (5.12) so that $\bar{\mathbf{x}} = \mathbf{x}^*$. *Thus, in a locally uniform gravitational field, the center of gravity and the center of mass of a body coincide, in which case the center of gravity shares all of the properties of the center of mass.*

Finally, we recall that sometimes the *weight density* $\gamma(P) \equiv \rho(P)g(P)$, the weight per unit volume of $\mathscr{B}$, is used in engineering analysis. In this case, we have $dw(P) = \gamma(P)dV(P)$. Thus, if the weight density of a body is constant, the weight of the body is the product of its weight density and its volume $V(\mathscr{B})$: $W(\mathscr{B}) = \gamma V(\mathscr{B})$. Hence, from (5.66) and (5.14), *the center of gravity of a body of uniform weight density is at its centroid. For a homogeneous body in a locally uniform, parallel gravitational field, ρ, g, and $\gamma = \rho g$ are constants, and hence in this important special case the center of gravity, the center of mass, and the centroid of the body are coincident points.* In general, however, they are not.

5.11. Coulomb's Laws of Friction

So far, our study has focused on one important kind of body force, the familiar force of gravity. We now consider a familiar kind of contact force, the frictional force that arises between pairs of separate bodies in their pending or relative sliding motion. Two physical laws, known as *Coulomb's laws*, govern the nature of this frictional force.

The first law of friction was known for a long time before Charles Coulomb (1736–1806), a senior captain in the French Royal Corps Engineers, verified it in 1781 during investigation of mechanical improvements for military gear. Historians, however, discovered long ago a statement of the first law in the notebooks of the famous Italian artist and inventor Leonardo da Vinci (1452–1519). From simple experiments, da Vinci concluded that *the amount of friction is proportional* *to the normal pressure between the contacting bodies and is independent of their area of contact.* Da Vinci's empirical proposition thus provided the first record in scientific writings of a law for sliding friction, an important contribution to mechanical science that was lost for nearly three centuries!

The notebooks, for several reasons, were virtually unknown prior to 1797. Translation of the manuscripts, language aside, was hampered by da Vinci's habit of writing in a reversed, left-directed fashion that required reading from a mirror, certainly an uninviting prospect. Though da Vinci apparently planned to assemble his voluminous notes for publication, this never happened. Upon his death in 1519, the encoded notebooks were passed to a close friend who guarded and preserved them until his own death in 1570; and from that time onward the manuscripts passed many hands, some parts being lost forever. Thirteen volumes survived and eventually were collected in the Ambrosian Library at Milan. But in the invasion of Italy in 1796, the documents were seized by Napoleon Bonaparte and carried to Paris, where for the first time they were studied by J. B. Venturi who later described them in an essay published in 1797. (See Hart, Chapters I and VII.)

It is no surprise, therefore, that da Vinci's law of friction was unknown to the French engineer Guillaume Amontons, who rediscovered it in 1699, nearly 200 years after da Vinci. It is astonishing, however, that the French Academy of Sciences, which expressed disbelief of the independence of the area of contact, received Amontons's rule with skepticism. Yet later, in 1781, the Academy awarded Coulomb a prize for essentially the same thing, though presented more thoroughly and in broader terms. (See Deresiewicz.) Coulomb's exemplary experiments established, not one, but two basic laws of friction that express a clear distinction between static friction and dynamic friction that went unnoticed by all others. These principles characterize the nature of the contact force between surfaces at rest and in relative sliding motion; they are the focus of the discussion that follows.

5.11.1. Contact Force between Bodies

A contact force is the mutual force acting at the interface between separate bodies that touch one another. At each interface point $\mathbf{q}$, the contact force $\boldsymbol{\gamma}(\mathbf{q})$, say, exerted by one body upon the other may be separated into component forces $\boldsymbol{\eta}(\mathbf{q})$ and $\boldsymbol{\tau}(\mathbf{q})$, respectively, normal and tangent to the interface at $\mathbf{q}$, so that $\boldsymbol{\gamma}(\mathbf{q}) = \boldsymbol{\eta}(\mathbf{q}) + \boldsymbol{\tau}(\mathbf{q})$. The normal component describes the mutual pulling (tension) or pushing (compression) of one body by the other perpendicular to the interface; it is called the *normal force*. If the contacting bodies $\mathscr{B}_1$ and $\mathscr{B}_2$ are subsets of the same body $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ separated by an imaginary material surface $\mathscr{A}$, the tangential component $\boldsymbol{\tau}(\mathbf{q})$ characterizes the mutual resistance to shearing of the two parts along $\mathscr{A}$, so that $\boldsymbol{\tau}(\mathbf{q})$ is named the *shear force*. These particular contact forces play a paramount role in the mechanics of deformable solids and fluids. We shall encounter them in a different setting in various problems ahead. If we wish, for example, to determine how the tension in the string of a pendulum varies as the pendulum swings to and fro, it is necessary to introduce an imaginary cut in the string, and show in its place in a free body diagram of the pendulum bob, the normal (tensile) force that the string exerts on the bob. On the other hand, when two bodies $\mathscr{B}_1$ and $\mathscr{B}_2$ are physically separate, to maintain their contact the normal component of the contact force must be compressive; and its tangential component characterizes the mutual resistance to sliding of one body surface over the other, a natural effect that everybody knows as *friction.* In this case, the tangential component is called the *frictional force.*

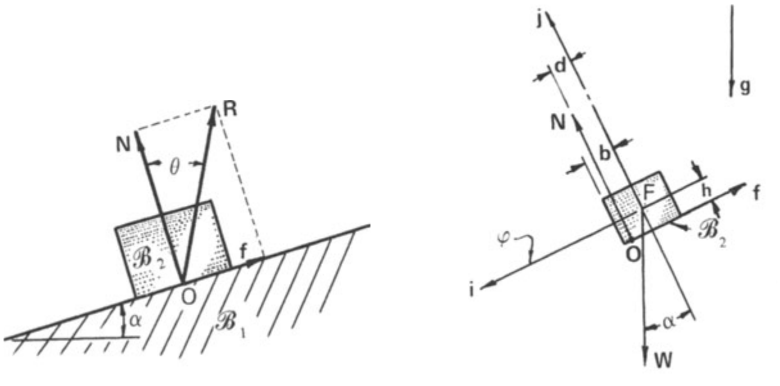
The description of the frictional force is far more complicated than suggested above. The normal and tangential components of the interfacial force are distributed over the area of contact. But the actual area of contact is unknown. Indeed, even the most carefully polished surfaces look under magnification like miniature mountain ranges with hills and valleys that are much larger than molecular dimensions, and the contacting surfaces press upon these tiny mountains. Therefore, the actual area of contact may be much smaller than the apparent area of contact described by the macroscopic dimensions of the interfacial region.

Although interlocking effects of the surface asperities play a role in the overall complex mechanism[‡] of sliding friction, it is known from sophisticated measurements that frictional force arises mainly from the force required to shear the mountain peaks. Moreover, these experiments reveal that the actual area of contact, accounting for the deformation, depends on the intensity of the normal force. This area, however, is very nearly independent of the apparent interfacial area of the sliding bodies. The intense pressure at the contact points increases the area of contact until it is large enough to support the load. But in observation of the frictional resistance, the growth in the real area of contact manifests itself through the increase in the applied normal thrust, and hence is independent of the apparent interfacial area of contact. These measurements confirm da Vinci's primary observations and support his law of sliding friction; and they are the foundation for Coulomb's laws.

The distribution of the contact force is also unknown. But information about this force is required before any problem that involves friction can be solved. To hurdle this obstacle, we adopt the advance strategy that the normal and tangential distributions of the contact force, whatever the actual area of contact may be, are equipollent to a resultant normal force **N** and a resultant frictional force **f** that acts to oppose the relative motion of two contacting separate bodies $\mathscr{B}_1$ and $\mathscr{B}_2$. Thus, if $\boldsymbol{\eta}(\mathbf{q})$ and $\boldsymbol{\tau}(\mathbf{q})$ denote the normal and tangential force distributions per unit area a of the apparent contact area A, then $\mathbf{N} = \int_A \boldsymbol{\eta}(\mathbf{q})da$, $\mathbf{f} = \int_A \boldsymbol{\tau}(\mathbf{q})da$; and the resultant contact force $\mathbf{R} = \int_A \boldsymbol{\gamma}(\mathbf{q})da$ exerted by $\mathscr{B}_1$ on $\mathscr{B}_2$ is

$$\mathbf{R} = \mathbf{N} + \mathbf{f}. \tag{5.69}$$

[‡] See the classical treatise by Bowden and Tabor. Contemporary molecular theories of friction and modern surface measurement techniques are discussed in the referenced article by Krim.



(a) Contact Forces on $\mathscr{B}_2$ by $\mathscr{B}_1$ (b) Free Body Diagram of $\mathscr{B}_2$

Figure 5.18. The contact forces exerted by the body $\mathscr{B}_1$ on the body $\mathscr{B}_2$, and the free body diagram of $\mathscr{B}_2$.

Thus, instead of having to deal with the unknown surface load distributions, we may work with their resultants in (5.69). The resultant contact forces exerted by a body $\mathscr{B}_1$ on another body $\mathscr{B}_2$ are shown in Fig. 5.18a. Other contact and body forces may act on $\mathscr{B}_2$, but these are not shown here. Of course, the contact forces exerted by $\mathscr{B}_2$ on $\mathscr{B}_1$ are opposite to those exerted by $\mathscr{B}_1$ on $\mathscr{B}_2$.

5.11.2. Governing Principles of Sliding Friction

Perfectly smooth, frictionless surfaces do not exist. Nonetheless, sometimes the surface asperity is so fine that the surface feels perfectly smooth to our sensation of touch. Therefore, in situations where frictional effects may be considered negligible or unimportant, we may sometimes consider an ideal model of smooth contacting surfaces that offer no sliding resistance whatever, a model that brings to mind the seemingly effortless, graceful motion of a skater on virtually frictionless ice. In this ideal case, the frictional force is zero and the contact force is normal to the interfacial tangent plane, that is, $\mathbf{f} = \mathbf{0}$ and $\mathbf{R} = \mathbf{N}$ in (5.69). This ideal property characterizes a so-called *smooth surface.*

When the surfaces are not perfectly smooth, it is intuitively clear that if the angle α of the inclination of the plane surface of the body $\mathscr{B}_1$ shown in Fig. 5.18a is sufficiently small, the body $\mathscr{B}_2$ will remain at rest on the plane. But as α is gradually increased, the magnitude of the frictional force must also increase gradually to restrain $\mathscr{B}_2$. Eventually, the *angle of repose* α will exceed a certain critical, maximum value α_c at which the frictional force can no longer sustain the equilibrium of $\mathscr{B}_2$, and $\mathscr{B}_2$ will begin to slide down the plane. Thus, the magnitude $f = |\mathbf{f}|$ of the *static frictional force* between the bodies eventually will reach a critical value $f = f_c$, called the *critical force, at which slip is imminent.* Of course, after sliding begins, friction continues to act between the contacting surfaces to oppose the relative motion. Clearly, this *dynamic frictional force* f_d cannot exceed the static, critical force; in fact, experiments show that $f_d < f_c$. These critical values of the frictional forces depend on the intensity of the normal contact force between the bodies. Indeed, we see readily that when lightly pressed together, our hands can be slid easily one upon the other; but when pressed tightly together, their relative sliding is rendered more difficult. The values of the critical, static and dynamic frictional forces are most simply related to the magnitude of the normal contact force between the bodies in accordance with the following basic and ideal principles of friction commonly known as **Coulomb's laws of friction**.

1. **The law of static friction:** *The critical magnitude f_c of the static frictional force between dry or lightly wetted surfaces that are at the verge of slipping relative to each other is proportional to the intensity N of the mutual, resultant normal force between them*:

$$f_c = \mu N. \tag{5.70}$$

   The constant μ, called the coefficient of static friction, is independent of the interfacial contact area; it depends only on the nature of the contacting surfaces.

2. **The law of dynamic friction:** *The magnitude f_d of the frictional force between two dry or lightly wetted surfaces sliding relative to one another is proportional to the intensity N of the mutual, resultant normal force between them*:

$$f_d = \nu N. \tag{5.71}$$

   The constant ν, named the coefficient of dynamic friction, is less than the static coefficient for the same conditions, $\nu < \mu$. Moreover, ν is independent of the interfacial area of contact and of the relative sliding speed of the surfaces; it depends only on the nature of the contacting surfaces.

The first law determines the *greatest* frictional force that can develop between contacting surfaces *before* sliding occurs, whereas the second law determines the magnitude of the frictional force that acts *during* the relative sliding motion. If a sliding motion between two bodies has not occurred and is not imminent, then the magnitude f of the frictional force is always less than the critical force f_c and may be determined by equilibrium considerations. These remarks are summarized schematically in Fig. 5.19 to illustrate the relations

$$\text{Static: } 0 \le f \le f_c = \mu N; \tag{5.72}$$

$$\text{Dynamic: } 0 \le f = f_d = \nu N < f_c. \tag{5.73}$$

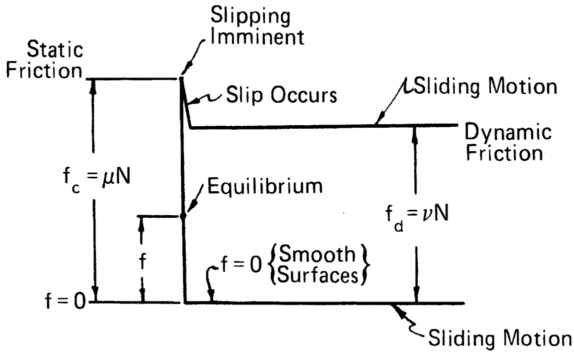


Figure 5.19. Graphical interpretation of Coulomb's laws of static and dynamic friction.

Note that $f = f_c$ holds in (5.72) only when relative slip is imminent; and $f = f_d$ holds in (5.73) only while sliding occurs. Further, $f = 0$ holds only for ideal, perfectly smooth surfaces for which $\mu = \nu = 0$. Otherwise, since $\nu < \mu$, once slip is achieved, a smaller force acts to retard the motion. These effects are assumed to be independent of the interfacial area of contact and of the relative speed of the dry or lightly wetted surfaces.

The static and dynamic coefficients of friction will depend only on the nature of the contacting material surfaces, that is, on the materials of which the bodies are made, their surface roughness quality, their degree of lubrication, their temperature, perhaps their chemical characteristics, and some other less important things. Clearly, the values of both μ and ν must be found by experiments. Also, when one body rolls on another, there is very little interfacial slip; but the bodies still experience mutual resistance to rolling, which is called *rolling friction*. Everyone knows that it is easier to roll than to slide a body on a flat surface; hence, rolling friction is considerably smaller than sliding friction. Further, when a layer of fluid, such as air or water, separates two surfaces, there is a resisting force exerted by the fluid which is called *drag* or *viscous friction*. Both rolling and viscous friction are determined by laws that are entirely different from Coulomb's rules of sliding friction. The effects of viscous friction are discussed in Chapter 6. The interested reader may consult the sources cited at the end of this chapter for details on these additional matters. We now turn to some examples.

5.11.3. Equilibrium of a Block on an Inclined Plane

Let us consider the familiar, elementary problem of equilibrium of a rigid block $\mathscr{B}_2$ shown in Fig. 5.18a at rest on an inclined plane $\mathscr{B}_1$. Our focus is on the general procedure for setting up and solving this problem. In addition, some elementary results of static friction are also reviewed.

First, choose $\mathscr{B}_2$ as a free body (the system to be investigated). Now identify all of the contact and body forces that act on $\mathscr{B}_2$ alone. We may ignore the contact force of the surrounding air. (Why?) Then the only body that touches $\mathscr{B}_2$ is the body $\mathscr{B}_1$, so the total contact force acting on $\mathscr{B}_2$ consists of the equipollent normal force $\mathbf{N}$ and frictional force $\mathbf{f}$ due to $\mathscr{B}_1$, or the equivalent reaction force $\mathbf{R}$. The Earth is the only body that exerts a significant body force on $\mathscr{B}_2$, hence the total body force acting on $\mathscr{B}_2$ is its weight $\mathbf{W}$. All of the forces that act on $\mathscr{B}_2$, whether it be in equilibrium or in motion in an assigned inertial frame are shown in the free body diagram in Fig. 5.18b. The direction of these forces must be consistent with the physical situation. In particular, $\mathbf{f}$ must act to retard the potential motion of $\mathscr{B}_2$, $\mathbf{N}$ must support $\mathscr{B}_2$, and $\mathbf{W}$ must be directed toward the center of the Earth. The vector $\mathbf{g}$ denotes in the figure the direction of the gravitational attraction of the Earth.

Any inertial frame may be introduced to formulate the problem, but one choice may be mathematically more convenient than another. The inertial frame $\varphi = \{F; \mathbf{i}_k\}$ shown in Fig. 5.18b is a good choice because the forces are most easily related to it. The free body diagram shows that the total force acting on the block $\mathscr{B}_2$ is

$$\mathbf{F}(\mathscr{B}_2, t) = \mathbf{W} + \mathbf{f} + \mathbf{N}. \tag{5.74a}$$

Next, express these forces in terms of their components in φ:

$$\mathbf{W} = W(\sin\alpha\mathbf{i} - \cos\alpha\mathbf{j}), \quad \mathbf{f} = -f\mathbf{i}, \quad \mathbf{N} = N\mathbf{j}. \tag{5.74b}$$

Here $W = mg$, f, and N denote the magnitudes of these forces. This completes the primary phase in the problem formulation.

5.11.3.1. *The Force Equilibrium Relations*

Since the block is in equilibrium in φ, in accordance with (5.45), the total force (5.74a) and the total moment about a point fixed in φ of the forces in (5.74b) must vanish for all times. First, consider the forces. Substitute (5.74b) into (5.74a), and write the force equilibrium equation,

$$\mathbf{F}(\mathscr{B}_2, t) = (W\sin\alpha - f)\mathbf{i} + (N - W\cos\alpha)\mathbf{j} = \mathbf{0}. \tag{5.74c}$$

Consequently, each vector component must be zero; and so the normal and frictional forces on $\mathscr{B}_2$ are determined by

$$N = W\cos\alpha, \qquad f = W\sin\alpha. \tag{5.74d}$$

5.11.3.2. *The Moment Equilibrium and No Tip Conditions*

The zero moment equation $\mathbf{M}_O(\mathscr{B}_2, t) = \mathbf{0}$, the second of the equilibrium conditions for a rigid body in (5.45), will fix the location d of the line of action of the resultant normal force $\mathbf{N}$. Since the resultants $\mathbf{N}$ and $\mathbf{f}$ in Fig. 5.18b are concurrent at a certain point O in the interfacial plane, it is clear that the moment $\mathbf{M}_O$ of the forces (5.74b) taken about this fixed point in φ will vanish if and only if the line of action of **W** also passes through O. Indeed, it is easily shown in general that three concurrent forces acting on a particle in equilibrium must be coplanar. (Is the same generally true for a particle in motion?) Hence, the concurrent forces **W**, **N**, and **f** lie in the vertical plane containing the interface point O and the center of gravity of the block at the height h above the plane surface. Clearly, the block will not tumble forward so long the line of action of **W** falls within the distance b to the leading edge of the block, and hence for $d \le b$ in Fig. 5.18b. The line of action of **W**, and hence the point O, is at the leading edge when the angle $\alpha = \alpha_t = \tan^{-1} b/h$. Therefore, if the plane's angle of inclination may be increased to the angle α_t without exciting slip so that $\alpha_t < \alpha_c$, the critical angle of sliding friction, the block will be at the verge of tipping over rather than sliding down the plane. The slightest further increase in the inclined angle α_t pushes the line of action of **W** ahead of the leading edge and the block will topple down the plane before sliding impends, because the moment of **W** about O at the leading edge of the block is no longer balanced. Henceforward, we shall suppose that the moment equilibrium condition $\alpha \le \alpha_t$ for no tipping of the block is respected. (See Problem 5.28.)

5.11.3.3. The No Slip Condition

From (5.74a), $\mathbf{R} = \mathbf{f} + \mathbf{N} = -\mathbf{W}$; and hence the resultant contact force must be opposite to the weight **W**. Indeed, the zero moment condition for equilibrium in (5.45) shows that **R** and **W** must be collinear. Hence, in consequence of equilibrium, the angle θ that **R** makes with the normal to the inclined plane surface in Fig. 5.18a is equal to the plane's angle α. From (5.74d) and (5.72), it is seen that the angle of repose must satisfy the inequality

$$\tan\alpha = \frac{f}{N} \le \mu, \tag{5.74e}$$

in order that f shall be less than the critical force f_c at which $\mathscr{B}_2$ will be at the verge of slipping. The greatest angle α_c for which (5.74e) holds is called the *critical angle of friction*; it is given by Coulomb's law (5.70) as expressed in (5.74e), that is,

$$\tan\alpha_c = \frac{f_c}{N} = \mu. \tag{5.74f}$$

Thus, the tangent of the angle of repose is critical when it reaches a value equal to the coefficient of static friction, a value that is independent of the weight of the block $\mathscr{B}_2$. When $\alpha = \alpha_c < \alpha_t$, the block will not topple over, but the slightest further increase in the plane's inclination will cause the block to slide, and our equilibrium analysis, no longer valid, must be replaced by analysis of the block's motion. □

The basic *free body formulation* procedure used above is applied almost invariably in the formulation of all problems in both statics and dynamics. Sometimes problems may be solved easily in direct vector notation, so the decomposition of the forces into components may not be necessary, but more often than not, for simplicity, it is. It cannot be too strongly emphasized that *the free body formulation for the total force is the same for both a statics and a dynamics problem; and it is important that the student become thoroughly familiar with this method.* The analysis of the block's motion follows.

5.11.4. Motion of a Block on an Inclined Plane

We now encounter our first application of dynamics in the analysis of the sliding motion of a block down an inclined plane. Let us continue from where we left off above and suppose that the plane's angle of inclination exceeds the critical angle of friction. Then the block slides down the plane without tumbling provided that $\alpha_c < \alpha < \alpha_t$ holds. The free body diagram for $\mathscr{B}_2$ is shown in Fig. 5.18b; it is the same as before. Consequently, the free body formulation for the dynamics problem is the same as that for the statics problem and leads again to (5.74a); but this time the block has a translational motion down the plane, and the appropriate dynamical equations of motion must be decided. Since the body $\mathscr{B}_2$ is rigid and does not tumble, its motion is determined by the Newton–Euler equation (5.43) for its center of mass:

$$\mathbf{F}(\mathscr{B}_2, t) = \mathbf{W} + \mathbf{f} + \mathbf{N} = m(\mathscr{B}_2)\mathbf{a}^*(\mathscr{B}_2, t). \tag{5.75a}$$

The next step is the formulation of the appropriate kinematics. Since the motion is along a straight line on and down the plane,

$$\mathbf{a}^*(\mathscr{B}_2, t) = \ddot{\mathbf{x}}^*(\mathscr{B}_2, t) = \ddot{x}^*\mathbf{i}, \tag{5.75b}$$

where $\mathbf{x}^*(\mathscr{B}_2, t) = x^*\mathbf{i}$ is the position vector of the center of mass of $\mathscr{B}_2$ from the fixed origin F in φ. Collecting the first equation in (5.74c) and (5.75b) in (5.75a), we have

$$(W\sin\alpha - f)\mathbf{i} + (N - W\cos\alpha)\mathbf{j} = m\ddot{x}^*\mathbf{i}. \tag{5.75c}$$

This yields the component equations of motion

$$m\ddot{x}^* = W\sin\alpha - f \quad \text{and} \quad N - W\cos\alpha = 0 \tag{5.75d}$$

to be solved for the normal force N and for the rectilinear motion $x^*(\mathscr{B}_2, t)$ of the center of mass of $\mathscr{B}_2$.

The second equation of (5.75d) determines the normal force $N = W\cos\alpha$, and Coulomb's second law (5.71) for the sliding motion gives

$$f = f_d = \nu N = \nu W\cos\alpha. \tag{5.75e}$$

Then with (5.75b) and $W = mg$, the first relation in (5.75d) yields

$$\mathbf{a}^* = \ddot{\mathbf{x}}^* = g(\sin\alpha - \nu\cos\alpha)\mathbf{i}. \tag{5.75f}$$

Thus, the acceleration of the center of mass, indeed the acceleration of every particle of the block in its parallel translation down the plane, is a constant vector.

The velocity and the motion of the center of mass point are now easily obtained by integration of (5.75f), subject to specified initial conditions. Let us suppose that the block is released from rest in φ so that $\mathbf{v}^*(\mathscr{B}_2, 0) = \mathbf{0}$ and $\mathbf{x}^*(\mathscr{B}_2, 0) = \mathbf{0}$ at $t = 0$. Then integration of (5.75f) yields

$$\mathbf{v}^*(\mathscr{B}_2, t) = gt(\sin\alpha - \nu\cos\alpha)\mathbf{i}, \qquad \text{then} \qquad \mathbf{x}^*(\mathscr{B}_2, t) = \tfrac{1}{2}gt^2(\sin\alpha - \nu\cos\alpha)\mathbf{i}. \tag{5.75g}$$

We thus find that *the sliding motion is independent of the mass of the body—it is the same for all bodies, both large and small, so long as* (5.71) *holds and the no tip constraint is satisfied.* This completes the analysis of the sliding translational motion of the block, but some additional points are noted in the exercise.

Exercise 5.6. Equation (5.75a) shows that in the dynamics problem the resultant contact force $\mathbf{R}$ on the block is *not opposite* to the weight $\mathbf{W}$. Consider at time t the moment equation (5.44) for the applied forces about the fixed origin F at the initial position of the center of mass of the block. (a) Prove that $\mathbf{M}_F = \mathbf{0}$, and thus show that $\mathbf{R}$ is *concurrent* with $\mathbf{W}$ through the center of mass. (See Example 5.5, page 23.) Therefore, in the absence of rotation, the moment of the forces about the moving center of mass point also vanishes. (b) Show that the same result follows when the fixed point F is in the contact plane at the initial position. What is $\dot{\mathbf{h}}_F$ in this case? (See Problem 5.28.) □

Our sliding block example illustrates for a simple translational motion the more complex nature of the motion analysis of bodies and the importance of the center of mass. The translational motion of the block is described completely by the motion of its center of mass particle, regardless of its location in the body. Notice that the actual identity of the center of mass was unimportant in (5.75f), and it remained anonymous in (5.75g)—its location (actually the center of gravity in this case) was important only in the discussion of potential rotational effects expressed by the no tip condition derived from the moment equation. The anonymity of the center of mass is typical of many rigid body problems in which rotational effects are absent.

5.12. Applications of Coulomb's Laws

Two problems that use Coulomb's laws in demonstration of the predictive value of the principles of mechanics are studied. The first example illustrates the

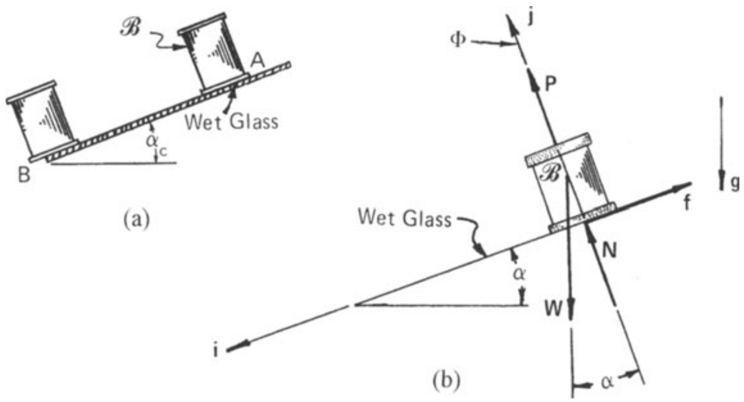


Figure 5.20. A simple experiment demonstrating the pressure induced, friction reduction principle.

phenomenon of pressure-induced friction reduction useful in a variety of engineering applications. The second example demonstrates the application of basic principles in providing the solution to a major technical problem during World War II.

5.12.1. The Sliding Can Experiment

An empty beverage can[§] $\mathscr{B}$ having identical top and bottom rims is shown in Fig. 5.20a. The can is placed at A on a sheet of slightly wetted glass, which is then gradually tilted until the critical angle α_c is attained at which sliding of the can is initiated. Since the can slides on its narrow rim, the critical angle is independent of whether the open or the closed end of the can is upward. Of course, upon reaching the edge of the glass at B, the can falls off. The experiment is conducted at room temperature and the measured critical angle of friction is about 17°. Coulomb's laws hold for slightly wetted surfaces, and (5.74f) thus determines the coefficient of static friction $\mu = \tan 17^\circ = 0.30$.

The empty can is then chilled and the test repeated by first placing the can on the wetted glass with its *open end upward*. The critical angle is found to be the same as before, thus showing for this case that μ is independent of the temperature. Finally, the can is chilled to the same temperature as before and placed on the wetted surface with its *open end downward*. Surprisingly, the can starts to slide when the critical angle α_c is only 1° or 2°; and it slides down the entire length of the glass held at this very small inclination. But it *stops* rather abruptly when the open end extends just beyond the edge of the sheet at B in Fig. 5.20a.

[§] Adapted from the article by M. K. Hubbert and W. W. Rubey cited in the chapter references. See also the related articles by M. B. Karelitz and by B. Noble reported therein.

This curious phenomenon occurs because after a few seconds the cold, trapped air expands as it begins to warm, causing the internal air pressure to increase. Because the surface area of the closed end of the can is greater than that of its open end, there is a resultant uplifting, internal normal pressure on the closed end that partially supports the weight of the can, and thus reduces the normal surface reaction force between the can and the glass. The can stops suddenly at the edge of the sheet because the pressure is abruptly released. To prove this hypothesis, we analyze the phenomenon.

5.12.1.1. The Equilibrium Analysis

We begin by showing in Fig. 5.20b the free body diagram of the chilled can placed on the glass with its open end downward in the inertial frame Φ. The body force is the weight $\mathbf{W}$ of the can. In addition to the normal and frictional contact forces $\mathbf{N}$ and $\mathbf{f}$, there is also a resultant internal contact force $\mathbf{P}$ on the closed end of the can due to excess of the internal air pressure over the outside air pressure. Thus, the total force acting on the can $\mathscr{B}$ is

$$\mathbf{F}(\mathscr{B}, t) = \mathbf{W} + \mathbf{N} + \mathbf{f} + \mathbf{P}. \tag{5.76a}$$

Introducing in (5.76a) the component representations for $\mathbf{W}$, $\mathbf{N}$, and $\mathbf{f}$ given in (5.74b), noting that $\mathbf{P} = P\mathbf{j}$, and equating each component to zero in the equilibrium equation $\mathbf{F}(\mathscr{B}, t) = \mathbf{0}$, we find the contact forces

$$f_d = W \sin \alpha_d, \qquad N_d = W \cos \alpha_d - P, \tag{5.76b}$$

in which the subscript notation should be evident. We see that N_d, the normal surface reaction force when the open end is down, is indeed reduced by the excess internal contact force P.

The case when the open end of the can is upward follows from (5.76b) in which we set $P = 0$, adjust the subscripts accordingly, and thus recover (5.74d). When α_u is increased gradually until sliding is imminent, (5.74f) yields

$$\mu \equiv f_{cu}/N_u = \tan \alpha_{cu}, \tag{5.76c}$$

α_{cu} denoting the critical angle of friction when the open end of the can is upward. This gives the coefficient of static friction μ between the can and the glass.

Now let us return to the case when the open end of the can is downward, and rewrite (5.76b) to obtain

$$\tan \alpha_d = (1 - p(\alpha_d)) \frac{f_d}{N_d}, \tag{5.76d}$$

in which $p(\alpha_d) \equiv P/W \cos \alpha_d$ is the ratio of the uplifting force P to the normal component $W \cos \alpha_d$ of the weight of the can. Hence, $0 \leq p(\alpha_d) \leq 1$. Suppose that α_d is gradually increased to the angle α_{cd} at which the can is at the verge of sliding down the plane. Now remember that in both instances the coefficient of static friction in (5.70) is defined by the ratio of the tangential surface frictional force to the normal surface reaction force; and since the coefficient of friction must be the same as before, by (5.76c), $f_{cd}/N_d = f_{cu}/N_u = \tan\alpha_{cu}$ holds, and (5.76d) yields the following relation for the *apparent critical angle* α_{cd} when the open end is downward:

$$\tan\alpha_{cd} = (1 - p(\alpha_{cd}))\tan\alpha_{cu}, \qquad \text{with} \quad p(\alpha_{cd}) \equiv P/W\cos\alpha_{cd}. \tag{5.76e}$$

Because $1 - p(\alpha_{cd}) < 1$, it follows that $\alpha_{cd} < \alpha_{cu}$, that is, the apparent critical angle of sliding when the open end of the can is downward is smaller, perhaps much smaller, than the actual critical angle when its open end is upward. Now, we know from the experimental data that $\mu = \tan\alpha_{cu} = \tan 17°$ and the largest critical angle $\alpha_{cd} = 2°$; therefore, (5.76e) yields $p(2°) = 1 - \tan 2°/\tan 17° = 0.886$, that is, the normal internal force on the closed end is very nearly 89% of the can's weight. The result (5.76e), therefore, confirms the hypothesis explaining the sliding beverage can phenomenon—the frictional effect is reduced due to the uplifting, internal air pressure.

To continue from here in the static case, we shall need to know the weight of a typical can, and then compare the predicted force $P = 0.89W$ with the value computed from thermodynamics on the basis of the volume and the initial temperature of the air trapped in the chilled can at room temperature. Without getting into this, however, we may ask instead—What can be learned about the subsequent motion of $\mathscr{B}$?

5.12.1.2. *The Motion Analysis*

The observation that the can stops abruptly when the open end extends just at the edge of the sheet is investigated. Singularity functions are used to describe the discontinuous behavior of **P** when the trapped air suddenly escapes. A similar analysis may be carried out without the use of singularity functions, an exercise left for the reader.

Let ℓ_o be the distance moved by the center of the can from its initial rest position at $x = 0$ to its position at B in Fig. 5.20a, where the trapped air is released. Afterwards the can will continue to move so that it extends beyond the edge of the glass an amount say, δ, but it does not fall off. To determine the value of δ compared with ℓ_o, we first find the speed of the can as a function of its position along the sheet.

Let $x^* = x$ denote the center of mass coordinate in the inertial frame Φ, and begin with the force analysis. The free body diagram of the can is shown in Fig. 5.20b. We suppose that the internal pressure is "turned on" at $x = 0$ when the can is placed on the glass with its open end downward, and later "shut off" at $x = \ell_o$ as the air suddenly escapes when the can reaches the edge of the sheet. Then, with the aid of the unit step function (1.117), we have

$$\mathbf{P} = \left[P < x - 0 >^0 - P < x - \ell_o >^0\right]\mathbf{i}. \tag{5.77a}$$

The total force on the can throughout its motion is given by (5.76a), and hence with (5.74b) and (5.77a), the equation of motion $\mathbf{F}(\mathscr{B}, t) = m\mathbf{a}^* = m\ddot{x}\mathbf{i}$ yields the scalar component relations for the sliding motion at the critical angle α_{cd}:

$$m\ddot{x} = W \sin\alpha_{cd} - f_{dd}, \qquad N_d = W \cos\alpha_{cd} - P\left(< x - 0 >^0 - < x - \ell_o >^0\right), \tag{5.77b}$$

wherein by Coulomb's second law (5.71), $f_{dd} = \nu N_d$ during the sliding motion. Then with $W = mg$ and $p(\alpha_{cd})$ in (5.76e), (5.77b) yields the equation of motion:

$$\ddot{x} = g\cos\alpha_{cd}\left[\tan\alpha_{cd} - \nu + \nu p(\alpha_{cd})(< x - 0 >^0 - < x - \ell_o >^0)\right]. \tag{5.77c}$$

To find the speed $\dot{x} = v(x)$ as a function of x, we write $\ddot{x} = v\,dv/dx = d(v^2/2)/dx$, and recall (1.132) for integration of the unit step function. Then use of the initial data $v(0) = 0$ at $x = 0$ in the integration of (5.77c) yields the squared speed of the can at its current position $x(t)$:

$$v^2(x) = 2g\cos\alpha_{cd}\left[x\left(\tan\alpha_{cd} - \nu\right) + \nu p(\alpha_{cd})\left(< x - 0 >^1 - < x - \ell_o >^1\right)\right]. \tag{5.77d}$$

Now consider the case when the can slides beyond the edge of the glass and stops at $x = \ell > \ell_o$. Recalling (1.127) for the unit slope function, setting $v(\ell) = 0$, and introducing $p(\alpha_{cd})$ from (5.76e), we find from (5.77d) the relation for δ/ℓ_o:

$$\frac{\ell}{\ell_o} = 1 + \frac{\delta}{\ell_o} = \frac{1 - (\tan\alpha_{cd})/\mu}{1 - (\tan\alpha_{cd})/\nu}, \tag{5.77e}$$

wherein $\delta \equiv \ell - \ell_o$ is the overhang distance at the edge of the sheet. The solution thus shows that the overhang δ is proportional to the length ℓ_o, and hence our analysis discloses an oversight in the experimental description. If the sheet were too long, δ might exceed the can's radius r, the critical overhang when the can slides beyond the edge of the glass; and the can would then fall off. An estimate of the critical length $\hat{\ell}_o$ of the sheet, i.e. the maximum initial distance of $\mathscr{B}$ from the edge in order that the can will not slide off the end, may be obtained from (5.77e) at $\delta = r$; we find

$$\frac{\ell_o}{\delta} = \frac{\hat{\ell}_o}{r} = \mu\frac{\nu - \tan\alpha_{cd}}{(\mu - \nu)\tan\alpha_{cd}}. \tag{5.77f}$$

Since $\tan\alpha_{cd}$ and $(\mu - \nu)$ are small quantities, it follows that the critical length may be rather large. Hence, for most practical experimental circumstances, our theoretical analysis predicts that the can generally will stop abruptly and not fall from the edge.

To get an idea of the size of $\hat{\ell}_o$, suppose that $\nu = 0.25 < \mu = 0.3$. Then for $\alpha_{cd} = 2°$, say, the critical length to can radius ratio, by (5.77f), is $\hat{\ell}_o/r = 36.95$, and for the same parameters the can's overhang ratio is $\delta/\ell_o = 0.027$. Thus, for a can of radius $r = 3.3$ cm (1.3 in), the critical distance would be about $\hat{\ell}_o =$ 1.22 m (4.00 ft). For a plate of length $\ell_o = 25$ cm (about 10 in.), say, the overhang will be $\delta = 0.68$ cm (0.27 in.), and for $\ell_o = 1$ m (39.4 in.), a value close to the length of plate reported for the experiment, $\delta = 2.7$ cm (1.1 in.). Both example values are much smaller than the can's radius. For a larger value of ν, or a smaller value of α_{cd}, the overhang will be even smaller while the critical length of the plate will grow larger. Thus, starting at a practicable distance from the edge, the can will travel beyond the edge only a small distance compared with its radius and will indeed stop rather suddenly.

5.12.1.3. Technical Applications of the Friction Reduction Principle

The idea that frictional effects may be reduced by an uplifting internal pressure has been applied to study other phenomena. The spectacular geological phenomenon in which huge masses of nearly horizontal rock formations are displaced great distances, sometimes as much as 10 to 50 miles or more, is an example. For sufficiently high interstitial fluid pressure in porous rock, fault blocks of rock may be pushed over a nearly horizontal subsurface. Like our can experiment, due to uplifting fluid pressure, the fault blocks slide under their own weight over very much smaller slopes than otherwise would be possible.

Another striking application of pressure induced friction reduction occurred in the mechanical design of bearings for the 200 inch telescope at the Mount Palomar Observatory. Frictional forces opposing the steady, precise rotation of the telescope in tracking the apparent motion of the stars relative to the Earth had to be very much less than those that would be produced by conventional bearings. Moreover, for these bearing devices, the torque required to turn the telescope would demand considerable horsepower, and the required loading would cause excessive deformation of the telescope's mounting yoke. The problem of supporting and moving precisely such a massive structure was solved by floating the telescope on a thin film of oil under pressure. The entire weight of the telescope, roughly one million pounds (455, 000 kg), was supported by bearing surfaces separated by a thin film of oil 0.005 in. (0.013 mm) thick and under pressure ranging from 200 to 500 psi (1.4 to 3.4×10^6 N/m^2). This design concept reduced considerably the power required to drive the massive telescope to only $1/12$ horsepower!

These examples underscore the utility of the friction reduction principle illustrated by the sliding can experiment. Our next example applies the principles of mechanics to explain critical U.S. Navy torpedo failures during World War II.

5.12.2. Damn the Torpedoes!

U.S. Navy submarine operations[¶] in the early months of World War II reported recurring instances of frustrating torpedo malfunction and detonation failures.

[¶] This narrative is adapted from the referenced articles by A. A. Bartlett, D. Murphy, and the book by T. Roscoe. All discuss the problem of torpedo failures in U.S. Navy submarine operations. See also S. E. Morison.

Faced with a shortage of torpedoes and state-of-the-art magnetic detonators that proved greatly unreliable, Admiral Charles A. Lockwood in Pearl Harbor ordered the magnetic detonators replaced with impact detonators. But in no time at all worrisome reports of torpedo failures continued to come in. More than a year passed with no solution in sight when good fortune in disguise appeared unexpectedly.

On July 24, 1943, the U.S. submarine *Tinosa* was patrolling west of Truk with 16 torpedoes aboard when Lieutenant Commander Lawrence R. Daspit sighted the unescorted oil tanker *Tonan Maru* No. 3, one of the largest in the Japanese fleet, at an unfavorable great range of 4000 yards (3658 m). Four torpedoes were fired in a fan pattern oblique to the tanker, actually an unfavorable angle of attack. Two found their target and exploded near the tanker's stern to slow the great ship. Two more were released. Daspit at the periscope, witnessed two explosions that brought the *Tonan Maru* to a stop, dead in the water, smoking and starting to settle by the stern, but not sinking. At the ideal range of about 875 yards (800 m) and now stationed for a perfect shot at 90° off the tanker's bow, Daspit setup for the kill. The *Tinosa* fired a single torpedo that struck normal to the side, nearly amidships of the giant tanker. The torpedo was heard to make a normal run, followed by silence. Daspit witnessed only a spray at the point of impact. The torpedo was a dud! Two more perfect shots followed—both duds. The remaining "tin fish" were pulled from their tubes and their settings checked, all in good order. Over the next few hours, six additional torpedoes were launched one at a time. Each failed to explode on impact. Damn the torpedoes—all duds! A frustrated Daspit returned to Pearl Harbor with his last torpedo, and Japanese salvage vessels from their naval base at Truk saved the *Tonan Maru*. The fact that many similar torpedo failures in the early months of the war slowed U.S. efforts to contain Japanese advances across the South Pacific islands and the Philippines, underscores the significance of this major technical problem.

The Germans experienced similar frustration with magnetic influence torpedo failures, many exploded prematurely, others missed their target, or failed to explode on impact. A particularly significant incident occurred on the morning of October 30, 1939, the day before Sir Winston Churchill's scheduled meeting aboard the battleship *Nelson* with Admiral Sir Charles Forbes, Commander-in-Chief, and Admiral of the Fleet Sir Dudley Pound. Two weeks earlier on October 14, the German U-boat commander, Lieutenant Commander Gunther Prien, slipped his *U-47* into the center of Britain's main naval harbor at the supposedly impregnable Scapa Flow. Prien maneuvered there on the surface, undetected, and around 1 a.m. attacked and sunk at anchor the magnificent British battleship *HMS Royal Oak*, afterwards escaping to become a celebrated naval hero.[‖] Following this disaster in

‖ On March 8, 1941, the destroyer *Wolverine* while escorting a convoy in the North Atlantic, sighted the *U-47* running initially on the surface, and attacked and sank her by depth charges. The remarkable and daring Lieutenant Commander Gunther Prien, age 33, and his entire crew lost their lives. See the book by G. S. Snyder in the chapter references for the full story of the *Royal Oak* disaster, including many tales of German submarine commander frustration with torpedo failures.

which 833 officers and men lost their lives, an urgent conference was arranged for October 31, between Churchill and his admirals aboard the *Nelson*, the flagship of Admiral Forbes. But another disaster was unfolding during the morning hours of the 30th, when *U-56*, commanded by Lieutenant Wilhelm Zahn, sighted the battleships *Nelson* and *Rodney,* accompanied by the battle cruiser *Hood* and a screen of ten destroyers. Zahn maneuvered within range and released a spread of three torpedoes on *Nelson.* Three impacting thumps against the battleship's side were heard in *U-56*, but no detonation. All duds! The angry Zhan turned away and reported his aborted attack to U-boat Command, unaware of the true significance of his failed attempt to sink the *Nelson*. Nearly every U-boat commander, including the celebrated "ace" Gunther Prien, reported torpedo failures; sometimes every "eel", whether set to explode on impact or set for magnetic detonation, was a dud.

5.12.2.1. Identifying the Problem

What was wrong with the German torpedoes? A special Torpedo Commission discovered that the fault was not with the torpedoes themselves, but with the depth at which they were set to pass beneath the target's hull, the point at which the magnetic pull of the victim was supposed to trigger the warhead. Errors of design caused the weapon to run too deep, and countermeasures applied by the British also may have contributed to the German problem. The delicate magnetic exploders eventually were replaced with dependable impact exploders. By the time the U.S. entered the war in Europe, the U-boats were scoring hit after hit with shocking efficiency. (I do not know of any studies on German torpedo defects responsible for impact failures reported above.)

What was wrong with the U.S. Navy's torpedoes? The torpedo returned by Daspit to Pearl Harbor, checked and later test fired at underwater cliffs of Kahoolawe Island in Hawaii, also was a dud. Examination of the torpedo's detonator mechanism revealed that the firing pin that would set off the warhead had released, but it failed to strike the primer cap with sufficient force to trigger it. Impact experiments were conducted to study the problem. To model a normal impact against the side of a ship, torpedoes loaded with cinder concrete rather than explosives were dropped from about 90 ft (27 m) onto a steel plate. Seventy percent of the tests revealed the same kind of trigger failure on normal impact. In actual submarine operations, however, an oblique impact was believed more likely to occur. To simulate this condition, the steel plate was set at an angle so that the torpedo would strike a glancing blow. It was found that the exploder mechanism generally functioned properly. The investigation now focused on the firing pin design, a small device weighing several ounces. When released, a spring drove the pin along parallel guide rods perpendicular to the torpedo axis. The perpendicular impact force of deceleration was found to be about 500g's, that is, 500 times the force of gravity, per unit mass. This force produced a guide rod Coulomb frictional component of nearly 190 lbs on the firing pin. The trigger spring was unable to overcome the frictional force and drive the firing pin with sufficient force against the primer

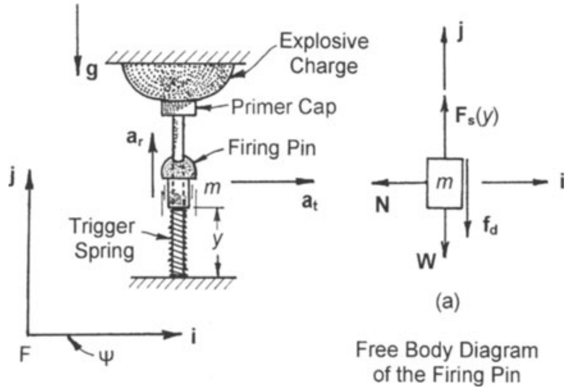


Figure 5.21. Model of a torpedo exploder mechanism.

cap. In an oblique, glancing impact, the frictional effect was less severe and the torpedoes often exploded on impact. So, nearly 2 years after the start of the war, between July and September 1943, as a fortuitous consequence of Daspit's failed attack on the *Tonan Maru*, the torpedo exploder mechanism problem was finally identified and solved.[§§]

5.12.2.2. The Model Analysis

The problem of U.S. Navy torpedo failures was finally explained by elementary principles of mechanics involving Coulomb friction. To explore this, consider the simple model of the exploder mechanism shown in Fig. 5.21. The free body diagram of the firing pin modeled as a block of weight $\mathbf{W} = m\mathbf{g}$ is shown in Fig. 5.21a. The actual direction of $\mathbf{g}$ may vary from that chosen in the example. The trigger spring driving force from its precompressed state is a known function $\mathbf{F}_s(y)$ of the firing pin displacement y; $\mathbf{N}$ denotes the normal (impulsive reaction) force exerted by the guide rods, and $\mathbf{f}_d$ is the dynamic friction force. So, the total force on the block in its sliding motion is $\mathbf{F} = \mathbf{F}_s + \mathbf{N} + \mathbf{W} + \mathbf{f}_d = -N\mathbf{i} + (F_s(y) - W - f_d)\mathbf{j}$, in which $f_d = \nu N$ and $W = mg$.

Here we have a motion of the mass m relative to the rapidly decelerating torpedo frame. Therefore, the total acceleration of m in the inertial frame $\Psi = \{F; \mathbf{i}_k\}$ is given by $\mathbf{a} = \mathbf{a}_r + \mathbf{a}_0 = \ddot{y}\mathbf{j} - a_t\mathbf{i}$, in which $\mathbf{a}_r \equiv \delta^2\mathbf{x}/\delta t^2 = \ddot{y}\mathbf{j}$ is the relative acceleration of the firing pin in the moving torpedo frame, and

[§§] The *Tinosa* soon returned to the hunt, and by the end of the war she had sunk 16 Japanese vessels, 64,655 tons in all, and survived. In both the number of ships and tonnage sunk in the Pacific theater, she ranked 19^{th} among the top 25 *pig boats* in the list of leading individual submarine scores. (See Roscoe, p. 446. According to this expert (p. 442), "submarines played the leading role in Japan's defeat. They wrecked Japan's merchant marine. They sank a sizeable chunk of the Imperial Navy. They bankrupted Japan's home economy with a blockade which established a new adage: viz., an island is a body of land surrounded by submarines.")

$\mathbf{a}_0 \equiv \mathbf{a}_t = -a_t\mathbf{i}$ is the rigid body deceleration of the torpedo in Ψ. Therefore, the corresponding scalar components in Newton's law (5.39) are $-N = -ma_t$ and $F_s(y) - mg - \nu N = m\ddot{y}$, from which the relative acceleration of the firing pin during the rapid deceleration period is given by

$$\ddot{y} = \frac{1}{m}F_s(y) - (g + \nu a_t).$$

This equation essentially determines the force with which the firing pin will strike the primer cap to detonate the warhead—it reveals both the problem and its easy solution. The contribution of g is negligible compared to νa_t. The spring force that drives the firing pin is effectively reduced by the increased frictional force arising from the large deceleration of the torpedo in its normal impact. Therefore, because of its reduced relative acceleration $\ddot{y}$, the firing pin is unable to strike the primer cap with sufficient force to trigger the warhead. The simplest direct solution is to increase the spring force, reduce the firing pin's mass and, if possible, reduce the coefficient of friction. The predictive value of the principles of mechanics demonstrated in this and in previous examples is repeated many times in future problems.

5.13. What is the Inertial Frame?

In addition to specifying a law of equilibrium for every material universe, Newton's first law provides the criterion for deciding whether a reference frame is an inertial frame. The inertial frame in Newton's laws is an undefined entity, a primitive concept, but its choice is not arbitrary; it must be a reference frame relative to which a uniform motion can be sustained without force. Otherwise, the laws are not applicable, in fact, they have no meaning until the inertial frame itself is identified. But the first law does not tell us which reference frame is the preferred referential frame, it merely assumes that such a reference frame exists. Therefore, what physical reference frame (or body) in the real world may be identified as Newton's preferential frame?

Plainly, every motion can be determined in a reference frame that is absolutely at rest. But a body can be identified as fixed in space only relative to other bodies known to be fixed in space, an evident irresolvable tautology. So, the idea of an inertial reference frame (or body) being fixed in space is meaningless. In its place, our most natural choice appears to be the Earth frame. We know, however, that the Earth's principal motion has a subtle, but demonstrable effect on the oscillations of a pendulum and on the trajectories of shells and falling bodies. Such relative motion effects preclude the possibility of an arbitrary uniform motion of a particle relative to the Earth without intervention of a controlling force, as we shall see shortly. Then what is the reference frame relative to which the Earth's motion may be referred, and under what circumstances may the Earth frame be used as a Newtonian frame?

It appears time after time that the remote stars visible in the night sky always are in their same place relative to the Sun. And these "fixed stars" are used to obtain a navigational fix on our motion. While sophisticated measurements reveal that the distant stars are, in fact, not fixed relative to each other, the so-called "fixed stars" are chosen as a physical model of an inertial reference system for the real world, because the remote stars comprise a set of objects (bodies) whose perceptible mutual distances have not changed significantly over countless centuries. Therefore, the astronomical frame of the fixed stars is a prime *candidate* for a reference system that may *approximate* an inertial frame to a precision sufficient for our needs. To evaluate the accuracy of this *assumption*, we may compare the observed physical behavior of bodies with theoretical predictions of that behavior based on Newton's laws in the astronomical frame. Well, it happens that theoretical predictions of the effects of the Earth's rotation on the swing of Foucault's pendulum, on the motion of missiles and falling bodies, and various other phenomena in the world, stand in sharp agreement with observations. Therefore, the real world, physical reference frame that corresponds to the ideal, abstract inertial reference frame in Newton's laws may be tentatively identified as a reference frame in the distant stars. The motion of the Earth relative to the astronomical frame is known, so we are now in a position to evaluate the effects of using the Earth as a first approximation to an inertial frame. The effect of the motion of a reference frame on the form of the laws of motion is described next.

5.14. The Second Law of Motion in a Noninertial Frame

Now, we are, after all, concerned mainly with motion relative to the noninertial Earth frame, or perhaps another convenient moving reference frame. Therefore, we shall need to express Newton's second law in terms of the acceleration $\delta^2\mathbf{x}/\delta t^2$ apparent to a moving observer. We thus recall (4.48) for the total acceleration of a particle referred to a moving frame and rewrite the second law (5.39) to obtain *the equation of motion for a particle of mass m having a motion relative to a moving frame* φ:

$$m\mathbf{a}_\varphi(P,t) = \mathbf{F} - m\left(\mathbf{a}_O + \boldsymbol{\omega}_f \times (\boldsymbol{\omega}_f \times \mathbf{x}) + \dot{\boldsymbol{\omega}}_f \times \mathbf{x} + 2\boldsymbol{\omega}_f \times \mathbf{v}_\varphi\right). \quad (5.78)$$

Here $\mathbf{F} = m\mathbf{a}_P$ is the force acting on the particle P whose absolute acceleration in the Newtonian frame Φ is $\mathbf{a}_P = \mathbf{a}(P,t)$; and $\mathbf{a}_\varphi(P,t) \equiv \delta^2\mathbf{x}/\delta t^2$ and $\mathbf{v}_\varphi = \mathbf{v}_\varphi(P,t) \equiv \delta\mathbf{x}/\delta t$ are the respective acceleration and velocity of P relative to φ.

The form of Newton's second law (5.78), in addition to the total force $\mathbf{F}$, exposes several "fictitious" forces apparent only to the moving observer in φ, to whom it appears that the particle is acted upon by a total force

$$\mathbf{F}_\varphi \equiv \mathbf{F} - m\left(\mathbf{a}_O + \boldsymbol{\omega}_f \times (\boldsymbol{\omega}_f \times \mathbf{x}) + \dot{\boldsymbol{\omega}}_f \times \mathbf{x} + 2\boldsymbol{\omega}_f \times \mathbf{v}_\varphi\right), \quad (5.79)$$

called the *apparent force*. The pseudoforces $-m\boldsymbol{\omega}_f \times (\boldsymbol{\omega}_f \times \mathbf{x})$ and $-2m\boldsymbol{\omega}_f \times \mathbf{v}_\varphi$ are called the *centrifugal force* and the *Coriolis force,* respectively. The total of the pseudoforces, namely,

$$\mathbf{F}_I \equiv -m\left(\mathbf{a}_O + \boldsymbol{\omega}_f \times (\boldsymbol{\omega}_f \times \mathbf{x}) + \dot{\boldsymbol{\omega}}_f \times \mathbf{x} + 2\boldsymbol{\omega}_f \times \mathbf{v}_\varphi\right), \tag{5.80}$$

is called the *inertial force.* Use of (5.79) in (5.78) now yields *Newton's second law of motion relative to any moving frame* φ, *including the Earth frame*:

$$\mathbf{F}_\varphi = m\mathbf{a}_\varphi(P,t) = m\frac{\delta^2\mathbf{x}}{\delta t^2}. \tag{5.81}$$

The basic difference between (5.81) and (5.39) is that the force $\mathbf{F}_\varphi$ in (5.79) is not the total of forces due purely to the interaction between pairs of bodies in the universe. The additional inertial force (5.80) arises solely from the motion of the moving observer's frame of reference. Therefore, to a moving observer, the actual forces that act on a body are not always what they may seem to be.

We are now positioned to show that there exists relative to the inertial frame infinitely many moving reference frames with respect to which Newton's laws hold unchanged. Hence, each of these frames is an inertial reference frame. Indeed, we need characterize only those frames for which the inertial force (5.80) vanishes for all motions relative to φ, i.e. those frames for which

$$\mathbf{a}_O + \boldsymbol{\omega}_f \times (\boldsymbol{\omega}_f \times \mathbf{x}) + \dot{\boldsymbol{\omega}}_f \times \mathbf{x} + 2\boldsymbol{\omega}_f \times \mathbf{v}_\varphi = \mathbf{0}, \tag{5.82}$$

for all $\mathbf{x}(P,t)$ and $\mathbf{v}_\varphi(P,t)$. This is possible when and only when both $\mathbf{a}_O \equiv \mathbf{0}$ and $\boldsymbol{\omega}_f \equiv \mathbf{0}$, that is, if and only if φ has a uniform translational motion relative to the inertial frame Φ. In this case, from (5.79) and (5.81), $\mathbf{F}_\varphi = \mathbf{F} = m\mathbf{a}_\varphi(P,t)$ holds for all motions of the particle P in the moving frame φ. In particular, $\mathbf{F}_\varphi = \mathbf{0}$ holds, if and only if the particle P has a uniform motion relative to φ, and hence φ is an inertial frame.

Now let us return momentarily to (5.81) and extend the definition of an equilibrium state to a particle in a moving frame φ. A particle P is in *equilibrium relative to* φ if and only if P is at rest or in uniform motion relative to φ. Then, by (5.81),

$$\text{equilibrium in } \varphi \Leftrightarrow \mathbf{a}_\varphi(P,t) = \mathbf{0} \Leftrightarrow \mathbf{F}_\varphi(P,t) = \mathbf{0}. \tag{5.83}$$

In this case, by (5.79), the force $\mathbf{F}$ applied to P to control its uniform motion in φ is balanced by the inertial force (5.80): $\mathbf{F} + \mathbf{F}_I = \mathbf{0}$. Hence, the frame φ is not an inertial frame. In general, a particle in equilibrium in φ will not be in equilibrium in the inertial frame Φ, and vice versa. In fact, by (5.38), the particle P may be in equilibrium simultaneously in Φ if and only if $-\mathbf{F}_I = \mathbf{F} = \mathbf{0}$ so that (5.82) holds for all uniform motions $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0 t$ relative to φ, where $\mathbf{x}_0$ and $\mathbf{v}_0 \equiv \mathbf{v}_\varphi(P)$ are constant vectors; but (5.82) holds when and only when frame φ has a uniform translational motion relative to the inertial frame in the distant stars.

In sum, *every nonrotating, uniformly translating reference frame is a Newtonian reference frame in which Newton's laws may be applied. Moreover, a particle*

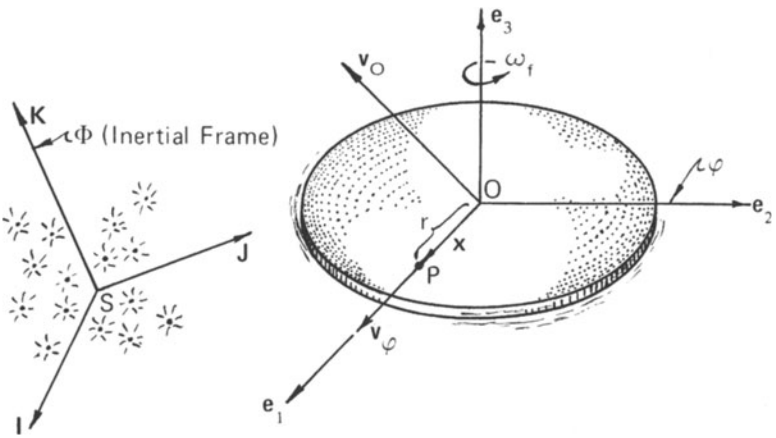


Figure 5.22. Uniform motion of a particle P relative to a moving frame $\varphi = \{O; \mathbf{e}_k\}$.

that is in equilibrium in one inertial frame Φ *is in equilibrium in every frame* φ *having only a uniform motion of translation relative to* Φ.

Example 5.9. A particle P in Fig. 5.22 has a radially directed, uniform motion relative to a frame $\varphi = \{O; \mathbf{e}_k\}$ that is rotating with angular velocity $\boldsymbol{\omega}_f$ relative to the inertial frame Φ fixed in the distant stars. The origin O has a constant velocity $\mathbf{v}_O$ in Φ. What is the force acting on the particle, and under what conditions does it vanish?

Solution. We wish to find $\mathbf{F} = \mathbf{F}(P, t)$ in (5.78). Since the motion of P relative to φ is uniform, the particle is in equilibrium relative to φ. Hence, (5.78) and (5.83) yield

$$\mathbf{F} = m\left(\mathbf{a}_O + \boldsymbol{\omega}_f \times (\boldsymbol{\omega}_f \times \mathbf{x}) + \dot{\boldsymbol{\omega}}_f \times \mathbf{x} + 2\boldsymbol{\omega}_f \times \mathbf{v}_\varphi\right). \tag{5.84a}$$

Moreover, the origin O has a constant velocity, so $\mathbf{a}_O = \mathbf{0}$. Further, with $\mathbf{x} = r\mathbf{e}_1$, we have $\mathbf{v}_\varphi = \delta\mathbf{x}/\delta t = \dot{r}\mathbf{e}_1$, which is constant relative to frame $\varphi = \{O; \mathbf{e}_k\}$, shown in Fig. 5.22. Thus, noting that $\boldsymbol{\omega}_f = \omega\mathbf{e}_3$ and $\dot{\boldsymbol{\omega}}_f = \dot{\omega}\mathbf{e}_3$ in the astronomical frame $\Phi = \{S; \mathbf{I}_k\}$, we find by (5.84a) the force that acts on the particle to control its uniform motion in the moving frame φ:

$$\mathbf{F}(P, \mathbf{t}) = -mr\omega^2\mathbf{e}_1 + m(r\dot{\omega} + 2\omega\dot{r})\mathbf{e}_2. \tag{5.84b}$$

Therefore, frame φ is not an inertial frame—the uniform motion in φ cannot be sustained without the application of force in the inertial frame Φ. Clearly, $\mathbf{F} = \mathbf{0}$ in Φ if and only if $\omega \equiv 0$, that is, when and only when the frame φ has a uniform translational motion with velocity $\mathbf{v}_O$ in the inertial frame Φ, then φ is an inertial frame too. □

5.15. Newton's Law in the Noninertial Earth Frame

Now let us consider the influence of the Earth's motion on the form of Newton's equation of motion for a particle moving relative to the noninertial Earth frame. Introduce an inertial frame $\Phi = \{F; \mathbf{A}, \mathbf{B}, \mathbf{C}\}$ fixed relative to the distant stars (see Fig. 5.23), and recall the notation used in (4.92) where $\boldsymbol{\omega}_f = \boldsymbol{\Omega}$ approximates the constant total angular velocity of the Earth frame $\varphi = \{O; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ relative to Φ, $\mathbf{x} = \mathbf{r}$ is the position vector from the Earth's center C to a particle P moving on or near the Earth's surface, and $\mathbf{a}_O = \mathbf{a}_C$ denotes the acceleration of C in Φ. Then the apparent force (5.79) acting on P in its motion relative to φ becomes

$$\mathbf{F}_\varphi = \mathbf{F} - m\left(\mathbf{a}_C + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) + 2\boldsymbol{\Omega} \times \mathbf{v}_\varphi\right). \tag{5.85}$$

First, determine $\mathbf{a}_C$ by using the second law in which the total force acting on the Earth as a center of mass object of mass m_E at C is to be estimated. All bodies in the universe exert a gravitational pull on the Earth, whose mass is estimated at 5.98×10^{24} kg (1.36×10^{22} tons). In view of the result (5.58) for spherical bodies, the gravitational actions of all bodies—the Sun and the Earth, the Earth and the Moon, the Earth and an apple—are modeled as the attractions of particles. Therefore, an estimate of the total gravitational force acting on the Earth may be obtained by regarding the Earth E as a free body, in Fig. 5.23, acted upon by the particle P, the moon M, and the sun S. The equation of motion for the center of mass of the Earth is thus given by

$$m_E \mathbf{a}_C = m_E(\mathbf{g}_S + \mathbf{g}_M + \mathbf{g}_P) + \mathbf{F}_E, \tag{5.86}$$

in which $\mathbf{g}_S$, $\mathbf{g}_M$, and $\mathbf{g}_P$ are the respective gravitational field strengths at C due to the principal surrounding bodies S, M, and P; and $\mathbf{F}_E$ is the resultant of all other forces that may act on E, including other weak gravitational forces and the contact force exerted by the Earth's atmosphere, for example. This estimates $\mathbf{a}_C$ in (5.85).

Now consider the free body diagram of the object P in Fig. 5.23. The total force acting on P is $\mathbf{F} = m(\mathbf{g}_1 + \mathbf{g}_2 + \mathbf{g}_3) + \mathbf{F}_O$, where $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ are the field strengths at P due to the bodies S, M, and E, respectively, $\mathbf{F}_O$ is the total of all other forces acting on P and $m_E \mathbf{g}_P = -m\mathbf{g}_3$ is the mutual gravitational force between E and P. Use of these relations and (5.86) in (5.85) yields the equation of motion (5.81) for the object P in the Earth frame φ:

$$\begin{aligned} m\mathbf{a}_\varphi = \mathbf{F}_O + m\mathbf{g}_3\left(1 + \frac{m}{m_E}\right) + m\left(\mathbf{g}_1 - \mathbf{g}_S\right) + m\left(\mathbf{g}_2 - \mathbf{g}_M\right) \\ - \frac{m}{m_E}\mathbf{F}_E - m\left(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times \mathbf{v}_\varphi\right). \end{aligned} \tag{5.87}$$

In view of the great distances separating the principal bodies, some further approximations are introduced to simplify (5.87). For the motion of P on or near the Earth's surface, we have $|\mathbf{r}| = r_3$ in Fig. 5.23. Hence, the other distances shown there may be approximated by $r_1 = r_S$ and $r_2 = r_M$ so that $\mathbf{g}_1 = \mathbf{g}_S$ and

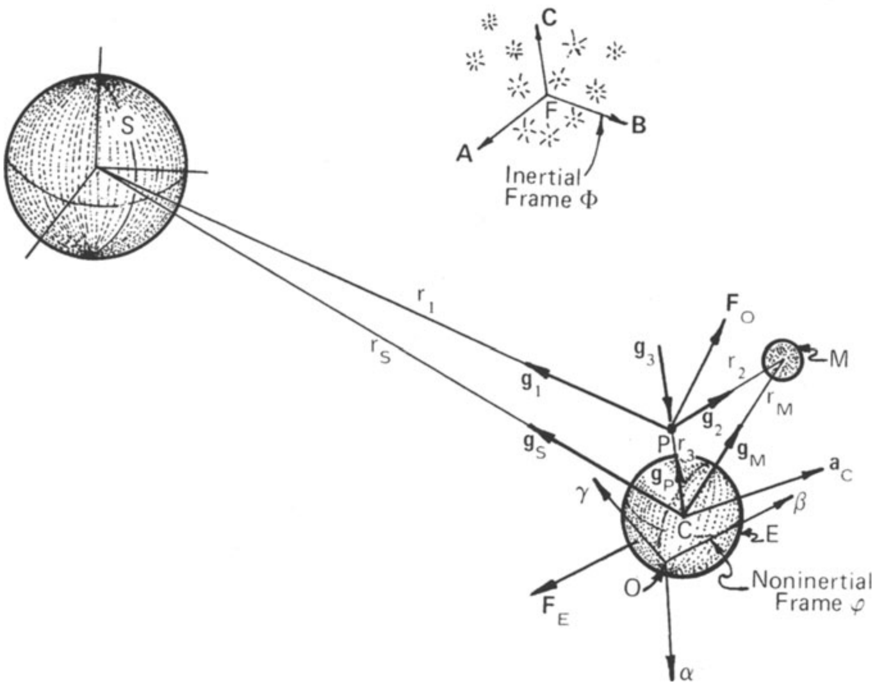


Figure 5.23. Free body diagram of a particle P and principal interacting bodies—the Earth, the Moon, and the Sun.

$\mathbf{g}_2 = \mathbf{g}_M$, very nearly. Clearly, the ratio m/m_E is infinitesimal, hence negligible compared with unity, and even though $|\mathbf{F}_E|$ may be large, we may assume that $m\,|\mathbf{F}_E|\,/m_E \ll |\mathbf{F}_O|$. Use of these further approximations in (5.87) yields the final reduced form of *Newton's equation of motion for a particle in the noninertial Earth frame*:

$$m\mathbf{a}_\varphi = m\mathbf{g}_3 + \mathbf{F}_O - m\left(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times \mathbf{v}_\varphi\right), \tag{5.88}$$

where $m\mathbf{g}_3$ is the gravitational force on P due to the Earth, $\mathbf{F}_O$ is the total of all contact and nongravitational body forces that act on P, and the other terms are inertial forces due to the Earth's rotation.

5.16. The Apparent Gravitational Field Strength of the Earth

The Earth's gravitational field strength apparent to an Earth observer is affected by the Earth's rotation and by the variation in its shape. To understand this and to learn how the real and apparent gravitational field strengths are related, let us consider an object P at rest relative to the Earth so that $\mathbf{v}_\varphi = \mathbf{0}$ and $\mathbf{a}_\varphi = \mathbf{0}$. Then (5.88) reduces to the equation of relative static equilibrium:

$$\mathbf{F}_O + m\,(\mathbf{g}_3 - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})) = \mathbf{0}. \tag{5.89}$$

Suppose P rests on the smooth, horizontal surface of a highly polished desk. Then $\mathbf{F}_O$ is the normal, desk top reaction force on P, and $\mathbf{F}_O + m\mathbf{g}_3 = (-F_O + mg_3)\mathbf{n}$, wherein $\mathbf{g}_3 = g_3\mathbf{n}$ and $\mathbf{n}$ is the central directed, unit normal vector to the Earth's spherical surface. Note, however, that the centrifugal force term in (5.89) is directed outward and perpendicular to the Earth's rotation vector $\boldsymbol{\Omega}$, so it has components both normal *and* tangential to the Earth's surface at the colatitude θ, namely,

$$-m\,\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = mr\Omega^2 \sin\theta(\cos\theta\ \mathbf{t} - \sin\theta\ \mathbf{n}), \tag{5.90}$$

where $\mathbf{t}$ is the southward directed, unit tangent vector to the surface at P. Because there is no other tangential component in (5.89) to balance the tangential component of the centrifugal force, we find the contradictory result $\Omega = 0$; otherwise, the relative equilibrium of an object at ease on a polished desk is not possible!

This dichotomy implies that the general equation (5.88) for the motion of a particle relative to the Earth cannot be a correct approximation. Review of earlier estimates used to obtain (5.88), however, suggests that the problem is more subtle than the possibility of error introduced by our treating the Earth, the Sun, and the Moon as particles separated by great distances and neglecting small terms in m/m_E. Suppose, on the other hand, that the gravitational force in (5.89) must have a small tangential component that balances the tangential centrifugal force component in (5.90). Though this correction addresses objections raised here, it implies that our spherical model of the Earth is inaccurate.

Let us consider the revised model shown in Fig. 5.24. Suppose that the attractive force $m\mathbf{g}_3$ of the Earth on P has a small northerly directed, tangential component $-mg_3 \sin\alpha\mathbf{t}$ to balance the tangential centrifugal force component $mr^2\Omega \sin\theta \cos\beta\mathbf{t}$ shown in Fig. 5.24a. If the gravitational force exerted by the Earth is directed toward its center C, while $\mathbf{F}_O$ is normal to its surface, as shown in Fig. 5.24, then the Earth must flatten somewhat at the poles and bulge slightly at the equator. In fact, geophysical theory and measurements show that the Earth is an oblate spheroid with a mean equatorial radius $r_E = 3963$ mile (6378 km) and a smaller mean polar radius $r_P = 3950$ mile (6357 km), approximately. The accepted international value for the *amount of flattening* at the pole is $\mu \equiv (r_E - r_P)/r_E = 1/297$. The centrifugal force arising from the Earth's rotation thus produces a measurable equatorial bulge of the Earth. Therefore, to derive a more precise equilibrium result and resolve earlier contradictions, it is necessary to account for the Earth's oblateness in computing the gravitational field strength for a spheroid. This is a difficult problem that we shall not need to discuss here. The interested reader may consult the chapter references by Heiskanen and Meinesz and by Ramsey for further details.

To account for polar flattening, let us suppose that the direction of the actual gravitational force $m\mathbf{g}_3$ due to the Earth is still directed toward its center C in

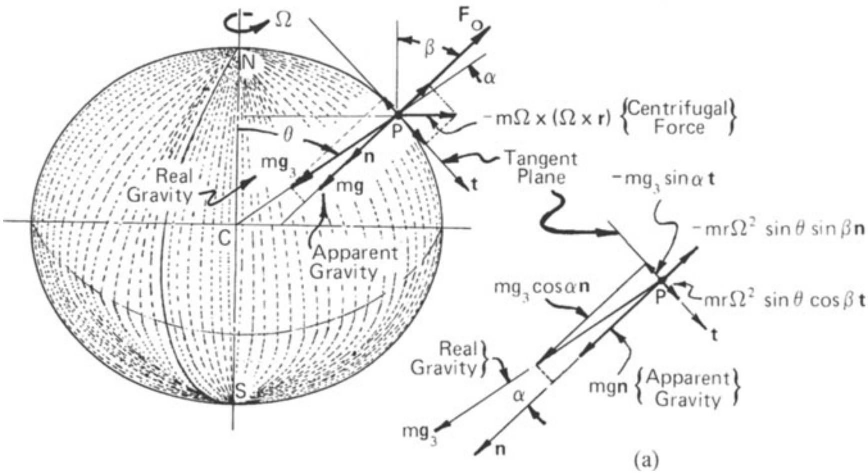


Figure 5.24. The real and apparent gravitational forces acting on a particle P at rest relative to a spheroidal Earth model.

Fig. 5.24. For equilibrium of P relative to the oblate spheroidal Earth, (5.89) now yields

$$\begin{aligned}&\left(-F_O + mg_3 \cos\alpha - mr\Omega^2 \sin\theta \sin\beta\right)\mathbf{n}\\&\quad + \left(-mg_3 \sin\alpha + mr\Omega^2 \sin\theta \cos\beta\right)\mathbf{t} = \mathbf{0}.\end{aligned} \tag{5.91}$$

In this equation, β is the *geographical colatitude angle*, the angle between the polar axis of rotation and the outward, normal vector to the surface; θ is the *geocentric colatitude angle*, the angle between the polar axis and the radial line through the Earth's center; and $\alpha \equiv \theta - \beta$ is their *angle of deviation*. (See Fig. 5.24.) Thus, the normal reaction force $\mathbf{F}_O$ in (5.89) balances the *apparent weight* $m\mathbf{g}$ of P, which varies slightly over the surface of the Earth. That is, $\mathbf{F}_O + m\mathbf{g} = \mathbf{0}$, wherein the *apparent gravitational field strength* $\mathbf{g}$ is defined by

$$\mathbf{g} \equiv \mathbf{g}_3 - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}). \tag{5.92}$$

This rule shows the effect of the Earth's rotation on the real gravitational field strength $\mathbf{g}_3$. The tangential component of $\mathbf{g}$ vanishes in accordance with (5.91):

$$-g_3 \sin\alpha + r\Omega^2 \sin\theta \ \cos(\theta - \alpha) = 0; \tag{5.93}$$

and (5.92) becomes

$$\mathbf{g} = g\mathbf{n} = \left(g_3 \cos\alpha - r\Omega^2 \sin\theta \sin(\theta - \alpha)\right)\mathbf{n}, \tag{5.94}$$

in which $\mathbf{n}$ is the inward directed unit normal vector to the Earth's surface. (See Fig. 5.24a.) This is named the *apparent acceleration of gravity*; it is the gravitational field strength apparent to an observer stationed at a point on the surface of the Earth at the geographic colatitude $\beta = \theta - \alpha$.

The apparent acceleration of gravity is always perpendicular to the Earth's surface. This is the direction $\mathbf{n}$ along which a plumb bob is attracted when freely suspended by a string. In this case, $\mathbf{F}_O$ is the tension in the line. The angle α of the plumb line's deviation from the direction of the real gravitational vector $\mathbf{g}_3$ in Fig. 5.24 may be determined by (5.93), but we must remember that g_3, r, and α vary with the angle θ. It can be shown by (5.93) and (5.94) that

$$g_3 = g\left(\cos\alpha + A\sin^2\theta\right), \tag{5.95}$$

in which $\sin\alpha = A\sin\theta\cos\theta$ and $A \equiv r\Omega^2/g$, with $r \equiv r(\theta) \in [r_P, r_E]$. Since A is very small (see (4.89)), the angle α is a very small quantity. Retaining only terms to the second order in α in (5.95), we derive the estimates

$$g_3 = g\left(1 + A\sin^2\theta - \frac{A^2}{8}\sin^2 2\theta\right), \qquad \alpha = \frac{A}{2}\sin 2\theta. \tag{5.96}$$

A final simplification of (5.96) in which terms of order A^2 and αA are omitted and r is approximated by its mean value R, say, is given by

$$g_3 = g + R\Omega^2\cos^2\lambda = g_E - R\Omega^2\sin^2\lambda, \qquad \alpha = \frac{R\Omega^2}{2g}\sin 2\lambda, \tag{5.97}$$

where $\lambda = \frac{\pi}{2} - \theta + \alpha$ is the *geographic latitude*, the angle between the equatorial plane and the outward normal to the Earth's surface. This simple estimate relates the values of the real and apparent field strengths as functions of the latitude λ, and it gives an estimate of the angle of deviation. In particular, $g_3 = g$ at the poles and $g_3 = g_E = g + R\Omega^2$ at the equator. Although g_3 is closely approximated by the apparent gravitational field strength g, we have not determined the variation of g as a function of θ or λ. This is given accurately by the international gravity formula.

When r and g are known as functions of θ or λ, the real gravitational field strength and its angle of deviation from the normal to the Earth's surface may be found. A more advanced analysis in potential theory is used to determine $g(\lambda)$, and ellipsoidal geometry is applied to determine $r(\lambda, \mu)$ in terms of the geographic latitude λ and the flatness factor μ. These details need not concern us. It turns out that the general formulas for the *Earth's variable radius* r and for the *apparent acceleration of gravity* g are given as

$$r(\lambda, \mu) = r_E\left(1 - \mu\sin^2\lambda + \frac{5\mu^2}{8}\sin^2 2\lambda\right), \tag{5.98}$$

$$g(\lambda) = g_E(1 + a\sin^2\lambda - b\sin^2 2\lambda), \tag{5.99}$$

wherein a and b are certain constants. It is seen that $r(0, \mu) = r_E$ and $g(0) = g_E$ are the respective equatorial values of $r(\lambda, \mu)$ and $g(\lambda)$. See the text by Heiskanen and Meinesz in the References.

The ellipsoidal shape function with $\mu = 1/297$ adopted by the International Geodetic Association at Madrid in 1924 is given by the *international ellipsoid formula*:

$$r = 6378.388(1 - 0.0033670 \sin^2 \lambda + 0.0000071 \sin^2 2\lambda) \text{ km}. \quad (5.100)$$

The constants g_E and a in (5.99) are obtained empirically from gravity measurements, but b is derived theoretically. The accepted values of these constants adopted by the General Assembly of the International Union of Geodesy and Geophysics at Stockholm in 1930, and reaffirmed unanimously at the Toronto, Canada Assembly in 1957, appear in the *international gravity formula*:

$$g = 9.780490(1 + 0.0052884 \sin^2 \lambda - 0.0000059 \sin^2 2\lambda) \text{ m/sec}^2. \quad (5.101)$$

This formula provides the apparent local acceleration of gravity as a function of the geographic latitude. The value of g varies from 9.83 m/sec^2 at the poles to 9.78 m/sec^2 at the equator. Our earlier rough calculation based on (5.61) for an ideal spherical Earth, namely, $g = 32.23$ ft/sec^2 $= 9.824$ m/sec^2, stands in excellent agreement with these extremes. The *standard value* adopted internationally for the apparent acceleration of gravity at sea level and at latitude $\lambda = 45°$ is $g = 32.1740$ ft/sec^2 $= 9.80665$ m/sec^2. It is customary to use the rounded value $g = 32.2$ ft/sec^2 $= 9.80$ m/sec^2 in numerical examples. In the sequel, however, we shall sometimes use $g = 32$ ft/sec^2 to simplify a numerical illustration.

The *apparent weight* $m\mathbf{g}$ of a body $\mathscr{B}$ is its weight apparent to an observer on the Earth; it is the weight, for example, that one measures when standing on a bathroom scale! We thus witness again that to a moving observer the actual force acting on a body is not always what it may seem to be. The difference between the apparent weight of $\mathscr{B}$ and its *absolute,* or *real weight relative to the Earth* in the inertial reference frame is quite small. Nevertheless, it is our custom to measure the weight of a body relative to our moving Earth frame, so no confusion should arise if, henceforward, the apparent weight of a body $\mathscr{B}$ relative to the Earth is called, briefly, the *weight* of $\mathscr{B}$. Then $\mathbf{g} = g\mathbf{n}$ in (5.62) is the apparent acceleration of gravity, and the weight of $\mathscr{B}$ is $\mathbf{W} = m\mathbf{g} = mg\mathbf{n}$, where $\mathbf{n}$ is the inward directed, unit normal vector to the Earth's surface.

5.17. Newton's Law in the Earth Frame

The foregoing analysis of the effect of the Earth's motion on the real weight of a body is based on static considerations. It is clear, however, that the terms in (5.92) are independent of the particle's motion relative to a fixed point Q on the Earth's surface, and the same terms may always be grouped in the same way in the dynamical equation (5.88) in which $\mathbf{r}$ is replaced by the current position vector of P from C, written as $\mathbf{x} = \mathbf{r} + \boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is the position vector of P from Q. Thus, for motion on or near the Earth's surface $|\boldsymbol{\rho}| \ll |\mathbf{r}|$, and hence the additional centripetal acceleration term $|\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \rho)| \ll |\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times r)|$ is negligible in comparison with all other terms in the equation. Therefore, in all cases of motion on or near the Earth's surface, the form of Newton's second law of motion (5.88) relative to the Earth frame simplifies to

$$m\mathbf{a}_\varphi = \mathbf{F} - 2m\boldsymbol{\Omega} \times \mathbf{v}_\varphi, \tag{5.102}$$

in which the total force $\mathbf{F}$ acting on the particle P includes its apparent weight $\mathbf{W} = m\mathbf{g}$ and the total $\mathbf{F}_O$ of all other forces that act on P. The Coriolis force in (5.102) is the only term that reflects directly the influence of the Earth's motion. Its maximum value, however, is about 1.6×10^{-4} sec^{-1} times the magnitude of the relative momentum $m\left|\mathbf{v}_\varphi\right|$, so its contribution is generally small in comparison with all other forces in (5.102). Consequently, very often the approximation of (5.102) to the classical Newtonian law in a noninertial Earth frame is used in engineering practice. Indeed, our examples demonstrate that excellent analytical predictions can be obtained by taking the Earth frame as the preferred frame. Nevertheless, Coriolis effects are sometimes surprising and difficult to predict without careful analysis, so use of (5.102) for the motion of a particle relative to the Earth is of interest. Some examples are explored in the next chapter.

In general, however, in problems of motion referred to a noninertial reference frame φ, Newton's law (5.81) may be used in φ provided that the total "force" $\mathbf{F}_\varphi$ defined in (5.79) includes all inertial forces and all applied forces. The inertial forces can be significant in noninertial frames other than the Earth frame, and they should never be thoughtlessly ignored.

This concludes the introduction to the foundation principles of classical mechanics created by great mathematicians of the seventeenth and eighteenth centuries. More about this grand and bountiful heritage follows in the chapters ahead. We end this chapter with an advanced topic borrowed from continuum mechanics. Here we focus on its application to the problem of the internal interaction between two particles. The result is useful in our study of the internal potential energy of a system of particles in Chapter 8. Study of this topic requires familiarity with the material in Chapters 3 and 4, the relevant parts of which are sketched below. The reader who may have omitted this material in a first reading, however, will suffer no significant loss of continuity in moving on to the next chapter.

5.18. Frame Indifference and the Law of Mutual Internal Action

Consider two reference frames $\varphi = \{O; \mathbf{i}_k\}$ and $\Phi = \{F; \mathbf{I}_k\}$, the frame Φ being the preferred frame so that $\mathbf{I}_k$ are independent of t, though this is not really essential. Recall the basis transformation tensor $\mathbf{Q}(t) = \mathbf{i}_k(t) \otimes \mathbf{I}_k$ so that $\mathbf{i}_k(t) = \mathbf{Q}(t)\mathbf{I}_k$ is the Euler rotation of the basis of frame Φ into the basis of frame φ. Of course, to an observer in frame φ, the bases vectors $\mathbf{i}_k$ are independent of t, as discussed in Chapter 4. Let $\mathbf{x}_\varphi(P, t)$ denote the position vector in Φ of a particle P from the origin of φ, but referred to φ so that $\mathbf{x}_\varphi(P, t) = x_k(P, t)\mathbf{i}_k(t) = \mathbf{Q}(t)[x_k(P, t)\mathbf{I}_k]$. We define the relative position vector $\mathbf{x}_\Phi(P, t) \equiv x_k(P, t)\mathbf{I}_k$ referred to Φ, and thus obtain the transformation rule relating the relative position vectors:

$$\mathbf{x}_\varphi(P, t) = \mathbf{Q}(t)\mathbf{x}_\Phi(P, t). \tag{5.103}$$

The relative position vectors have the same time dependent components $x_k(P, t)$ in both frames. Therefore, a transformation of this kind is said to be *frame indifferent*, or *objective*. (Here and below, see Chapter 4, pages 313–317.)

5.18.1. Change of Reference Frame

A change of reference frame is characterized by an orthogonal linear transformation that preserves distances and angles, and for which all observers use the same universal clock so that trivial, constant time shifts may be ignored. The change of frame is exhibited in terms of the position vectors $\mathbf{X}_\Phi(P, t)$ and $\mathbf{x}_\varphi(P, t)$ of the same particle from the origins F and O of the respective frames Φ and φ in accordance with

$$\mathbf{X}_\Phi(P, t) = \mathbf{B}_\Phi(O, t) + \mathbf{x}_\varphi(P, t) = \mathbf{B}_\Phi(O, t) + \mathbf{Q}(t)\mathbf{x}_\Phi(P, t), \tag{5.104}$$

where $\mathbf{B}_\Phi(O, t)$ is the position vector of O from F and we recall (5.103). Henceforward, for simplicity of notation, let us write $\mathbf{x}'(P, t) \equiv \mathbf{X}_\Phi(P, t)$, $\mathbf{c(t)} \equiv \mathbf{B}_\Phi(O, t)$, and $\mathbf{x}(P, t) \equiv \mathbf{x}_\Phi(P, t)$ so that the change of reference frame is given by

$$\mathbf{x}'(P, t) = \gamma(\mathbf{x}, t) \equiv \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}(P, t). \tag{5.105}$$

Thus, $\mathbf{c}(t)$ is the position vector of O in frame Φ and $\mathbf{Q}(t)$ is an orthogonal tensor that specifies the rigid rotation of frame φ relative to frame Φ. It is easy to verify that the change of frame preserves distance between points and angles between lines.

From now on, let us consider (5.105) as a general change of reference frame mapping $\varphi = \{O; \mathbf{e}_k\}$ into $\varphi' = \{O'; \mathbf{e}'_k\}$. Then $\mathbf{x}$ and $\mathbf{x}'$ are the respective position vectors of the same particle P from the origins O and O' at time t, and $\gamma(\mathbf{x}, t) : \varphi \to \varphi'$ is shorthand for the right-hand side of (5.105). We may exclude trivial rigid body rotations of $2n\pi$ rad, for $n = 1, 2, \ldots$. For all of these and for a null rotation, $\mathbf{Q} = \mathbf{1}$. A pure translation is thus described by $\mathbf{Q} \equiv \mathbf{1}$ so that $\gamma(\mathbf{x}, t) = \mathbf{c}(t) + \mathbf{x}(P, t)$. Also, we recall from (3.88) that a rotation tensor $\mathbf{Q}$ preserves the axis of rotation $\mathbf{e}$, and hence all points $\mathbf{u} = u\mathbf{e}$ along that axis, that is, $\mathbf{Qu} = \mathbf{u}$. Therefore, $\mathbf{Qv(u)} = \mathbf{v(u)}$ holds if and only if the vector $\mathbf{v(u)}$ is parallel to $\mathbf{u}$, and hence $\mathbf{v(u)} = g(\mathbf{u})\mathbf{u}$, where $g(\mathbf{u})$ is a scalar-valued function of $\mathbf{u}$. These results are needed below.

5.18.2. The Principle of Material Frame Indifference

It is commonly assumed, without actually saying so, that the internal force in a spring is independent of the particular situation in which the spring might be used. We take for granted that the same extension of the same spring in a fixed reference frame and in any other reference frame having an arbitrary motion, gives rise to the same internal spring force and vice versa. Accordingly, the internal force-extension law of the spring (introduced in the next chapter) is the same at the top of a high mountain, the bottom of a deep mine, in fact at any place of rest, and on a rotating table in a laboratory or in a vehicle speeding along a tortuous highway. In fact, the idea of invariance of the spring law under translations was adopted by Hooke in 1675 in a proposal to use the spring to measure gravity. Thus, it is commonly assumed that the law relating the internal force to the extension depends only on the extension of the spring relative to itself, and it is not affected in any manner by arbitrary superimposed rigid body motions of translation and rotation, the latter altering only the relative direction of the spring force. This is an example of the important classical principle of invariance of internal material response to external superimposed rigid body motions, called, briefly, the *principle of material frame indifference*. The principle[**] has been widely applied in works on material response of deformable bodies, though often indirectly. In 1955, however, the general principle of material frame indifference for deformable bodies was given new motivation by Noll in its application to the constitutive theory of materials in continuum mechanics. This rule is stated in Noll's terms[††] as follows.

The principle of material frame indifference: *The constitutive laws governing the internal interactions between the parts of a physical system do not depend on whatever external frame of reference is used to describe them.*

It is emphasized that the principle applies only to *internal interactions between parts of a system*, not to actions of the external world on the system and its parts. It does not apply to actions on a body that arise, for example, from inertial forces induced by the motion of the reference frame. These are frame dependent actions of the external environment on the system, actions that arise as a consequence of the noninertial nature of the reference frame, and which vanish only when an inertial frame is used. The choice of the external frame of reference is a matter of convenience. The internal interactions may be mechanical, gravitational, thermodynamical, electromagnetic, for example. Here we apply the principle to study the nature of the internal force between a pair of particles, an illustration due to Noll.

[**] A history of this principle is traced in the remarkable treatise by Truesdell and Noll cited in the References.

[††] The presentation below, in somewhat different notation and without use of the language and mathematical rigor of finite dimensional spaces, parallels that due to W. Noll in unpublished articles described in the References. I thank Professor Noll for providing a copy of his papers and for his permission to use the example.

5.18.3. The Law of Mutual Internal Action

Newton's law of universal gravitational interaction between any two particles in (5.46) postulates that the force exerted by one particle on another at any given instant depends only on their positions, such that (i) the force is directed along their common line; and (ii) the magnitude of the force depends only on the distance between them. We are going to show, as Noll proved, that both conditions are consequences of the principle of material frame indifference.

Consider a system of two particles P_1 and P_2 at a given fixed time t; and let us assume that the mutual force $\mathbf{F}_{21}$ exerted on the particle P_2 by P_1 depends only on the positions $\mathbf{y}$ and $\mathbf{x}$ of the two particles at that instant, so that

$$\mathbf{F}_{21} = \hat{\mathbf{F}}(\mathbf{x}, \mathbf{y}). \tag{5.106}$$

Of course, we consider only distinct material points: $\mathbf{x} \neq \mathbf{y}$. Now, after a change of frame (5.105), or an equivalent superimposed rigid body motion of the system, the particles appear at the positions $\mathbf{x}' = \gamma(\mathbf{x})$, $\mathbf{y}' = \gamma(\mathbf{y})$ and the force appears to be rotated into $\mathbf{F}'_{21} = \mathbf{Q}\mathbf{F}_{21}$, where $\mathbf{Q}$ is the orthogonal tensor in (5.105). Then according to the principle of frame indifference, the same function $\hat{\mathbf{F}}$ should also describe the dependence of the force on the positions $\mathbf{x}'$, $\mathbf{y}'$ after the change of frame, so that $\mathbf{Q}\mathbf{F}_{21} = \mathbf{F}'_{21} = \hat{\mathbf{F}}(\mathbf{x}', \mathbf{y}')$. This means that the function $\hat{\mathbf{F}}$ must satisfy

$$\mathbf{Q}\hat{\mathbf{F}}(\mathbf{x}, \mathbf{y}) = \hat{\mathbf{F}}(\mathbf{x}', \mathbf{y}'), \tag{5.107}$$

for every change of frame (5.105) and for all points $\mathbf{x}$ and $\mathbf{y} \in \varphi$ at the instant t.

Let $\mathbf{x}, \mathbf{y}$ be given, choose a point at $\mathbf{q} \in \varphi$ arbitrarily, and consider a pure translation for which $\mathbf{Q} = \mathbf{1}$ and $\gamma(\mathbf{x}) = \mathbf{x} + \mathbf{c}$ translates $\mathbf{x}$ to $\mathbf{x}' = \mathbf{q}$. Then $\mathbf{y}' = \mathbf{c} + \mathbf{y} = \mathbf{q} + (\mathbf{y} - \mathbf{x})$, and hence (5.107) reduces in a pure translation to

$$\hat{\mathbf{F}}(\mathbf{x}, \mathbf{y}) = \hat{\mathbf{F}}(\mathbf{q}, \mathbf{q} + (\mathbf{y} - \mathbf{x})).$$

In particular, we may take $\mathbf{q} = \mathbf{0}$, which is equivalent to our choosing $\mathbf{c} = -\mathbf{x}$. This relation, however, must hold regardless of what $\mathbf{q}$ may be chosen. Therefore, we find that the function $\hat{\mathbf{F}}$ must have the form

$$\hat{\mathbf{F}}(\mathbf{x}, \mathbf{y}) = \hat{\mathbf{G}}(\mathbf{y} - \mathbf{x}), \tag{5.108}$$

for all $\mathbf{x}, \mathbf{y}$. Returning to (5.107) and using (5.108), we have

$$\mathbf{Q}\hat{\mathbf{G}}(\mathbf{y} - \mathbf{x}) = \hat{\mathbf{G}}(\mathbf{y}' - \mathbf{x}'), \tag{5.109}$$

for all orthogonal $\mathbf{Q}$ and for all positions $\mathbf{x}, \mathbf{y}$.

Recalling from (5.105) that $\mathbf{y}' - \mathbf{x}' = \mathbf{Q}(\mathbf{y} - \mathbf{x})$ holds for all rotations $\mathbf{Q}$ and for all $\mathbf{x}, \mathbf{y}$, by (5.109), we have $\mathbf{Q}\hat{\mathbf{G}}(\mathbf{y} - \mathbf{x}) = \hat{\mathbf{G}}(\mathbf{Q}(\mathbf{y} - \mathbf{x}))$, that is, with $\mathbf{r} \equiv \mathbf{y} - \mathbf{x}$, the position vector of the particle P_2 relative to the particle P_1,

$$\mathbf{Q}\hat{\mathbf{G}}(\mathbf{r}) = \hat{\mathbf{G}}(\mathbf{Q}\mathbf{r}). \tag{5.110}$$

This must hold for all vectors $\mathbf{r}$ and for all orthogonal transformations $\mathbf{Q}$. Given $\mathbf{r}$, (5.110) must hold, in particular, for all rotations $\mathbf{Q}$ about the axis $\mathbf{r}$ so that $\mathbf{Q}\mathbf{r} = \mathbf{r}$.

Then, by (5.110), $\mathbf{Q}\hat{\mathbf{G}}(\mathbf{r}) = \hat{\mathbf{G}}(\mathbf{r})$, and hence these $\mathbf{Q}$ leave $\hat{\mathbf{G}}(\mathbf{r})$ unchanged. This means that $\hat{\mathbf{G}}(\mathbf{r})$ must be parallel to $\mathbf{r}$, the axis of rotation. Hence, there exists a scalar-valued function $g(\mathbf{r})$ such that

$$\hat{\mathbf{G}}(\mathbf{r}) = g(\mathbf{r})\mathbf{r}, \tag{5.111}$$

for all $\mathbf{r}$. But the condition (5.110) requires that $g(\mathbf{r})\mathbf{Q}\mathbf{r} = g(\mathbf{Q}\mathbf{r})\mathbf{Q}\mathbf{r}$, that is,

$$g(\mathbf{r}) = g(\mathbf{Q}\mathbf{r}) \quad \text{for all orthogonal } \mathbf{Q}. \tag{5.112}$$

Given $\mathbf{r} = r\mathbf{e}$, where $r = |\mathbf{r}| = \sqrt{\mathbf{r}\cdot\mathbf{r}}$, introduce $\mathbf{e}' = \mathbf{Q}\mathbf{e}$ and note that $\mathbf{Q}\mathbf{r} = \mathbf{Q}r\mathbf{e} = r\mathbf{e}'$. Then, by (5.112), $g(r\mathbf{e}) = g(r\mathbf{e}')$ for an arbitrary direction $\mathbf{e}'$. Thus, choose $\mathbf{e}' = -\mathbf{e}$ to obtain $g(r\mathbf{e}) = g(-r\mathbf{e})$. Therefore, the scalar-valued function $g(\mathbf{r})$ must be an even function of $\mathbf{r}$ and independent of its direction. Hence, $g(\mathbf{r})$ is a scalar-valued function of r alone, defined by $g(\mathbf{r}) \equiv h(r)$, and now (5.111) becomes

$$\hat{\mathbf{G}}(\mathbf{r}) = h(r)\mathbf{r}. \tag{5.113}$$

Recalling (5.108) and noting in (5.113) that $\mathbf{r} = \mathbf{y} - \mathbf{x}$, we have

$$\hat{\mathbf{F}}(\mathbf{x}, \mathbf{y}) = h(|\mathbf{y} - \mathbf{x}|)(\mathbf{y} - \mathbf{x}). \tag{5.114}$$

We thus find that the dependence of the force $\mathbf{F}_{21}$ in (5.106) on the positions $\mathbf{x}$ and $\mathbf{y}$ must reduce to the specific form

$$\mathbf{F}_{21} = h(r)\mathbf{r}, \tag{5.115}$$

where $r = |\mathbf{r}|$ and $\mathbf{r} = r\mathbf{e} = \mathbf{y} - \mathbf{x}$ is the position vector of particle P_2 from P_1. *This is the most general form of the law of mutual internal action that satisfies the principle of material frame indifference.* Moreover, from (5.114), $\hat{\mathbf{F}}(\mathbf{y}, \mathbf{x}) = -\hat{\mathbf{F}}(\mathbf{x}, \mathbf{y})$, that is, $\mathbf{F}_{12} = -\mathbf{F}_{21}$. This is Newton's third law of mutual action. Thus, the principle of frame indifference applied to the internal force between two particles that depends only on their positions, shows that their mutual internal force is a function of the distance of their separation and is directed along their common line.

Exercise 5.7. Begin with (5.115) and show that (5.107) is satisfied for an arbitrary change of frame (5.105). This will conclude the proof of Noll's theorem: *The internal force* (5.106) *between two particles that depends only on their positions is frame indifferent if and only if it has the form* (5.115). □

Newton's law (5.46) for the mutual gravitational attraction of a pair of particles is obtained from (5.115) with $h(r) \equiv -Gm_1m_2/r^3$. Similarly, Coulomb's law for the electrostatic force between two particles with electrical charges q_1 and q_2, studied in the next chapter, follows from (5.115) with $h(r) \equiv kq_1q_2/r^3$, in which k is a constant. The general rule (5.115) is also useful in characterizing the total internal potential energy of a system of particles in Chapter 8.

References

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Royal, George Airy, blocked further communication and publication of Adams's more precise prediction of Neptune's location, while details of Le Verrier's work appeared later in the *Comptes Rendus* of the French Academy of Sciences. Le Verrier's final paper on the topic reached England on September 29, 1846; it gave Neptune's mass and coordinates within only a few degrees of Adams's prediction. The concluding irony of the story, however, is that Galileo had twice recorded in his notebooks during the period December 1612 to January 1613, almost 234 years earlier, diagrams of telescopic observations that show a "fixed star" drawn on a directed line from Jupiter in the plane of its satellites. A more recent review by astronomers in 1980 of Galileo's observations revealed that he had actually discovered Neptune. Due to the poor resolution of his telescope, however, he identified it as a star.

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Appendix: Measure Units in Mechanics

In numerical examples, exercises, and problems where measure units are not explicit, consistent measure units always are understood. It makes no difference in theoretical mechanics what measure units may be used to express numerical results. But all countries throughout the world have agreed to adopt in scientific work the International System of Units, called SI units. Some SI units used in mechanics are listed in the Table 5.1.

Table 5.1. Systems of measure units

Measure	SI units	Engineering units	English units
Mass	kilogram (kg)	slug ($\text{lb} \cdot \text{sec}^2/\text{ft}$)	pound (lb_m)
Length	meter (m)	feet (ft)	feet (ft)
Time	second (sec)	second (sec)	second (sec)
Force	Newton (N)	pound (lb)	poundal (lb_l)

Universal conversion to the SI system, even at this date, is incomplete, and, of course, many important earlier reference works employ other systems of units, including the Engineering system which still enjoys wide use throughout the United States and to a lesser extent in Great Britain. The English system, now largely abandoned, is another scheme that has been used by engineers in these countries. Table 5.1 identifies for these systems the measure units of force based on Newton's second law:

$$1\ \text{N} = 1\ \text{kg} \cdot \text{m/sec}^2, \qquad 1\ \text{lb} = 1\ \text{slug} \cdot \text{ft/sec}^2, \qquad 1\ \text{lb}_l = 1\ \text{lb}_m \cdot\ \text{ft/sec}^2 .$$

The following conversion factors may be used to relate SI and engineering units:

$$1\ \text{N} = 0.225\ \text{lb}, \qquad 1\ \text{m} = 3.281\ \text{ft}, \qquad 1\ \text{slug} = 14.58\ \text{kg}.$$

The Engineering and the English units of mass are related by a dimensionless conversion factor g_o whose numerical value is equal to the standard value of the acceleration of gravity at a specified point on the Earth. By definition, the mass of a *standard one pound body* is 1 lb_m and its weight is 1 lb, thus $W = 1$ lb $= 1$ slug $\cdot$ ft/sec^2 $= mg_o = 1\ \text{lb}_m \cdot g_o$ ft/sec^2. Then with $g_o = 32.2$, say, 1 slug $= 32.2\ \text{lb}_m$. Similarly, the pound is defined as the unit of force that will impart to a 1 lb_m an acceleration equal to g_o. Then with force measured in pounds (engineering units) and mass measured in pounds mass (English units), Newton's law would become $\mathbf{F} = m\mathbf{a}/g_o$. We may be thankful that this practice is no longer fashionable. Though we shall have no need in this book to prefer one system over another, in numerical work only Engineering and SI units are used.

Problems

It is essential that throughout the study of this text the student should work a variety of problems in order to grow familiar with use of the notation, concepts, and definitions; to cultivate, test, and expand one's understanding of the subject matter; to learn the general methods of mechanics; and to master various techniques of problem solving. Moreover, it is important that the problems be approached in a spirit and manner similar to that expressed in the examples, namely by the use of vector methods so far as may be reasonable and, in large measure, without the aid of a computing device. Instances where use of a computer is desirable to promote practice with some numerical calculations will be evident. In general, however, numerical values usually will serve only to simplify an analysis and to lay bare the relevant aspects of the illustration. Therefore, the majority of problems in this book have been constructed to avoid senseless use of a computer so that the student's skills with direct calculations and with manipulations of analytical relations may be reinforced and sharpened to further develop the student's ability to handle fundamental aspects of analytic geometry, trigonometry, calculus, vector methods, and differential equations, all essential to the modern demands of engineering practice.

5.1. Three particles of mass $m_1 = 3$ kg, $m_2 = 2$ kg, $m_3 = 5$ kg are initially located in $\Phi = \{F; \mathbf{i}_k\}$ at $\mathbf{x}_1 = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ m, $\mathbf{x}_2 = 2\mathbf{i} - 3\mathbf{k}$ m, $\mathbf{x}_3 = -\mathbf{i} + 4\mathbf{k}$ m, respectively, and their corresponding initial velocities are given by $\mathbf{v}_1 = \mathbf{i} - 2\mathbf{k}$ m/sec, $\mathbf{v}_2 = 2\mathbf{i} - 3\mathbf{j}$ m/sec, $\mathbf{v}_3 = -2\mathbf{k}$ m/sec. Determine for the initial instant (a) the position and velocity of the center of mass and (b) the momentum of the system.

5.2. Consider a system $\beta = \{P_k\}$ of n particles P_k with mass m'_k, and introduce the normalized mass $m_k \equiv m'_k/m(\beta)$ in which $m(\beta)$ is the mass of the system. Let $\mathbf{x}_k = \mathbf{x}^* + \hat{\mathbf{x}}_k$ and $\hat{\mathbf{x}}_k$ denote the respective position vectors of P_k from point O and from the center of mass C in frame $\Psi = \{F; \mathbf{e}_k\}$. Then, by (5.5), the position vector of C from O is given by

$$\mathbf{x}^* = \sum_{k=1}^{n} m_k \mathbf{x}_k \quad \text{with} \quad \sum_{k=1}^{n} m_k = 1. \tag{P5.2a}$$

Lagrange observed that the location of the center of mass C of a system of particles is determined uniquely by their relative positions, that is, by their mutual distances of separation d_{jk}. He thus posed the problem of finding C in terms of only these mutual distances. To see how this may be done, first (a) prove *Lagrange's Lemma* (1783):

$$\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} m_j m_k d_{jk}^2 = \sum_{k=1}^{n} m_k \hat{d}_k^2, \tag{P5.2b}$$

where $\hat{d}_k$ is the distance from C to the particle P_k and d_{jk} denotes the distance between the particles with mass m_j and m_k. Hint: Note that the vector $\hat{\mathbf{x}}_k - \hat{\mathbf{x}}_j = \mathbf{x}_k - \mathbf{x}_j$ from m_j to m_k determines the squared distance $d_{jk}^2 = d_{kj}^2$. (b) Apply (P5.2b) to prove *Lagrange's Theorem*[‡‡] *on the center of mass* (1783):

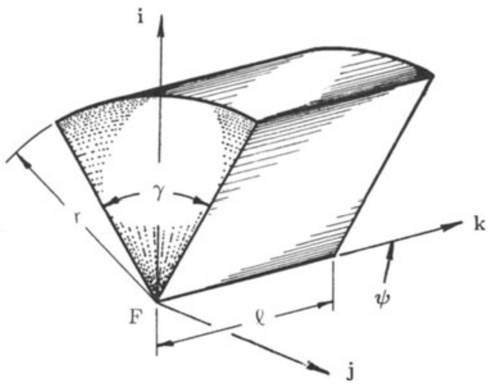
$$d_C^2 = \sum_{k=1}^{n} m_k d_k^2 - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} m_j m_k d_{jk}^2, \tag{P5.2c}$$

wherein d_k and d_C are the respective distances of the particle P_k and of the center of mass C from any specified point O. Hint: Determine $\sum_{k=1}^{n} m_k \hat{d}_k^2 = \sum_{k=1}^{n} m_k(\hat{\mathbf{x}}_k \cdot \hat{\mathbf{x}}_k)$. The result follows from here. Because O is an arbitrary point, it may be chosen at any of the particle locations so that the distance of C from any three noncoplanar and noncoaxial particles can be found from (P5.2c). Therefore, the location of C may be found when *only* the mutual distances of separation of the particles are known.

5.3. Lagrange's method described in the previous problem generally involves some rather tedious calculations in its application, but it gives an easy solution in some cases. To grasp the idea of the theorem, consider a system of two identical particles separated by a distance a. Apply Lagrange's theorem to find the center of mass, and describe carefully how its location is fixed.

5.4. Four identical particles are situated at the vertices of an equilateral pyramid with edge lengths a and height h. Find the center of mass C of the system (a) by use of Lagrange's theorem in Problem 5.2 and (b) by the usual method expressed in the normalized form (P5.2a). (c) Show that C is the intersection point of the pyramid altitude lines at distance $d_C = 3h/4$ from each particle.

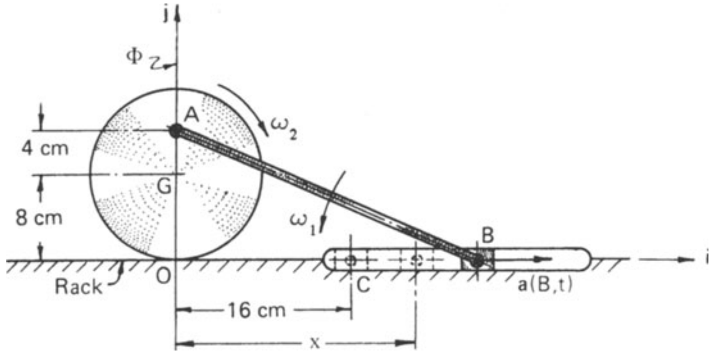
5.5. Find the center of mass of a homogeneous right circular cone of base radius r and height h. What is the mass of the cone?



Problem 5.6.

5.6. A homogeneous cylindrical wedge of radius r, length ℓ, and central angle γ is shown in the figure. Determine the mass of the wedge, and find its center of mass in $\psi = \{F; \mathbf{i}_k\}$. Locate the center of mass of a homogeneous half cylinder.

[‡‡] A special case of Lagrange's theorem applied to a molecular chain configuration of n atoms of equal mass is presented by P. J. Flory, *Statistical Mechanics of Chain Molecules*, Hanser, New York, pp. 5, 383–4, 1988. See also M. F. Beatty, Lagrange's theorem on the center of mass of a system of particles, *American Journal of Physics* **40**, 205–7 (1972).

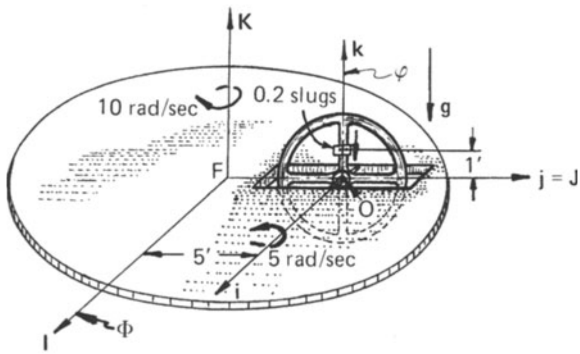


Problem 5.7.

5.7. One end of a connecting link AB is hinged at A to a gear G of radius 8 cm; the other end is hinged at B to a slider block of mass $m = 100$ gm. The gear rolls on a fixed horizontal rack. In 2 sec, the slider block moves from its initial rest position at C in frame $\Phi = \{O; \mathbf{i}_k\}$ to the position shown in the diagram. During the interval of interest, the slider has acceleration $\mathbf{a}_B = 18\sqrt[3]{(x-16)/3}\,\mathbf{i}$ cm/sec^2 in Φ. Determine the momentum of the slider block at the instant shown in the figure. What is the moment of momentum of B about points at O and A at the instant of interest?

5.8. At a moment of interest t_0, a particle P of mass 2 kg has the velocity $\mathbf{v}(P, t_0) = 16\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$ m/sec at the place $\mathbf{X}(P, t_0) = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ m in frame $\Phi = \{F; \mathbf{i}_k\}$. (a) Determine the momentum of P and find its moment about F at the time t_0. (b) What is the instantaneous moment of momentum of P about the point O at $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ m in Φ when (i) O is fixed in Φ and (ii) O is moving in Φ with the velocity $\mathbf{v}_O = 4\mathbf{i} - 6\mathbf{j}$ m/sec?

5.9. Water issuing from the nozzles of the garden sprinkler described in Problem 4.66, Volume 1, causes it to turn with an angular velocity $\omega(t)$ as shown. Compute the moment of momentum about O of a fluid particle P of mass m as it exits the nozzle at E with a constant speed v relative to the nozzle. What is the absolute time rate of change of the moment of momentum of P at E?



Problem 5.10.

5.10. The flywheel shown in the figure has a constant, counterclockwise angular speed of 5 rad/sec relative to a platform turning with a constant angular speed of 10 rad/sec, as indicated. A small slider block of mass 0.2 slug is moving along a wheel spoke toward the center O. At the instant t_o shown, the slider block is 1 ft from O and has a speed of 20 ft/sec that is increasing at the rate of 10 ft/sec^2 relative to the flywheel frame $\varphi = \{O; \mathbf{i}_k\}$. (a) What is the linear momentum of the block in the ground frame $\Phi = \{F; \mathbf{I}_k\}$ at t_o? (b) What is its corresponding moment of momentum about O? (c) Determine at t_o the moment of momentum of the slider about F in Φ.

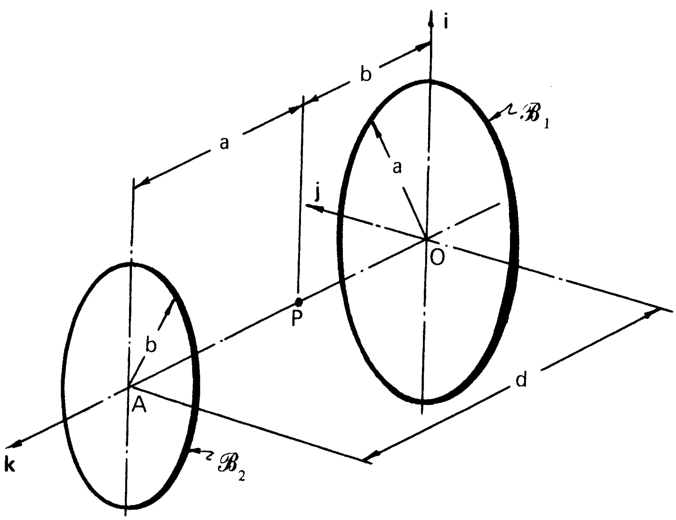
5.11. For the data provided in Problem 5.1, determine for the initial instant the moment of momentum of the system about F. What is the moment of momentum of the system about another fixed point O at $\mathbf{X}_O = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ m in Φ? How is the moment about O of the momentum of the system affected when O has the initial velocity $\mathbf{v}_O = 4\mathbf{i} - 13\mathbf{j} + 6\mathbf{k}$ m/sec?

5.12. Three particles of mass m, $2m$, and $3m$ occupy the respective initial positions $\mathbf{x}_1 = 6\mathbf{j}$ ft, $\mathbf{x}_2 = \mathbf{0}$, $\mathbf{x}_3 = -2\mathbf{j}$ ft, and they have the constant velocities $\mathbf{v}_1 = 6\mathbf{i} + 3\mathbf{j}$, $\mathbf{v}_2 = 6\mathbf{i} - 3\mathbf{j}$, $\mathbf{v}_3 = 4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ (all in ft/sec), respectively, in frame $\Phi = \{O; \mathbf{i}_k\}$. Determine (a) the velocity of the center of mass particle and (b) the momentum of the system. (c) Find the motion of the center of mass particle as a function of time t, and describe its path. (d) What is the moment of momentum of the system about O initially? (e) What is the moment about O of the momentum of the center of mass particle?

5.13. A loaded balloon of total weight W is falling vertically with a constant acceleration $\mathbf{a}$. Neglect wind effects and air resistance, but account for the buoyant force of the air, and find the amount of ballast weight w that must be discarded to give the balloon an upward acceleration $-\mathbf{a}$.

5.14. Three particles of mass m, $2m$, and $3m$ are stationary at the respective points (0, 0, 0), (1, 2, 3), and (3, 2, 1) in frame $\Phi = \{O; \mathbf{i}_k\}$. Find the resultant gravitational force exerted on the particle of mass m.

5.15. A particle P of mass μ is at the central point of a homogeneous, semicircular, thin wire of radius b and mass density σ per unit length. Determine the gravitational force exerted on P by the wire.



Problem 5.16.

5.16. Two thin, homogeneous circular wires $\mathscr{B}_1$ and $\mathscr{B}_2$ of radii a and b, respectively, are positioned in parallel planes distance d apart. The mass density of $\mathscr{B}_2$, per unit length, is twice that of $\mathscr{B}_1$. A particle P of unit mass is situated as shown in the figure on the normal line OA through their centers. (a) Apply (5.54d) to find the total gravitational force on P due to both rings. (b) Show that the gravitational force due to $\mathscr{B}_1$ alone vanishes at the center of the ring at O and at infinity, hence a maximum value of this force exists. Find the location b^* of P where the intensity of the gravitational force of $\mathscr{B}_1$ on P is greatest. (c) Repeat part (a) for the case $b = b^*$. What is the mass ratio m_2/m_1 of the rings?

5.17. A thin, flat annular body $\mathscr{B}$ has an inner radius R_1, an outer radius R_2, and uniform mass density σ per unit of area. (a) What gravitational field strength does $\mathscr{B}$ produce at a point P on the line normal to the plane of $\mathscr{B}$ through its center O, at distance X from O? (b) Determine the field strength at O due to $\mathscr{B}$. (c) Show that for $X \gg R_2$ the field strength of $\mathscr{B}$ is $\mathbf{g}(\mathbf{X}) = -Gm/X^2\mathbf{k}$, wherein $m = m(\mathscr{B})$, and hence in its gravitational attraction at a sufficiently great distance $\mathbf{X}$, the ring behaves like a particle in accordance with (5.47).

5.18. A particle P of mass β is situated at a distance $X > a$ from the center, and along the axis of a homogeneous thin rod of length $2a$ and mass density σ per unit length. Find the gravitational force acting on P due to the rod.

5.19. A particle P of mass β is located at a distance X on the center line perpendicular to the axis of a homogeneous thin rod of mass m and length $2a$, both lying in the xz-plane. The origin is at the center of the rod with its axis along $\mathbf{k}$. Show that the gravitational force that the rod exerts on P is

$$\mathbf{F}(P;\mathbf{X}) = -\frac{Gm\beta}{X\sqrt{X^2+a^2}}\mathbf{i}. \tag{P5.19}$$

5.20. A particle of mass m is placed at an external point on the axis of a homogeneous, right circular cylinder at a distance α from one end. (Choose a frame with origin at the particle and the cylinder axis as $\mathbf{k}$.) The cylinder has radius R, length L, and mass M. Find the attractive force it exerts on the particle.

5.21. Determine the gravitational field strength at the central point Q of a homogeneous, thin hemispherical shell of radius R and mass m. What is the field strength at Q for a complete spherical shell?

5.22. Show that the gravitational field strength of a spherical Earth model with radius R and mass density $\rho = \rho(R)$ varies with the normal altitude h from its surface in accordance with the relation

$$\mathbf{g}(\mathbf{X}) \equiv \hat{\mathbf{g}}(h) = \frac{\mathbf{g}(\mathbf{R})}{(1+h/R)^2}, \tag{P5.22}$$

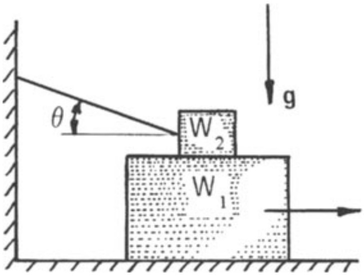
where $\mathbf{g}(\mathbf{R})$ denotes the field strength at the surface.

5.23. A homogeneous thin rod R_1 of length $2b$ and mass M is placed with its axis along the center line perpendicular to the axis of a similar rod R_2 of mass m and length $2a$, in the xy-plane. The center of R_1 is at $\mathbf{c} = c\mathbf{j}$ from the center of R_2. Determine the gravitational force that the rod R_2 exerts on R_1. (See Problem 5.19.) Tables of integrals may be needed.

5.24. A homogeneous, thin rod of length ℓ and mass m is positioned with its axis on the line through the center O and perpendicular to the plane of a homogeneous, thin circular disk of radius R and mass m. The end of the rod near the disk is at $\mathbf{a} = a\mathbf{k}$ from point O. Find the total gravitational force exerted on the rod by the disk. What gravitational force does the rod exert on the disk?

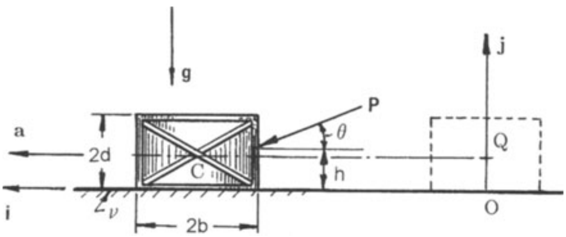
5.25. The moon has a mean diameter of about 2160 miles, while that of the Earth is roughly 7910 miles. The ratio of the mass of the moon to that of the Earth is about 3/250. What is the acceleration of gravity on or near the surface of the moon? Compare your weight relative to the Earth and the Moon.

5.26. Determine the gravitational force between two identical spheres of diameter d when they touch each other. What is the ratio of the magnitude W_o of their mutual attraction to the magnitude W of the attractive force exerted on each of them by the Earth? Evaluate the result for lead spheres with $d = 2$ ft and $\rho = 22.5$ slug/ft^3.



Problem 5.27.

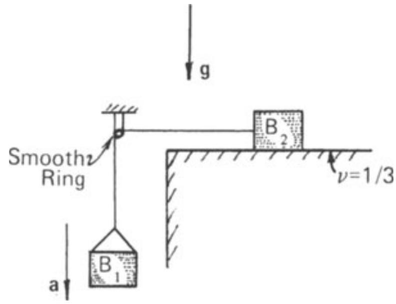
5.27. A block of weight W_1 supports a smaller block of weight $W_2 = \frac{1}{2} W_1$ constrained by a light wire inclined at an angle θ, as shown. (a) Find the horizontal force **P** required to just start the block of weight W_1 moving toward the right. (b) Find the tension in the cable after slip has occurred. Assume that all surfaces have the same coefficients of static and dynamic friction, and express the results in terms of $\tan\theta$.



Problem 5.28.

5.28. A homogeneous crate of mass m rests on a horizontal surface where the coefficient of dynamic friction is ν. (a) Find the magnitude of the inclined force **P** required to give the crate a constant acceleration **a** in the direction shown. (b) Apply Euler's second law (5.44) to find the distance from the center of mass to the line of action of the normal surface reaction force **N**. Do this in three ways. (i) Prove that $\mathbf{M}_Q = \mathbf{0}$ about a fixed point Q at the initial position of the center of mass of the crate, and thus solve for the location of **N**. (ii) Repeat the analysis for the torque $\mathbf{M}_O = \dot{\mathbf{h}}_O$ about a fixed point O in the contact plane at the initial position. (iii) Prove that the total torque $\mathbf{M}_C$ about the moving center of mass must vanish and thus locate the action line of **N**. (c) What is the critical angle θ_c for impending tip expressed in terms of assigned quantities only?

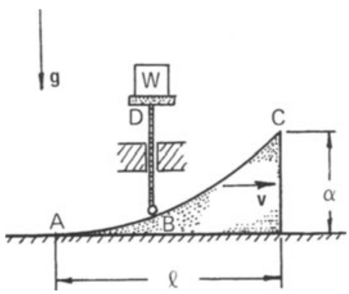
5.29. The wedge body $\mathscr{B}_1$ in Fig 5.18a, page 53, is accelerated at a constant rate a toward the right. The block $\mathscr{B}_2$ maintains contact with the plane throughout the motion. The gravitational force acts downward in the figure. Show that $\mathscr{B}_2$ will slide down the inclined surface if $a > g\tan(\alpha - \psi)$, where $\tan\alpha = \mu$ is the coefficient of static friction for the two surfaces and $\psi < \alpha$.



Problem 5.30.

5.30. The figure shows a block B_1 of weight W_1 attached by an inextensible cable to a block B_2 of weight W_2. The weight ratio $W_1/W_2 = 5/6$. The cable is supported by a smooth ring, and B_2 rests on a rough horizontal surface where $\mu = 2/5$ and $\nu = 1/3$. (a) Determine the critical weight ratio W_1/W_2 for which motion is imminent, and thus show that the system must move if released from rest. (b) Find the acceleration $\mathbf{a}$ of the block B_1 as a function of the weight ratio, and determine its value for the assigned data.

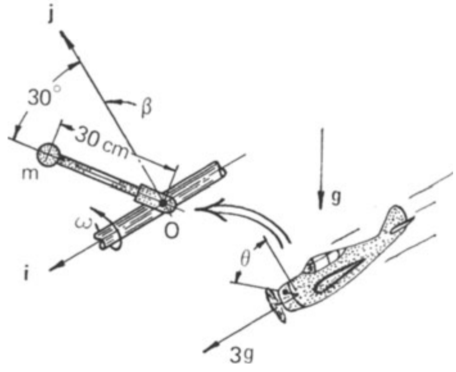
5.31. A body P of mass $m = 5$ slug has weight $\mathbf{W} = 160\mathbf{j}$ lb relative to the planet Φ. (a) Suppose that P is at rest on a scale in a nonrotating frame $\varphi = \{O; \mathbf{i}_k\}$ which has an acceleration $\mathbf{a}_O = 20\mathbf{j}$ ft/sec^2 relative to Φ. What is the weight of P apparent to an observer in φ? What is its apparent weight when φ has the opposite acceleration $\mathbf{a}_O = -20\mathbf{j}$ ft/sec^2 in Φ? Find the acceleration of φ for which the apparent weight of P vanishes. (b) Now suppose that P is dropped from a state of rest in Φ so that the only force that acts on P is its weight relative to Φ. Address the previous question for the observer in φ. (c) Discuss the results and compare the observations in φ with those in Φ.



Problem 5.32.

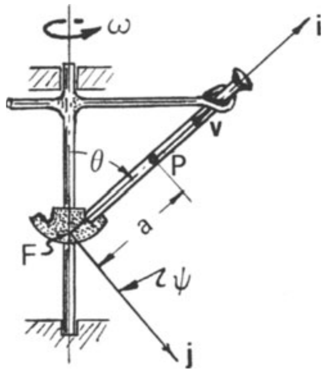
5.32. During an interval of interest, the vertical motion of a load W is controlled by a parabolic cam ABC that moves horizontally with a constant velocity $\mathbf{v}$ directed as shown. Draw a free body diagram of the block. Determine the compressive force in the push rod BD in terms of the load and the assigned quantities. Neglect friction.

5.33. A part in an aircraft engine consists of a 0.10 kg mass m attached by a 30 cm rod to the propeller drive shaft. The shaft turns, as shown, with an angular velocity $\boldsymbol{\omega} = 100\omega(t)\mathbf{i}$ rad/sec. During a dive, the aircraft accelerates at $3g$, and the rod is inclined at a fixed angle $\theta = 30°$ in the frame $\beta = \{O; \mathbf{i}_k\}$ fixed in the propeller shaft. Determine the total force acting on m.



Problem 5.33.

5.34. A test tube is held at a fixed angle θ in a centrifuge spinning, as shown, with a constant angular velocity $\boldsymbol{\omega}$ about a fixed vertical axis. A fluid particle of mass m, initially near the bottom at F, is moving outward in the tube with a constant relative velocity $\mathbf{v} = v\mathbf{i}$. Identify the time dependent variables, and determine as a function of time the total force that acts on P, referred to the tube frame $\psi = \{F; \mathbf{i}_k\}$.



Problem 5.34.

5.35. A system of three forces $\mathbf{F}_1 = 6\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ N, $\mathbf{F}_2 = -2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ N, $\mathbf{F}_3 = 5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ N act at the respective points $\mathbf{x}_1 = (1, 0, 0)$ m, $\mathbf{x}_2 = (0, 1, 0)$ m, $\mathbf{x}_3 = (0, 0, 1)$ m in frame $\Phi = \{Q; \mathbf{i}_k\}$. (a) Find the equipollent system with force $\mathbf{F}^A = \mathbf{P}$ and torque $\mathbf{M}_Q^A$ with respect to Q. (b) Is $\mathbf{F}^A \cdot \mathbf{M}_Q^A = 0$? (c) Find the equations that describe the line of action of the single force $\mathbf{P}$. (d) Determine the center of force $\mathbf{x}_Q^*$ with respect to the origin Q. (e) Determine the center of force $\mathbf{x}_O^*$ with respect to the point O at $\mathbf{x}_1$ in Φ, and confirm your result by showing that $\mathbf{x}_O^* \times \mathbf{P} = \mathbf{M}_O^A$ for the original system of forces.

6

Dynamics of a Particle

6.1. Introduction

We have seen that in an inertial reference frame, Euler's first law (5.43) for the motion of the center of mass "particle" of a rigid body $\mathscr{B}$, a fictitious material point of mass $m(\mathscr{B})$ that moves with the body, has the same form as Newton's second law (5.39) for the motion of a particle P of mass $m(P)$. Hence, the motion of any such "material point" or "particle" is governed by the Newton–Euler law of motion, here written in its various forms as

$$\mathbf{F} = \dot{\mathbf{p}} = m\mathbf{a} = m\dot{\mathbf{v}} = m\ddot{\mathbf{x}}, \tag{6.1}$$

in which m is the mass of the "particle," $\mathbf{p} = m\mathbf{v}$, and $\mathbf{x}$, $\mathbf{v}$, and $\mathbf{a}$ are its respective current position, velocity, and acceleration in an inertial reference frame.

Our objective now is to study a variety of physical applications and solutions of the Newton–Euler equation of motion of a particle for various kinds of forces and motions and thus demonstrate its predictive value. In some examples, the principal body of interest may be small in some sense. An electron, a grain of sand, and a fluid droplet are typical examples of infinitesimal or small bodies commonly modeled as particles. Larger bodies like a ball, a pendulum bob, a crate, a person, and an automobile are modeled as center of mass objects of rigid bodies. So long as the rigid body has no rotation itself, there is no intrinsic difference between these two models. In fact, in many such problems in which the body is replaced by its center of mass "particle," precise identification of the center of mass point is not necessary; the mass distribution and the specific body geometry play no major roles; and the actual points of application of the resultant forces that act on the body are unimportant—they act on the particle. All of these virtually inconsequential matters, however, have great importance later when rotational effects of a rigid body are introduced. We recall, for example, the simple problem of a block sliding down an inclined plane without tipping over. In this case, the body's physical and geometrical properties, the location of the points of application of forces that act on it, and their moments were all very important to the description and analysis of the block's motion. These sorts of underlying potential complications are avoided when rotational effects are absent and a rigid body is modeled as a particle.

The study of particle dynamics thus deals with the analysis of the vector differential equation (6.1) for the motion of a particle and the forces that produce it. When the motion, the velocity, or the acceleration is known either as a function of time or as a function of a time dependent parameter, such as arc length along a path, the force required to produce the motion is readily determined by (6.1). The converse problem, to determine the motion of a particle under various kinds of assigned forces, however, is more difficult, because it involves the integration of (6.1). Moreover, the specification of some forces together with some components of acceleration, velocity, or position leads to a mixed variety of problem types. Some easy methods of integration useful in the analysis of (6.1) were studied in earlier chapters. Additional methods and several new concepts will be introduced as our study unfolds.

6.2. Component Forms of the Newton–Euler Law

We recall that the motion of a particle may be described in terms of different coordinate systems that offer special advantages in applications; and, clearly, in applications of (6.1), the force vector and the motion eventually must be represented in the same reference basis. For handy reference, the vector representations of the Newton–Euler law in four familiar kinds of reference bases are provided below.

Rectangular Cartesian reference frame $\Phi = \{O; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$: The acceleration is given by (1.12) and (6.1) may be written as

$$\mathbf{F} \equiv F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} = m(\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}). \tag{6.2}$$

Intrinsic reference frame $\psi = \{P; \mathbf{t}, \mathbf{n}, \mathbf{b}\}$: Equation (1.71) provides the acceleration and (6.1) becomes

$$\mathbf{F} \equiv F_t\mathbf{t} + F_n\mathbf{n} = m(\ddot{s}\mathbf{t} + \kappa\dot{s}^2\mathbf{n}). \tag{6.3}$$

Notice that there can be no intrinsic force component F_b normal to the osculating plane. Hence, if the motion is constrained to a plane, the total force component perpendicular to the plane must vanish. This is a property of every plane motion.

Cylindrical reference frame $\varphi = \{O; \mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$: The Newton–Euler law (6.1) and the acceleration vector in (4.60) yield the representation

$$\mathbf{F} \equiv F_r\mathbf{e}_r + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z = m\left[(\ddot{r} - r\dot{\phi}^2)\mathbf{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\phi})\mathbf{e}_\phi + \ddot{z}\mathbf{e}_z\right]. \tag{6.4}$$

Spherical reference frame $\varphi = \{O; \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$: The acceleration components are defined in (4.71). Hence, (6.1) becomes

$$\mathbf{F} \equiv F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi = m\left[(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta)\mathbf{e}_r + \left(\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r\dot{\phi}^2 \sin\theta\cos\theta\right)\mathbf{e}_\theta + \left(\frac{1}{r}\frac{d}{dt}(r^2\dot{\phi}\sin\theta) + r\dot{\phi}\dot{\theta}\cos\theta\right)\mathbf{e}_\phi\right]. \tag{6.5}$$

The left-hand expressions in (6.2) through (6.5) define the respective component forms of the total force. The force components are then related to the acceleration components by equating their corresponding scalar components in these expressions. The intrinsic force components F_t and F_n in (6.3), for example, are thus related to the intrinsic acceleration components by $F_t = m\ddot{s}$, $F_n = m\kappa\dot{s}^2$. The procedure is the same for the others. The component equations are called the *scalar equations of motion*. In general, however, we first formulate each problem in its vector form, and afterwards identify the corresponding scalar equations of motion.

It is not always necessary to introduce a specific component form of (6.1). Sometimes it is possible to solve a problem in direct vector form without mention of any components, but more often than not this approach proves tedious and impractical; therefore, the component forms find wider use in applications.

6.3. Some Introductory Examples and Additional Concepts

We shall begin with several introductory examples that employ the foregoing representations in some problems where the motion is essentially known and certain force conditions are to be determined. Some earlier concepts are reviewed, and some new concepts are introduced as the examples progress. The importance of the Newton–Euler law in its generic form (5.34) is underscored in characterizing the motion of a relativistic particle.

6.3.1. Some Applications in a Rectangular Cartesian Reference Frame

Three problems that use a rectangular Cartesian reference frame are solved. The first is an easy application of (6.1) in which the acceleration is known and a certain force is to be found. The example demonstrates the importance of our distinguishing the inertial reference frame in applications of the Newton–Euler law. The second exercise illustrates an application in which the acceleration of one body is known, and a Coulomb condition for relative sliding of another contacting body is to be determined. The results will be used in the third example to illustrate

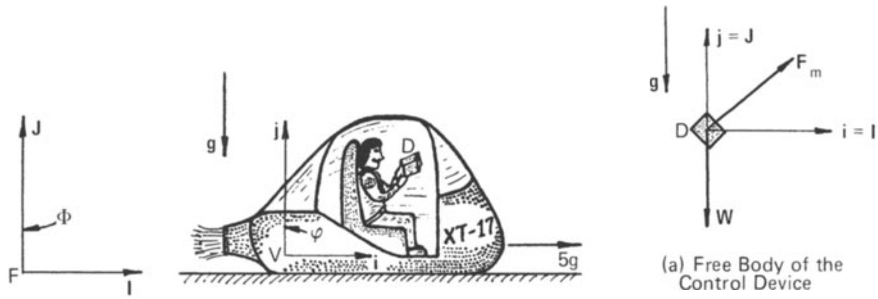


Figure 6.1. Motion in an accelerating reference frame.

the converse problem in which the forces are known and information about the motion is to be obtained. The form of the law in (6.2) is evident in the applications.

Example 6.1. A rocket propelled test vehicle V in Fig. 6.1 is used to study man's ability to function at high rates of acceleration and deceleration.* (a) Suppose the vehicle is accelerating at $5g$ along a straight track in the inertial frame $\Phi = \{F; \mathbf{I}_k\}$. What force does the operator need to apply to a 2 lb control device D to impart to its center of mass a relative acceleration $\mathbf{a}_{DV} = 16\mathbf{i} + 80\mathbf{j}$ ft/sec^2 in the vehicle frame $\varphi = \{V; \mathbf{i}_k\}$? (b) Compare the result with the force required to perform the same task when the vehicle has a uniform motion in Φ. Assume that the local acceleration of gravity is 32 ft/sec^2.

* The example brings to mind the daring exploits of U.S. Air Force Colonel John P. Stapp, MD, Ph.D., the biomedical engineering pioneer, who in December 1947, at Edwards (then Muroc) Air Force Base, California, became the first human to ride a rocket propelled test sled to study human tolerance to severe decelerations of the sort sustained in the crash of an automobile or aircraft. Based on Stapp's research studies, appropriate safety harnesses, helmets, restraints, and other essential equipment could be developed. Stapp demonstrated firsthand that a properly harnessed and protected driver, pilot, or astronaut could indeed survive an incredible impact, the wind blast, and deceleration of ejection from an aircraft traveling at supersonic speeds at great altitudes, or the large acceleration of a rocket lift-off, himself having withstood test sled decelerations of 25 to more than 40 times the acceleration of gravity. With new facilities at the Holloman Air Force Base, New Mexico, where subsequently he set up and directed his biomedical engineering and crash research programs, in 1954 Stapp rode the rocket vehicle "Sonic Wind" from 632 mph to a dead stop in 1.4 sec, suffering only minor injuries in a deceleration of more than 40 gs! A 2200 lb (1000 kg) automobile smashing into a brick wall at 50 mph (≈80 kmph) would subject its driver to roughly the same impulsive shock. Other human volunteers in his program tested the security of safety belts in decelerations that exceeded 25 gs. See *Time*, *The Weekly Newsmagazine*, Volume **66**, No. 11, September 12, (1955), 80–2, 85–6, 88. Stapp's adventures, his sense of humor, and his generosity to others are portrayed here. Dr. Stapp, then dubbed "the fastest man on earth," died at his New Mexico home on November 13, 1999, at age 89. I thank Professor O. W. Dillon, who during the early 1950s was stationed at Holloman when Stapp was directing these research programs, and upon reading the manuscript reminded me of Stapp's heroic feats.

Solution of (a). We begin with the problem kinematics. The absolute acceleration of the vehicle in the inertial frame Φ is given as $\mathbf{a}_{VF} = 5g\mathbf{I}$, where $g = 32$ ft/sec^2. Thus, recalling the simple relative acceleration rule (4.50) and the assigned center of mass acceleration $\mathbf{a}_{DV} = 16\mathbf{i} + 80\mathbf{j}$ ft/sec^2 of D in the vehicle frame φ in which $\mathbf{i}_k = \mathbf{I}_k$, we determine the absolute acceleration of D in frame Φ:

$$\mathbf{a}_{DF} = \mathbf{a}_{DV} + \mathbf{a}_{VF} = 176\mathbf{I} + 80\mathbf{J} \text{ ft/sec}^2. \tag{6.6a}$$

This completes the kinematical analysis.

We now turn to the force analysis. The free body diagram of D is shown in Fig. 6.1a. As usual, we shall assume that the contact force due to the surrounding air is self-equilibrated to zero. Then the total force $\mathbf{F}(D, t)$ acting on D is the sum of its weight $\mathbf{W}$ and the force $\mathbf{F}_m$ exerted by the operator. Hence, the Newton–Euler law (6.1) applied to D in the inertial frame Φ yields

$$\mathbf{F}(D, t) = \mathbf{W} + \mathbf{F}_m = m(D)\mathbf{a}_{DF}, \tag{6.6b}$$

in which $\mathbf{W} = -mg\mathbf{J} = -2\mathbf{J}$ lb and $m(D) = 1/16$ slug. The kinematics in (6.6a) is now coupled with the force analysis in (6.6b) to yield the solution

$$\mathbf{F}_m = 11\mathbf{I} + 7\mathbf{J} \text{ lb}. \tag{6.6c}$$

Solution of (b). We note from (6.6c) that $|\mathbf{F}_m| = \sqrt{170} \approx 13.04$ lb. We wish to compare this result with the force needed to perform the same task when the vehicle has a uniform motion in Φ. To impart the same acceleration to the device when the vehicle has a constant velocity or may be at rest in Φ so that now $\mathbf{a}_{VF} = \mathbf{0}$ and $\mathbf{a}_{DF} = \mathbf{a}_{DV}$, we find from (6.6b) that the operator must apply a force $\mathbf{F}_m = m\mathbf{a}_{DV} - \mathbf{W} = \mathbf{I} + 7\mathbf{J}$ lb. Hence, $|\mathbf{F}_m| = 5\sqrt{2} \approx 7.07$ lb. Therefore, if the Newton–Euler law were applied in the accelerating reference frame, the operator would conclude incorrectly that a force of about 7 lb is needed, while the task actually requires nearly twice that. We thus learn that *when the operator works in the accelerating vehicle, nearly twice the effort must be expended to perform the assigned task.* □

This example demonstrates the important role of the inertial reference frame in applications of the Newton–Euler law. The next problem concerns the prediction of relative sliding of a body in contact with an accelerating surface.

Example 6.2. A truck carrying a crated load W is moving down a 15° grade in Fig. 6.2. The driver suddenly applies the brakes and the truck decelerates at the steady rate of 4 ft/sec^2 along its straight path. The coefficient of static friction between the crate and the trailer bed is $\mu = 0.3$. Determine for the given values of the parameters whether the crate will slide or remain stationary relative to the trailer.

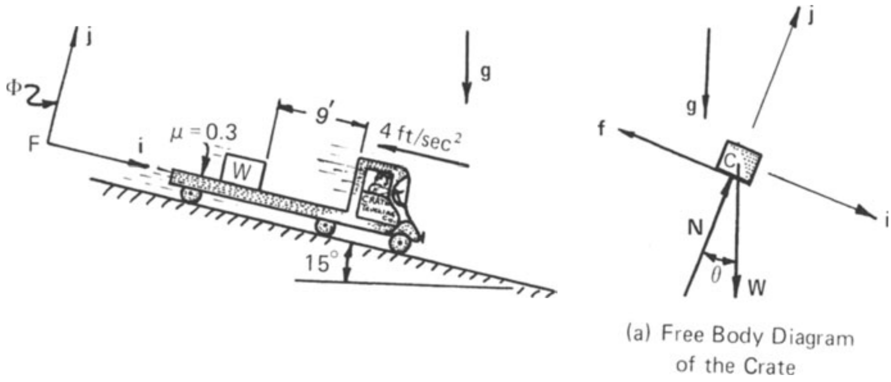


Figure 6.2. Relative motion of a crate on an accelerating truck.

Solution. We shall assume initially that the crate does not slide relative to the truck and seek a Coulomb condition sufficient to assure this. If this condition fails for the assigned data, we then know that the crate will slide. This strategy will enable us to decide the issue.

To investigate the motion of the crate C, we first draw its free body diagram in Fig. 6.2a. To simplify matters, all contact forces due to the Earth's atmosphere, including air flow effects due to the truck's motion and other wind effects, are neglected. Then the total force $\mathbf{F}(C,t)$ acting on C is approximated by its weight $\mathbf{W}$ and the resultant normal and tangential contact forces $\mathbf{N}$ and $\mathbf{f}$ exerted by the trailer bed. The equation of motion (6.1) for C becomes

$$\mathbf{F}(C,t) = \mathbf{W} + \mathbf{f} + \mathbf{N} = m\mathbf{a}_{CF}, \tag{6.7a}$$

wherein $m = m(C)$ is the total mass of C and $\mathbf{a}_{CF}$ is its total rectilinear acceleration in the inertial ground frame $\Phi = \{F; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. The vectors in (6.7a) are given by

$$\mathbf{W} = W(\sin\theta\mathbf{i} - \cos\theta\mathbf{j}), \qquad \mathbf{f} = -f\mathbf{i}, \qquad \mathbf{N} = N\mathbf{j}, \qquad \mathbf{a}_{CF} = a_C\mathbf{i}, \tag{6.7b}$$

and hence

$$(W\sin\theta - f)\mathbf{i} + (N - W\cos\theta)\mathbf{j} = ma_C\mathbf{i}. \tag{6.7c}$$

Therefore, the scalar equations of motion for the crate are

$$ma_C = W\sin\theta - f, \qquad N - W\cos\theta = 0. \tag{6.7d}$$

When a_C is known, equations (6.7d) determine the unknown forces N and f. Thus, with $W = mg$,

$$N = W\cos\theta, \qquad f = W(\sin\theta - a_C/g). \tag{6.7e}$$

Recalling the strategy proposed earlier, we note that the crate will not slip if the frictional force f is smaller than its critical static Coulomb value (5.70), that is, provided that $f < f_c = \mu N$. (See also (5.72).) In this case, because the crate is assumed not to slip, its acceleration is the same as that of the truck, namely, $\mathbf{a}_{TF} = a_T\mathbf{i}$. Thus, with the aid of (6.7e) and $a_C = a_T$, the Coulomb no slip criterion is

$$\sin\theta - \frac{a_T}{g} < \mu\cos\theta. \tag{6.7f}$$

This conclusion is independent of the weight, the size, and the shape of the crate. Actually, however, we have tacitly assumed in (6.7f) that the crate geometry is consistent with the no tip condition, which imposes limitations on the crate geometry. The reader may confirm, for example, that for a rectangular box of height $2h$ and a square cross section of side $2b$, the crate will not topple before slip occurs, if it occurs at all, provided that $b/h > \mu$.

The crate will not slide if (6.7f) holds for the assigned data; otherwise, it will. We now test (6.7f) for the assigned values $a_T = -4$ ft/sec^2, $g = 32.2$ ft/sec^2, $\mu = 0.3$, and $\theta = 15°$. The terms on the left side of (6.7f) yield the value $l \equiv 0.383$ while those on right give $r \equiv 0.290$. Since $l > r$, (6.7f) does *not* hold, and the crate will slide. For an alternative approach, the reader may show that the critical acceleration $\hat{a}_T$ of the truck for which sliding of the crate is imminent is given by $\hat{a}_T = g(\sin\theta - \mu\cos\theta) = -1$ ft/sec^2, the condition for equality in (6.7f). Since $|a_T| = 4$ ft/sec$^2 > |\hat{a}_T|$, the crate will slide, as concluded previously. □

The simple relative motion of the crate on the truck bed is examined next in illustration of the converse problem in which the forces are known and the velocity and the motion of the crate are to be found.

Example 6.3. The coefficient of dynamic friction between the crate and the trailer bed is $\nu = 0.25$. What is the rectilinear acceleration of the crate relative to the trailer? Determine the distance on the bed traveled by the crate after 1 sec and after 2 sec.

Solution. The crate C has a rectilinear acceleration $\mathbf{a}_{CT}$ relative to the truck T given by

$$\mathbf{a}_{CT} = \mathbf{a}_{CF} - \mathbf{a}_{TF}, \tag{6.8a}$$

wherein $\mathbf{a}_{TF} = a_T\mathbf{i}$ is the known absolute acceleration of the truck in the inertial frame Φ. We need to find $\mathbf{a}_{CF}$, the total acceleration of the crate in Φ.

The vector equation for the sliding motion of the crate is the same as (6.7a), and hence the scalar equations of motion for the crate in Φ are given in (6.7d). But this time, because the crate is sliding on the trailer bed, the Coulomb frictional force is given by (5.71). (See also (5.73).) Thus, with the last of (6.7d), we have $f = f_d = \nu N = \nu W\cos\theta$, and use of this relation in the first equation in (6.7d) yields a_C. That is, $\mathbf{a}_{CF} = a_C\mathbf{i} = g(\sin\theta - \nu\cos\theta)\mathbf{i}$. Hence, (6.8a) delivers the first of the desired results:

$$\mathbf{a}_{CT} = a_{CT}\mathbf{i} = [g(\sin\theta - \nu\cos\theta) - a_T]\mathbf{i}. \tag{6.8b}$$

Therefore, the rectilinear acceleration of the crate relative to the truck is independent of the weight, the size, and shape of the crate, consistent with the no tip condition.

The relative acceleration (6.8b) is a constant vector. With $a_T = -4$ ft/sec^2, $g = 32.2$ ft/sec^2, $\nu = 0.25$, and $\theta = 15°$, we find $\mathbf{a}_{CT} = 4.56\mathbf{i}$ ft/sec^2. To determine the distance traveled by the crate on the bed, we first integrate the differential equation $\delta\mathbf{v}_{CT}/\delta t = \mathbf{a}_{CT}$ with the initial condition $\mathbf{v}_{CT}(0) = \mathbf{0}$ to obtain $\mathbf{v}_{CT} = \mathbf{a}_{CT}t = 4.56t\mathbf{i}$. Hence, the relative speed of C is $\dot{s}(t) = 4.56t$; and with $s(0) = 0$, the distance traveled by the crate is $s(t) = 2.28t^2$. Therefore, after 1 sec the crate has moved a distance $s(1) = 2.28$ ft. After 2 secs, $s(2) = 9.12$ ft, and the crate, regardless of its physical features, slams into the cab, initially only 9 ft away in Fig. 6.2. □

6.3.2. Intrinsic Equation of Motion for a Relativistic Particle

In this section, the intrinsic equation of motion for a relativistic particle whose "effective" mass varies with its speed is derived, and the result is applied to examine the nature of a purely normal force that acts on the particle in its motion along a smooth curved path. The Newton–Euler law in the form (6.1), however, cannot be used in problems where the mass of the particle is variable; so we return to the basic law (5.34).

In relativistic mechanics, the *relativistic mass* m of a particle P in a frame Φ varies with its speed $\dot{s}$ relative to Φ in accordance with the rule

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1-\beta^2}} \qquad \text{with} \quad \beta \equiv \frac{\dot{s}}{c}. \tag{6.9}$$

The constant m_0, the invariant mass of the particle, is called the *rest mass* of P in Φ and the constant c is the speed of light in a vacuum. The relativistic mass m is not the intrinsic mass of P. Rather, the concept of mass is retained as an invariant, intrinsic property of an object, and hence m_0 is identified as the invariant mass of the object, the same for all observers and for all times. The principle of conservation of mass applies to m_0, not to m. Although nowadays it is unfashionable to refer to m as the relativistic mass, it is convenient in this text to retain the symbolic relation $m \equiv \gamma m_0$ defined by (6.9) and continue to call it the relativistic mass.

These semantics aside, the *relativistic momentum* of P is defined by $\mathbf{p} \equiv m\mathbf{v} = \gamma m_0\mathbf{v}$, where $\mathbf{v} = d\mathbf{x}/dt$ is the usual time derivative; and the rule governing the motion of P is retained in the general Newtonian form $\mathbf{F} = d\mathbf{p}/dt$ stated in (5.34). Although m changes with $\dot{s}$, it is easy to show that $\mathbf{F} = \mathbf{0}$ holds if and only if the motion is uniform in Φ. This conforms with the condition set by the first law, i.e. $\mathbf{F} = \mathbf{0} \Longleftrightarrow \mathbf{a} = \mathbf{0}$. Otherwise, in view of (6.9), the second law becomes

$$\mathbf{F} = m\mathbf{a} + \frac{dm}{dt}\mathbf{v}. \tag{6.10}$$

Now, with the aid of (6.9) and $\mathbf{v} = \dot{s}\mathbf{t} = c\beta\mathbf{t}$, we find

$$\frac{dm}{dt}\mathbf{v} = \frac{m_0 \beta\dot{\beta}\mathbf{v}}{(1-\beta^2)^{3/2}} = \frac{m\beta^2}{1-\beta^2}\ddot{s}\mathbf{t}.$$

Therefore, use of this result and (1.71) for the intrinsic acceleration in (6.10) leads to the *intrinsic equation of motion for a relativistic particle*:

$$\mathbf{F} = m\left(\frac{\ddot{s}}{1-\beta^2}\mathbf{t} + \kappa\dot{s}^2\mathbf{n}\right). \tag{6.11}$$

When $\dot{s} \ll c$ so that $\beta \ll 1$, (6.9) reduces approximately to $m = m_0$ and we recover from (6.11) the classical, nonrelativistic intrinsic equation in (6.3). It follows from (6.11) that *the total force* $\mathbf{F}$ *acting on a particle may be normal to its path, hence perpendicular to its velocity vector* $\mathbf{v} = \dot{s}\mathbf{t}$, *if and only if its speed is constant*. (See Problem 1.5, Volume 1.) This is illustrated below.

Example 6.4. A particle P, free from gravitational force, experiences a relativistic motion in a smooth, spatially curved tube. Find the force exerted on the particle by the tube and characterize the tube geometry in order that the force may have a constant magnitude.

Solution. The reader's free body diagram of P will show that the total force on P is simply the normal reaction force exerted by the smooth tube. Hence, use of $\mathbf{F} = \mathbf{N} = N\mathbf{n}$ in (6.11) yields the desired information:

$$N = m\kappa\dot{s}^2 \quad \text{and} \quad \ddot{s} = 0. \tag{6.12}$$

Indeed, the second of these equations shows that the particle speed must be constant; and hence the relativistic mass in (6.9) must be constant too. Therefore, the first relation in (6.12) shows that *in a smooth motion with constant speed, the normal reaction force intensity at each point along the path is proportional to the curvature and is directed toward the center of curvature*. Clearly, $N = 0$ if and only if the motion is uniform, in which case the tube must be straight. *In general, N may be constant if and only if the tube has a constant curvature.* A cylindrical helix is a familiar example of a space curve having a constant curvature. (See Example 1.14.) If the motion is a plane motion, the tube must be circular. The following further example is left for the reader. □

Exercise 6.1. A particle P moves on a smooth surface S so that the only force on P is the normal surface reaction force $\mathbf{R}$. Prove that the principal normal vector **n** must be perpendicular to S at each point of the trajectory of P and hence the path is a geodesic on S. (See Example 1.16 in Volume 1.) □

The results for the motion of a relativistic particle in a smooth tube hold independently of relativistic considerations when $\beta \ll 1$. It is shown later that the same behavior occurs when an electrically charged particle, relativistic or not, moves in a uniform magnetic field.

6.3.3. Electric and Magnetic Forces on a Charged Particle

Two basic laws that describe electric and magnetic body forces are introduced. Afterwards, the trajectory of an electrically charged particle moving in a steady and uniform magnetic field is described.

First, consider the mutual force of attraction or repulsion between two particles with electric charges q_1 and q_2 respectively situated at $\mathbf{X}_1$ and $\mathbf{X}_2$ in an arbitrarily assigned reference frame so that the distance between them is $r = |\mathbf{X}_2 - \mathbf{X}_1|$. Let $\mathbf{F}_{12}$ denote the force exerted on q_1 by q_2, and write $\mathbf{e}$ for the unit vector directed from q_2, the source of the action, toward q_1. The force exerted on q_2 by q_1 is equal and oppositely directed so that $\mathbf{F}_{21} = -\mathbf{F}_{12}$. Experiments support the following principle governing the mutual interaction of electrically charged particles.

Coulomb's law of electrostatics: *Between any two charged particles in the world, there exists a mutual electrostatic force which is directly proportional to the product of the charges, inversely proportional to the square of the distance between them, and directed along their common line in the sense of mutual repulsion or attraction according as the charges are of the same or opposite kind, respectively; that is,*

$$\mathbf{F}_{12} = \frac{kq_1q_2}{r^2}\mathbf{e}. \tag{6.13}$$

The value of the positive constant k depends on the nature of the medium in which the charges are placed. The physical dimensions of k are fixed by (6.13): $[k] = [FL^2Q^{-2}]$, where $[Q] = [q]$ denotes the physical dimension of electric charge. The metric measure unit of q is named the *coulomb*. Experiments on charges in vacuum show that $k = 9 \times 10^9$ N $\cdot$ m^2/coulomb^2. Notice that only the relative position vector $\mathbf{r} = r\mathbf{e}$ of q_1 from q_2 is important.

The rule (6.13) is a particular example of Noll's general rule (5.115) governing the internal force between any pair of particles, in this case charged particles; and the formal similarity of (6.13) with Newton's law of gravitation (5.46) is evident. We thus introduce the parallel idea of an electric field $\mathscr{E}$ that arises from the existence of a charged particle situated in space. And when a particle of charge q is placed in this space, it experiences a force of attraction or repulsion determined by (6.13). An *electric field* $\mathscr{E}$ is said to exist throughout space due to a particle of positive charge q_0, called the *source* of the electric field, whenever a force is felt by another charged "test" particle placed anywhere in $\mathscr{E}$. Thus, the *electric field strength* $\mathbf{E}$ at the place $\mathbf{X}$ due to q_0 is defined by

$$\mathbf{E}(\mathbf{X}) = \frac{kq_0}{r^2}\mathbf{e}(\mathbf{X}), \tag{6.14}$$

where $\mathbf{e}$ is the unit vector directed from q_0 toward the field point $\mathbf{X}$ at r from q_0. Hence, the *electric force* $\mathbf{F}_e$ that acts on a particle P of charge q at the place $\mathbf{X}$ is a body force given by

$$\mathbf{F}_e(P; \mathbf{X}) = q(P)\mathbf{E}(\mathbf{X}). \tag{6.15}$$

The same rule holds when the charged particle moves in the electrostatic field $\mathscr{E}$.

The electric body force is in the direction of $\mathbf{E}$ (repulsive) when q is positive and opposite to $\mathbf{E}$ (attractive) when q is negative. Hence, the action of this force alone will move a charged particle in a straight line in the direction of $\mathbf{E}$ if $q > 0$, oppositely if $q < 0$. The *principle of conservation of electric charge* asserts that the total charge Q for a closed system of n charges q_k is a constant equal to their algebraic sum: $Q = \sum_{k=1}^{n} q_k$. Thus, in a manner parallel to that demonstrated for a gravitational field, the resultant electric force on a particle of charge q placed in the electric field of a system of charged particles or, similarly, in the field of a charged continuum is given by the fundamental law (6.15). In general, then, the electric force acting on a particle of charge q having a motion $\mathbf{X}(q, t)$ in an electrostatic field of strength $\mathbf{E}(\mathbf{X})$ is given by (6.15).

A *magnetic field of strength* $\mathbf{B}$ arises in a similar way from the existence in space of some kind of magnetic object. When a charged particle moves with a velocity $\mathbf{v}$ in a time independent magnetic field $\mathbf{B}$, it experiences a body force $\mathbf{F}_m$, the *magnetic force*, given by

$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}. \tag{6.16}$$

This equation shows that the magnetic body force $\mathbf{F}_m$ on a charged particle is always perpendicular to $\mathbf{v}$, and hence to the particle's path. Under the action of this force alone the particle, from (6.12), must move with a constant speed v_0, say; so, the magnitude of its momentum $|\mathbf{p}| = mv_0$ is constant.

Example 6.5. Consider a relativistic charged particle of rest mass m_0 moving in a constant magnetic field of strength $\mathbf{B}$. (a) Prove that the charge moves in a circular helix, a curve of constant curvature, and hence $\mathbf{F}_m$ has a constant magnitude. (b) Derive the equation of the path for a plane motion perpendicular to a constant magnetic field $\mathbf{B} = B\mathbf{k}$.

Solution of (a). To determine the trajectory of a particle of charge q moving in a magnetic field of constant strength $\mathbf{B}$, we recall Newton's law in (5.34) and consider the relation

$$\frac{d}{dt}(\mathbf{p}\cdot\mathbf{B}) = \frac{d\mathbf{p}}{dt}\cdot\mathbf{B} = \mathbf{F}_m\cdot\mathbf{B} = 0, \tag{6.17a}$$

wherein (6.16) is the total force on q. Therefore, the component of the momentum in the direction of $\mathbf{B}$ is constant:

$$\mathbf{p}\cdot\mathbf{B} = m\mathbf{v}\cdot\mathbf{B} = C, \text{ a constant.} \tag{6.17b}$$

Since the magnitudes of $\mathbf{p}$ and $\mathbf{B}$ are constant, (6.17b) implies that the angle between the fixed axis of $\mathbf{B}$ and the tangent to the space curve along which q moves is constant everywhere along the path. Consequently, as described in Example 1.14, the path is a circular helix, a space curve of constant curvature; therefore, $|\mathbf{F}_m| = qv_0B\sin\langle\mathbf{v},\mathbf{B}\rangle$ is constant. Conversely, it follows from (6.16) that if $\mathbf{F}_m$ has a constant magnitude, $\sin\langle\mathbf{v},\mathbf{B}\rangle$ is constant and hence the path is a circular helix.

The initial velocity $\mathbf{v}_0$ may be considered arbitrary. If the velocity is initially perpendicular to $\mathbf{B}$, then, by (6.17b), $\mathbf{p}\cdot\mathbf{B} = 0$ always, and the path is a circle in the plane perpendicular to $\mathbf{B}$. If the initial velocity $\mathbf{v}_0$ is parallel to $\mathbf{B}$, the constant force $\mathbf{F}_m = \mathbf{0}$; the motion is uniform and the path is a straight line along the axis of $\mathbf{B}$. The circle and the line are degenerate kinds of helices. In summary, *the trajectory of a charged particle which is given an arbitrary initial velocity in a constant magnetic field is a circular helix.*

Solution of (b). The path of a charge q in a plane motion perpendicular to the constant vector $\mathbf{B}$ is a circle. To describe this circle, we apply Newton's law in (6.16) to write $d\mathbf{p}/dt = d(q\mathbf{x}\times\mathbf{B})/dt$. Integration yields $\mathbf{p} - \mathbf{qx}\times\mathbf{B} = \mathbf{A}$, a constant vector. Let $\mathbf{B} = B\mathbf{k}$ and consider a plane motion perpendicular to $\mathbf{B}$, so that $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$. Then $\mathbf{p} = (A_1 + qBy)\mathbf{i} + (A_2 - qBx)\mathbf{j}$, and with $\mathbf{p}\cdot\mathbf{p} = |\mathbf{p}|^2 = m^2v_0^2$, a constant, this yields the equation of a circular orbit of radius $R \equiv mv_0/qB$:

$$\left(x - \frac{A_2}{qB}\right)^2 + \left(y + \frac{A_1}{qB}\right)^2 = R^2. \tag{6.17c}$$

We thus find with (6.9) that a charged relativistic particle in a uniform magnetic field moves on a circular orbit with angular speed $\omega = v_0/R = qB/m = (qB/m_0)\sqrt{1-\beta^2}$. This is known as the *circular cyclotron frequency.* □

When both fields (6.15) and (6.16) are present, the total *electromagnetic force,* known as the *Lorentz force,* is

$$\mathbf{F}_e + \mathbf{F}_m = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}. \tag{6.18}$$

Many interesting effects may be produced by an electromagnetic field. In some cases of physical interest an electromagnetic force is used to accelerate atomic

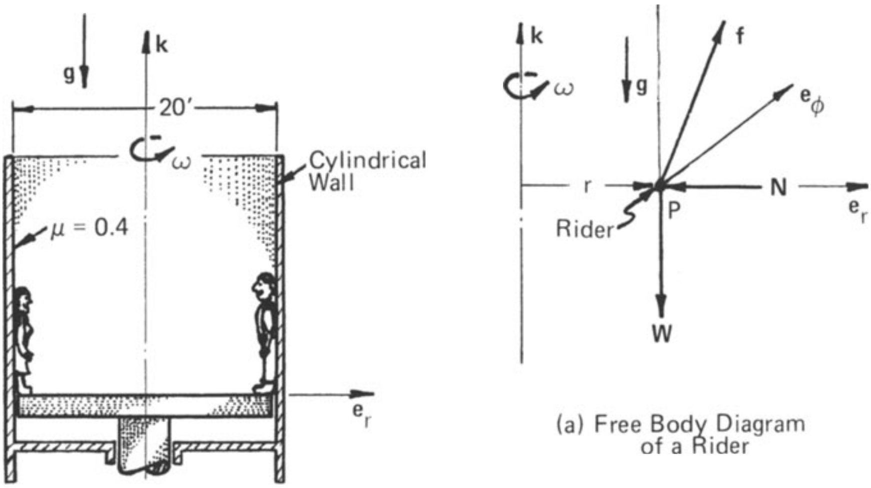


Figure 6.3. Relative equilibrium of passengers in an amusement park centrifuge.

particles in a cyclotron to speeds nearly as great as the speed of light. In these applications the electromagnetic force on the particle is considerably greater than the usual gravitational force, which is ignored. In further applications presented below, unless explicitly stated otherwise, it will be assumed that the speed of the particle is small compared with the speed of light so that the classical, Newton–Euler form (6.1) of the equation of motion for a particle or center of mass object is appropriate.

6.3.4. Fun at the Amusement Park

Our final illustration in this section concerns a design analysis of an amusement park ride to assess the safety of its occupants during its rotational motion. The cylindrical coordinate representation (6.4) for the equation of motion is illustrated.

Example 6.6. An amusement park ride shown in Fig. 6.3 consists of a 20 ft diameter cylindrical room that turns about its axis. People stand against the rough cylindrical wall. After the room has reached a certain angular speed, the floor drops from under the riders. What must be the angular speed of the room to assure that a person will not slide on the wall? The design coefficient of static friction is $\mu = 0.4$.

Solution. To assess the safe angular speed design, we seek a no-slip Coulomb condition sufficient to assure that a rider does not slide on the wall of the rotating room. The free body diagram of a rider represented as a center of mass object P is shown in Fig 6.3a. The rider's weight is $\mathbf{W} = -W\mathbf{k}$, and $\mathbf{N} = -N\mathbf{e}_r$ and $\mathbf{f} = f_\phi \mathbf{e}_\phi + f_z \mathbf{k}$ are the normal and the tangential frictional forces exerted by the wall. Thus, the total force $\mathbf{F}$ on a rider in a cylindrical frame that turns with the room is

$$\mathbf{F}(P, t) = \mathbf{N} + \mathbf{f} + \mathbf{W} = -N\mathbf{e}_r + f_\phi \mathbf{e}_\phi + (f_z - W)\mathbf{k}. \tag{6.19a}$$

For the safety of a rider, we require that the rider remain at rest relative to the wall. Then by (6.4) in which $\dot{\phi} = \omega$, or by (4.48) in which $\boldsymbol{\omega}_f = \omega \mathbf{k}$, it follows that $m\mathbf{a}_P = -mr\omega^2 \mathbf{e}_r$. Equating this to the force in (6.19a), we obtain the scalar equations of motion

$$N = mr\omega^2, \qquad W = f_z, \qquad f_\phi = 0. \tag{6.19b}$$

In the steady rotation of the room, no circumferential component f_ϕ of the frictional force is exerted on the rider by the wall; and the second of these relations shows that the rider will not slide down the wall if the Coulomb condition $W = f_z \leq f_c = \mu N$ holds. Therefore, with the first equation in (6.19b), the design criterion for safety of the riders is given by $\mu m r \omega^2 \geq W$. That is,

$$\omega \geq \sqrt{\frac{g}{r\mu}}, \tag{6.19c}$$

equality holding when slip is imminent; the smallest value $\omega^* = \sqrt{g/r\mu}$ being the critical angular speed of the room. *The result is independent of the weight of the rider; so all persons, fat or thin, will stay on the wall, provided that their coefficient of friction with the wall is not less than the design value chosen for* μ.

For the given conditions $r = 10$ ft and $\mu = 0.4$, the critical angular speed is $\omega^* = 2.84$ rad/sec, which is about 27 rpm. Thus, to secure the safety of the riders, the room must spin at a rate greater than 27 rpm. □

6.3.5. Formulation of the Particle Dynamics Problem

The foregoing examples show that when information about the motion is known, various questions involving the nature of the applied forces may be addressed. Some unanticipated physical conclusions are also pointed out, and the predictive value of the classical principles of mechanics is demonstrated. A review of the methods used in these examples reveals a fairly orderly arrangement of steps followed in the formulation and in the solution procedure applied to the particle dynamics problem; namely,

1. To begin, identify and express the data and the unknown quantities in mathematical form, and ask the key question: what relations connect the given data to the information to be found? Write these down and decide upon an initial problem attack strategy; but be prepared to modify your strategy as the attack advances and additional data is revealed.
2. To continue, construct a free body diagram that shows all of the properly directed contact forces and body forces that act on the free body in an appropriate reference frame.

3. Write down the total, **F**, of all forces identified in the free body diagram and express these various forces by their vector component representations in the chosen reference basis.
4. Determine the absolute acceleration **a** of the particle in the inertial frame but referred to the reference basis used above.
5. Assemble the results of steps 3 and 4 into the vector differential equation of motion: $\mathbf{F} = m\mathbf{a}$.
6. Equate the corresponding scalar components to obtain the scalar equations of motion, and proceed to solve these equations subject to the assigned data. Other laws appropriate to the problem, such as Newton's third law or Coulomb's laws, should be recalled and included here.

This basic procedural model is encountered repeatedly throughout our work. The outlined program, however, is not rigid. The examples suggest that sometimes it is useful, or simply a matter of personal preference, to begin with the kinematics in step 4 and then advance to the formulation of the force relations described in steps 2 and 3. Sometimes the vector equation in step 5, as shown in Example 6.5, page 105, may be solved directly without decomposing the vectors into their scalar components, eliminating steps 3 and 6. The student must be prepared to modify this schedule as other methods are introduced below. But the primary organizational step 1 always should be considered first and revisited as the solution unfolds.

With these ideas in mind, we shall begin the study of a variety of situations in which certain forces are prescribed functions and information concerning the motion and other forces is to be determined. This will require integration of the vector equation of motion (6.1). Some new forces of nature will be introduced along the way. We begin with some familiar examples.

6.4. Analysis of Motion for Time Dependent and Constant Forces

Problems of the motion of a particle under time varying and constant forces are readily solved by the method of separation of variables, a familiar approach used often in earlier examples. The formal solution of problems in this class, first presented as kinematical problems in Chapter 1, is reviewed next. The results are then applied in some elementary examples.

6.4.1. Motion under a Time Varying Force

Let us consider a total force $\mathbf{F} = \mathbf{F}(P, t)$ acting on a particle P in an inertial frame, given as a specified function of time. Then (6.1) yields $\mathbf{a}(P, t) = \mathbf{F}(P, t)/m(P)$, a known function of time. Hence, with $d\mathbf{v} = \mathbf{a}dt$, this differential equation is readily integrated in direct vector form to obtain the velocity of P:

$$\mathbf{v}(P,t) = \frac{1}{m}\int \mathbf{F}(P,t)dt + \mathbf{c}_1, \tag{6.20}$$

in which $\mathbf{c}_1$ is a constant vector of integration.

A second integration with $d\mathbf{x} = \mathbf{v}dt$ gives the motion of P:

$$\mathbf{x}(P,t) = \int \mathbf{v}(P,t)dt + \mathbf{c}_2, \tag{6.21}$$

wherein $\mathbf{c}_2$ is another constant vector of integration. The constants $\mathbf{c}_1$ and $\mathbf{c}_2$ are fixed by the assigned initial data. The reader will notice that (6.20) and (6.21) are respectively equivalent to the kinematical equations (1.24) and (1.23). A typical example follows. (See also Example 1.7 in Volume 1.)

Example 6.7. A particle P in an inertial reference frame has an initial velocity $\mathbf{v}_0$ at the place $\mathbf{x}_0$, and subsequently moves under the influence of a force that is proportional to the time and acts in a fixed direction $\mathbf{e}$. Find the position and velocity of P at time t.

Solution. The force on P is given by $\mathbf{F}(P,t) = kt\mathbf{e}$, where k is a constant and $\mathbf{e}$ is a constant unit vector. Use of this relation in (6.1) and integration of the result as shown in (6.20) with the initial value $\mathbf{v}(P,0) = \mathbf{v}_0$ yields the velocity $\mathbf{v}(P,t) = kt^2/2m\mathbf{e} + \mathbf{v}_0$. With the initial value $\mathbf{x}(P,0) = \mathbf{x}_0$, a second integration described by (6.21) yields the motion $\mathbf{x}(P,t) = kt^3/6m\mathbf{e} + \mathbf{v}_0 t + \mathbf{x}_0$. Let the reader show that if P starts at the origin with velocity $\mathbf{v}_0 = v_0\mathbf{j}$ and the force acts in the direction $\mathbf{e} = \mathbf{i}$, the path of P is a cubic parabola $x = cy^3$. Identify the constant c. □

6.4.2. Motion under a Constant Force

In the special case when $\mathbf{F}(P,t) = \mathbf{F}_0$ is a constant force, the acceleration $\mathbf{a}(P,t) = \mathbf{F}_0/m$ is also a constant vector. Hence, (6.20) reduces to

$$\mathbf{v}(P,t) = \frac{\mathbf{F}_0}{m}t + \mathbf{v}_0 \tag{6.22}$$

with $\mathbf{c}_1 = \mathbf{v}(P,0) \equiv \mathbf{v}_0$. Integration of (6.22) in accordance with (6.21) and use of $\mathbf{c}_2 = \mathbf{x}(P,0) \equiv \mathbf{x}_0$ delivers the motion

$$\mathbf{x}(P,t) = \frac{\mathbf{F}_0}{2m}t^2 + \mathbf{v}_0 t + \mathbf{x}_0. \tag{6.23}$$

These elementary formulas are applied below to study projectile motion and the motion of a particle that falls from rest relative to the Earth. To simplify matters, the spin of the Earth and aerodynamic and atmospheric drag effects are neglected. Then the two problems are similar because they occur under the same constant gravitational force $\mathbf{F}_0 = \mathbf{W} = m\mathbf{g}$, while only the initial conditions are different. Any motion under gravity alone is called *free fall*.

6.4.2.1. *Galileo's Principle for Free Fall of a Particle*

The initial conditions in the free fall problem of a particle P released from rest at the origin are $\mathbf{v}_0 = \mathbf{0}$, $\mathbf{x}_0 = \mathbf{0}$, and (6.22) and (6.23) thus yield the familiar elementary *equations for the free fall motion, velocity, and acceleration of the particle*:

$$\mathbf{x}(P,t) = \frac{1}{2}\mathbf{g}t^2, \qquad \mathbf{v}(P,t) = \mathbf{g}t, \qquad \mathbf{a}(P,t) = \mathbf{g}. \tag{6.24}$$

The results (6.24) *are independent of the mass or any other property of the object, and hence, for the same circumstances, we learn that all bodies fall with the same speed along the plumb line of* $\mathbf{g}$. This is known as *Galileo's principle*. Accordingly, if two balls, one made of cast iron and the other of wood, were simultaneously released from the summit of the Leaning Tower of Pisa, an experiment *alleged*† to have been done in 1590 by the famous Italian scientist, Galileo Galilei (1564–1642), then together they would fall, and together they would strike the ground. Of course, common experience with feathers and stones contradicts this principle. But this happens because the physical attributes of the feather are not accurately modeled by the assumptions—specifically, the primary assumption of negligible air resistance which is plainly essential to our physical interpretation of the theoretical results. On the contrary, experiments conducted on bodies falling in a vacuum, including feathers and stones, lend support to Galileo's principle, which otherwise is especially altered by air resistance and to a lesser extent by the rotation of the Earth, effects that are investigated later.

6.4.2.2. *Motion of a Relativistic Particle under Constant Force*

Many elementary but interesting problems concern the motion of a particle when the total force is either a constant vector or an elementary function of time. It is not intended, however, that any of the foregoing formulas should be memorized. On the contrary, the examples serve to review procedures used often in Volume 1 to obtain solutions to similar problems by the easy method of separation of variables. While the same basic procedure may be applied to investigate the motion of a relativistic particle, for example, the formulas derived above cannot be used at all. This is illustrated next. Afterwards, the results are compared with those in (6.22) and (6.23) when $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{v}_0 = \mathbf{0}$.

† See Cooper's study described in the References.

Example 6.8. A relativistic particle P, initially at rest at the origin in frame ψ, is moving along a straight line under a constant force $\mathbf{F}_0$. Determine the relativistic speed and the distance traveled by P as functions of time.

Solution. The equation of motion for the relativistic particle is given by (5.34) in which $\mathbf{F}(P, t) = \mathbf{F}_0$ is a constant force and (6.9) is to be used. Hence, separation of the variables and integration of $\mathbf{F}_0 dt = d(m\mathbf{v}) = d(\gamma m_0 \mathbf{v})$, with the initial values $\mathbf{v}(P, 0) = \mathbf{0}$ and $\gamma = 1$, yields $m\mathbf{v} = \mathbf{F}_0 t$. Thus, recalling (6.9) and noting that $\mathbf{v} = v\mathbf{t}$ and $\mathbf{F}_0 = F_0\mathbf{t}$ are parallel vectors, we have only one nontrivial component equation: $m_0 v/(1 - v^2/c^2)^{1/2} = F_0 t$. This scalar equation yields the rectilinear, relativistic speed

$$v(P, t) = \frac{ckt}{\sqrt{1 + (kt)^2}} \qquad \text{with} \quad k \equiv \frac{F_0}{m_0 c}. \tag{6.25a}$$

Introducing $v = \dot{s}$ into (6.25a), separating the variables, and integrating $ds = v dt$ with the initial value $s(0) = 0$, we obtain the rectilinear distance traveled by P:

$$s(P, t) = \frac{c}{k}\left(\sqrt{1 + (kt)^2} - 1\right). \tag{6.25b}$$

Notice in (6.25a) that $v/c < 1$ for all t, and $v/c \to 1$ as $t \to \infty$; that is, *under a constant force, the relativistic particle speed cannot exceed the speed of light c.* This result is quite different from the corresponding speed $v = F_0 t/m_0$ described by (6.22) for a Newtonian particle of mass $m = m_0$ initially at rest and subject to a constant force F_0; in this case $v \to \infty$ with t. If $m_0 c$ is large compared with $F_0 t$ so that $kt \ll 1$, then (6.25a) and (6.25b) reduce approximately to

$$v(P, t) = ckt = \frac{F_0}{m_0} t, \qquad s(P, t) = \frac{1}{2} ckt^2 = \frac{F_0}{2m_0} t^2. \tag{6.25c}$$

These are the Newtonian formulas described by (6.22) and (6.23) for the corresponding rectilinear motion of a particle of mass m_0 initially at rest at the origin and acted upon by a constant force F_0. In the present relativistic approximation, however, these results are valid for only a *sufficiently small* time for which $v/c = kt \ll 1$. □

6.4.2.3. *Elements of Projectile Motion*

Equations (6.22) and (6.23) are applied next in two examples involving projectile motion and the simultaneous rectilinear or free fall motion of another target body. Afterwards, a fascinating technological application of a controlled projectile motion is studied. In addition to earlier assumptions, frictional effects are ignored.

Example 6.9. Percy Panther is snoozing in an open-top artillery truck when he senses the presence of the mischievous Arnold Aardvark lurking beneath. He

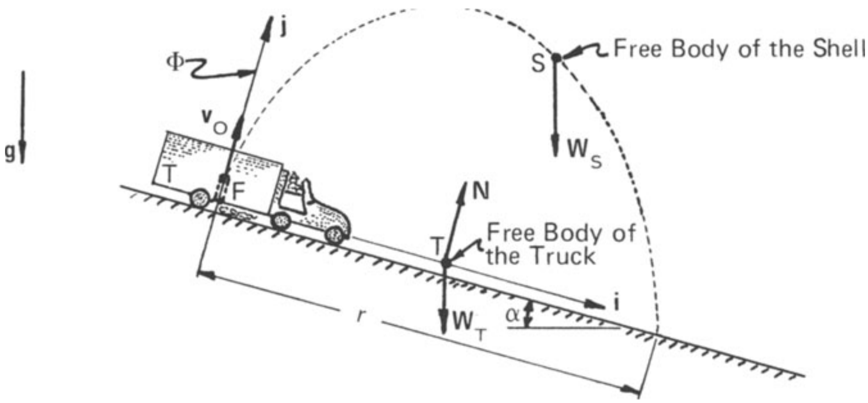


Figure 6.4. Projectile motion in an inertial reference frame without friction.

quietly releases the handbrake to escape down the hill inclined at an angle α. Arnold Aardvark having quietly rigged a remote trigger, immediately fires the gun, launching a shell of mass m straight up from the truck, as shown in Fig. 6.4. The gun has a muzzle velocity $\mathbf{v}_0$, and the total mass of the truck and its strange driver is M. Determine the time and the location at which the shell impacts the ground, and find the location of Percy Panther at that time.

Solution. First, we determine the motion of the shell S, whose free body diagram is shown in Fig. 6.4. The total force acting on S is its weight $\mathbf{W}_S = m\mathbf{g}$. Thus, in the inertial frame $\Phi = \{F; \mathbf{i}_k\}$ fixed in the ground, the constant force in (6.22) and (6.23) is $\mathbf{F}_0 = \mathbf{W}_S = mg(\sin\alpha\mathbf{i} - \cos\alpha\mathbf{j})$; and with $\mathbf{v}_0 = v_0\mathbf{j}$ and $\mathbf{x}_0 = \mathbf{0}$ initially, we obtain, in evident notation,

$$\mathbf{v}_S(t) = v_0\mathbf{j} + gt(\sin\alpha\mathbf{i} - \cos\alpha\mathbf{j}), \tag{6.26a}$$

$$\mathbf{x}_S(t) = \frac{1}{2}gt^2\sin\alpha\mathbf{i} + \left(v_0 t - \frac{1}{2}gt^2\cos\alpha\right)\mathbf{j}. \tag{6.26b}$$

Let the reader derive these results starting from (6.1), determine the maximum height reached by S, and show that its trajectory is a parabola.

The shell returns to the ground after a time t^* when $\mathbf{x}_S(t^*) = r\mathbf{i}$ in Fig. 6.4, and hence by (6.26b),

$$r = \frac{1}{2}gt^{*2}\sin\alpha, \qquad t^* = \frac{2v_0}{g\cos\alpha}. \tag{6.26c}$$

The results are independent of the mass or any other property of the shell. Elimination of t^* from the first of (6.26c) yields the impact range r in terms of the muzzle speed v_0 and the angle α that the gun makes with the vertical axis of $\mathbf{g}$:

$$r = \frac{2v_0^2 \tan\alpha}{g\cos\alpha}. \tag{6.26d}$$

Now consider the free body diagram of the truck in Fig. 6.4. The total force $\mathbf{F}_T$ acting on the truck is its total weight $\mathbf{W}_T$ and the normal surface reaction force $\mathbf{N}$. Without frictional effects, (6.1) becomes

$$\mathbf{F}_T = \mathbf{N} + \mathbf{W}_T = N\mathbf{j} + Mg(\sin\alpha\mathbf{i} - \cos\alpha\mathbf{j}) = M\mathbf{a}_T. \tag{6.26e}$$

Since the truck accelerates along the $\mathbf{i}$ direction, $N = Mg\cos\alpha$ and $\mathbf{a}_T = g\sin\alpha\mathbf{i}$. Hence, two easy integrations with $\mathbf{v}_0 = \mathbf{0}$ and $\mathbf{x}_0 = \mathbf{0}$ yield

$$\mathbf{v}_T(t) = gt\sin\alpha\mathbf{i}, \tag{6.26f}$$

$$\mathbf{x}_T(t) = \frac{1}{2}gt^2\sin\alpha\mathbf{i}. \tag{6.26g}$$

Comparison of the $\mathbf{i}$ components in (6.26b) and (6.26g) parallel to the truck's motion reveals that the shell at each instant is directly above the truck, now coasting toward the ultimate *surprise*! But a few tenths of a second before the impending catastrophe, Percy Panther spots the converging shell and slams on the brakes. The shell explodes violently in front of the truck, destroying it. Through the smoky haze, Arnold Aardvark spies the black, whisker-singed and disheveled driver crawling safely away to seek revenge another day. □

Example 6.10. Arnold Aardvark is sunbathing on a lookout platform at $\mathbf{x}_0 = a\mathbf{i} + b\mathbf{j}$ in the frame $\Phi = \{O; \mathbf{i}_k\}$ when he spots Percy Panther at O preparing to fire an artillery gun pointed directly toward the platform, as shown in Fig. 6.5. The gun has a muzzle velocity $\mathbf{v}_0$ and the tower is well within its range r. At the moment the gun is fired, Arnold Aardvark, sensing impending danger, grabs his umbrella, steps through a hole in the platform, and falls freely in pursuit of safety toward the ground. Determine the distance d that separates Arnold Aardvark and the shell at the instant t^* when it crosses his line of fall.

Solution. The free body diagrams of the shell S and Arnold Aardvark B are shown in Fig. 6.5, in which $\mathbf{W}_S = m_S\mathbf{g}$ and $\mathbf{W}_B = m_B\mathbf{g}$ denote their respective weights. Their free fall equations of motion, in evident notation, are

$$\mathbf{F}_B = m_B\mathbf{g} = m_B\mathbf{a}_B, \qquad \mathbf{F}_S = m_S\mathbf{g} = m_S\mathbf{a}_S. \tag{6.27a}$$

Therefore, B and S have the same constant, free fall acceleration, $\mathbf{a}_B = \mathbf{a}_S = \mathbf{g}$, but their respective initial conditions differ. Integration of this equation, i.e. $d\mathbf{v}_B = d\mathbf{v}_S$, with $\mathbf{v}_B(0) = \mathbf{0}$ and $\mathbf{v}_S(0) = \mathbf{v}_0$, the muzzle velocity of the gun, gives

$$\mathbf{v}_B = \mathbf{v}_S - \mathbf{v}_0 \quad \text{with} \quad \mathbf{v}_0 = v_0(\cos\beta\mathbf{i} + \sin\beta\mathbf{j}). \tag{6.27b}$$

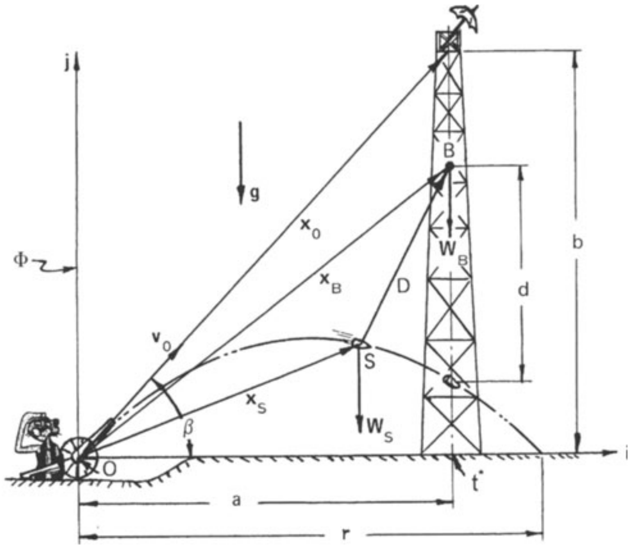


Figure 6.5. An unusual lesson on projectile motion.

A second integration with $\mathbf{x}_B(0) = \mathbf{x}_0$ and $\mathbf{x}_S(0) = \mathbf{0}$ yields the relative position vector $\mathbf{D} \equiv \mathbf{x}_B - \mathbf{x}_S$ of B from S at any time t:

$$\mathbf{D} = \mathbf{x}_0 - \mathbf{v}_0 t \qquad \text{with} \quad \mathbf{x}_0 = a\mathbf{i} + b\mathbf{j}. \tag{6.27c}$$

At the instant t^* when the shell crosses Arnold Aardvark's line of escape $x = a < r$, $\mathbf{D} = d\mathbf{j}$. Thus, with $\mathbf{v}_0$ given in (6.27b), (6.27c) yields $d\mathbf{j} = (a - v_0 t^* \cos\beta)\mathbf{i} + (b - v_0 t^* \sin\beta)\mathbf{j}$. The $\mathbf{i}$ component determines t^*, and the $\mathbf{j}$ component yields

$$d = b - a\tan\beta \tag{6.27d}$$

for the distance separating Arnold Aardvark and the unyielding shell at t^*. But Percy Panther had directed the gun on the line toward the platform with $\tan\beta = b/a$; so, Arnold Aardvark is headed straight toward an unpleasant *surprise* at the instant t^*! But a few moments before disaster strikes, he spies the approaching shell and quickly fixes the crook-handled umbrella to a tower beam, instantly arresting his fall. The shell explodes violently beneath him, destroying the tower. Arnold Aardvark, his snout scorched and twisted, escapes the assault with renewed mischief in mind.

So long as the tower is within the gun's range, the result is independent of the muzzle speed and of the masses of the objects involved; it depends only on the initial coordinates of B and the angle of elevation of the gun. Explain why Arnold Aardvark, living in a world where this solution is meaningful, was wise not to have used the umbrella as a parachute. □

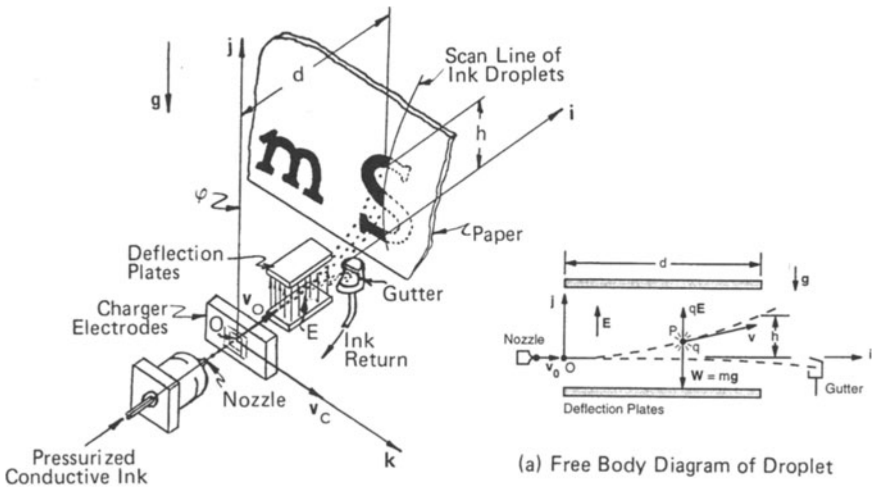


Figure 6.6. Schema of the IBM ink jet printing process. Copyright 1977 by International Business Machines Corporation; reprinted by permission.

6.4.2.4. Ink Jet Printing Technology

The same projectile ideas together with the basic law (6.15) for the electric force on a charged particle have a fascinating application in ink jet printing technology. An ink jet printer, illustrated schematically in Fig. 6.6, produces an image from tiny, charged spherical droplets of electrically conductive ink fired from a drop generating nozzle, approximately $1/1000$ in. diameter, at the rate of 117,000 drops per second. The conductive droplets pass between charging electrodes where they are selectively charged electrostatically by command from programmed electronic control circuits that describe the image characters in terms of charge–no charge language. Moving at roughly 40 mph initially, the charged droplets pass through a constant electric field that directs them onto the paper. As vertical scanning occurs, an electromechanical control mechanism moves the printer carriage parallel to the paper at a constant speed of 7.7 in./sec. In this way, the ink jet printer quietly composes characters of high quality at a rate of about 80 characters per second, a full line of type across a standard page in about 1 sec. Of course, these operating rates will vary with printer design and evolving technology.

To understand its fundamental working principle, we shall determine the relative motion of a droplet P of mass m and charge q having an initial velocity $\mathbf{v}_0$ relative to the printer carriage. Since the carriage has a uniform velocity $\mathbf{v}_C$, as indicated in Fig. 6.6, the reference frame $\varphi = \{O; \mathbf{i}_k\}$ fixed in the charger at O is an inertial frame in which Newton's law may be applied. For simplicity, aerodynamic drag and wake effects, and the influence of electric repulsive forces between the charged droplets are neglected. Then, as shown in the free body diagram in Fig. 6.6a, the total force $\mathbf{F}(P,t) = \mathbf{F}_e + \mathbf{W}$ acting on a drop P is due to its weight $\mathbf{W} = -mg\mathbf{j}$ and the constant applied electric force $\mathbf{F}_e = q\mathbf{E} = qE\mathbf{j}$. Hence, $\mathbf{F}(P,t) = (qE - mg)\mathbf{j}$ is a constant force. From (6.1) and the initial condition $\mathbf{v}_0 = v_0\mathbf{i}$, we obtain the velocity of the drop relative to the printer carriage whose constant velocity is $\mathbf{v}_C = v_C\mathbf{k}$:

$$\mathbf{v}(P,t) = v_0\mathbf{i} + (cE - g)t\mathbf{j} \qquad \text{with} \quad c \equiv q/m. \tag{6.28a}$$

With $\mathbf{x}_0 = \mathbf{0}$ initially, integration of (6.28a) yields the motion of a droplet relative to the printer carriage:

$$\mathbf{x}(P,t) = v_0 t\mathbf{i} + \frac{1}{2}(cE - g)t^2\mathbf{j}. \tag{6.28b}$$

Hence, the path of the droplet relative to the carriage is a parabola

$$y(x) = \frac{1}{2v_0^2}(cE - g)x^2. \tag{6.28c}$$

Let us imagine for simplicity that the deflection plates of length d extend from the origin at the charger to the paper surface, as suggested in Fig. 6.6. Then (6.28c) holds for $0 \le x \le d$. (See Problem 6.22.) Therefore, at $x = d$, the droplet deflection or scan height $h \equiv y(d)$ at the paper surface is determined by

$$h = \frac{d^2}{2v_0^2}(cE - g). \tag{6.28d}$$

The result (6.28d) shows that when an electrostatically charged drop enters the uniform electric field, the electric force alters its free fall trajectory and deflects it vertically by an amount proportional to its charge. An uncharged drop is collected in a gutter that returns the unused ink to its reservoir as shown in Fig. 6.6. A charged drop impacts the paper. Alphabetic or any other characters, shown schematically in Fig. 6.6, are formed by directing the ink dots onto the paper in patterns determined by the printer electronics. The decision to charge or not to charge is made automatically 117,000 times each second. The formula (6.28d) shows that the character height is inversely proportional to the square of the stream speed v_0 which is controlled by the pump pressure. The printer controls the character height automatically by its pump control circuit. In this way, the ink jet printer is able to rapidly generate various characters of high quality. Some interesting style effects may be produced by varying the carriage rate.

A remarkable stroboscopic microphotograph of droplets of ink emerging from an ink jet printer is reproduced[‡] in Fig. 6.7. A jet of ink that originated in the drop generator to the right has dissociated into spherical droplets. The lower line of drops

[‡] This extraordinary photograph by Mr. Carl Lindberg was adapted from the color photograph on the cover of the Number 1 issue of the 1977 IBM Journal of Research and Development. Copyright 1977 by International Business Machines Corporation; reprinted by permission. In Fig. 6.7, however, the intensity of the droplets has been enhanced for greater clarity.

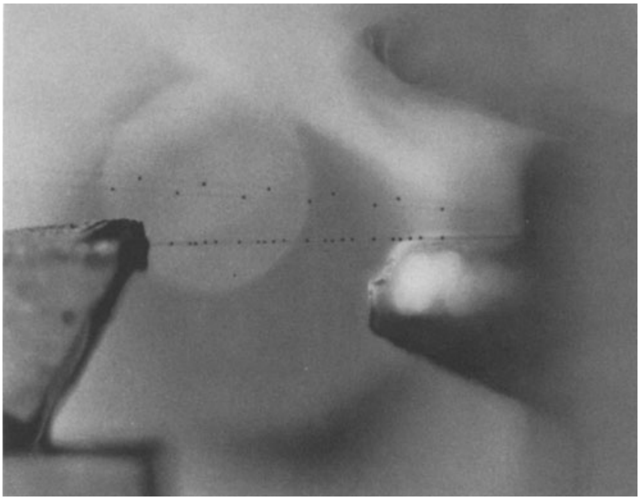


Figure 6.7. Stroboscopic microphotograph of ink drops in a jet printer. Copyright 1977 by International Business Machines Corporation; reprinted by permission.

were not charged, so these are moving toward the ink gutter to the left. The larger gaps between these uncharged drops are the vacated positions formerly occupied by the field deflected, charged drops that are traveling on the trajectories above.

The same ink jet technique was first applied in a similar way in the construction of a strip chart recorder, a high speed device for recording rapidly changing electrical signals on a moving paper chart. The disintegrating fluid jet concept has found other applications that include the sorting of cells in blood samples and the atomization of fuels for combustion. The deflection of a charged particle by an electric field also is used to control the motion of an electron stream in an oscilloscope and to produce images on a television screen or a computer monitor. Technological advances in electronic imaging, however, have led to the replacement of cathode ray tube devices by liquid crystal and high resolution plasma display systems whose basic operating principles are altogether different, and far more complex. The practical use of liquid crystal technology, for example, is evident in its increasingly diverse applications to computer and television screens, computer games, digital cameras, calculators, cellular phones, digital clocks and watches, microwave ovens, and a great host of other consumer and military electronic products.

6.5. Motion under Velocity Dependent Force

So far, complications due to air resistance have been ignored. Realistically, however, a projectile experiences atmospheric drag forces that slow it down and alter its trajectory. The same is true of an aircraft, a sky diver, and a raindrop; and water behaves similarly to retard the motion of swimmers, water skiers, and ships. Experience in such situations shows that the retarding force varies with the speed of the body.

For objects moving slowly through the air, the resistance is roughly proportional to the speed; but this simple rule breaks down at speeds typical of low velocity projectiles for which the air resistance varies roughly with the square of the speed. For an aircraft or a rocket whose velocity may approach the speed of sound, the drag force increases in proportion to some higher power of the speed, and so on. The retarding force is also a function of the density of air and hence varies with the altitude. Of course, aerodynamic design plays an important role too. These complications aside, we may gain physical insight into the nature of air and water resistance by study of special, ideal models.

6.5.1. Stokes's Law of Resistance

The simplest model used to study the nature of phenomena arising from drag effects of air and water on an object moving at low speeds is described by *Stokes's law*: *The drag force* $\mathbf{F}_D$ *on a particle is oppositely directed and proportional to its velocity* $\mathbf{v}$:

$$\mathbf{F}_D = -c\mathbf{v}. \tag{6.29}$$

The constant $c > 0$ is called the *drag* or *damping coefficient*. This model is applied later to investigate the motion of a projectile and of a particle falling with air resistance. First, however, we formulate the problem for a more general model for which the drag force is an unspecified function of the speed.

6.5.2. Formulation of the Resistance Problem

Figure 6.8 shows a particle P moving in the vertical plane of frame $\varphi = \{O; \mathbf{i}_k\}$, under a total force $\mathbf{F}(P, t) = \mathbf{W} + \mathbf{F}_D$ consisting of its weight $\mathbf{W} = m\mathbf{g}$

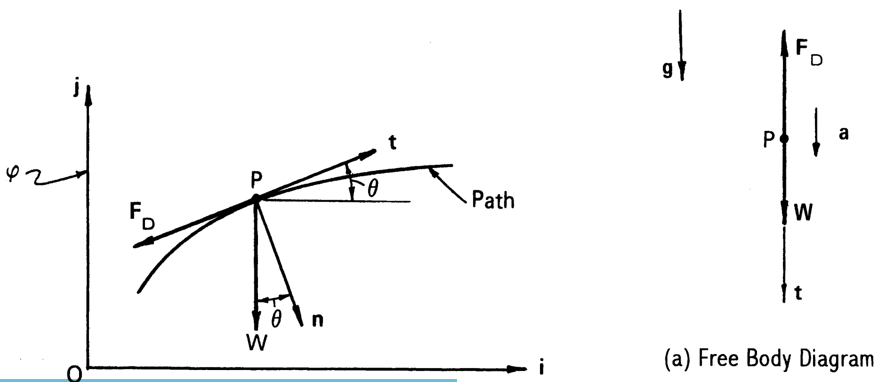


Figure 6.8. Motion of a particle under a drag force $\mathbf{F}_D$.

and the drag force $\mathbf{F}_D \equiv -R(v)\mathbf{t}$, where $R(v)$ is an unspecified, positive-valued function of the particle speed v. The equation of motion, by (6.1), is

$$m\mathbf{g} - R(v)\mathbf{t} = m\mathbf{a}(P, t). \tag{6.30}$$

Two cases are considered—rectilinear motion and plane motion.

6.5.2.1. Rectilinear Motion with Resistance

Let us consider a vertical rectilinear motion in the direction of $\mathbf{g} = g\mathbf{t}$ in Fig. 6.8a. Then with $\mathbf{a}(P, t) = \dot{v}\mathbf{t}$, (6.30) becomes

$$\dot{v} = g - \frac{R(v)}{m} \equiv F(v). \tag{6.31}$$

Integration of (6.31) yields the travel time as a function of the particle velocity in the resisting medium,

$$t = \int \frac{dv}{F(v)} + c_0, \tag{6.32}$$

where c_0 is a constant. Theoretically, this equation will yield $v(t) = ds/dt$ which may be solved to find the distance $s(t)$ traveled in time t. Alternatively, using $\dot{v} = v\,dv/ds$ in (6.31), we find the distance traveled as a function of the speed,

$$s = \int \frac{v\,dv}{F(v)} + c_1, \tag{6.33}$$

in which c_1 is another constant of integration. In principle, the integrals in (6.32) and (6.33) can be computed when the resistance function $R(v)$ is specified in (6.31). The following example illustrates these ideas for Stokes's linear rule (6.29).

Example 6.11. *Falling body with air resistance.* A particle of mass m, a raindrop for example, falls from rest through the atmosphere. Neglect the Earth's motion, wind effects, and the buoyant force of air, and adopt Stokes's law to model the air resistance. Determine as functions of time the rectilinear speed and the distance traveled by the particle.

Solution. The solution may be read from the foregoing results in which the drag force is modeled by Stokes's law (6.29) so that $\mathbf{F}_D = -R(v)\mathbf{t} = -\,cv\mathbf{t}$. Hence, use of $R(v) = cv$ in (6.31) gives

$$\frac{dv}{dt} = g - \nu v \equiv F(v) \qquad \text{with} \quad \nu \equiv \frac{c}{m}. \tag{6.34a}$$

For the initial condition $v(0) = 0$, we find by (6.32)

$$t = \int_0^v \frac{dv}{g - \nu v} = -\frac{1}{\nu}\log\left(1 - \frac{\nu v}{g}\right);$$

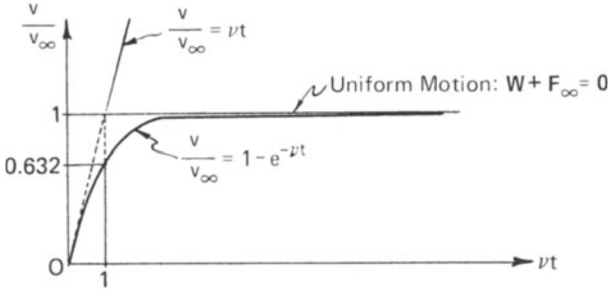


Figure 6.9. Graph of the normalized speed versus the normalized time for the vertical motion of a particle falling with resistance proportional to its speed.

so, the rectilinear speed of the particle in its fall from rest is

$$v(t) = v_\infty(1 - e^{-\nu t}) \qquad \text{with} \quad v_\infty \equiv \frac{g}{\nu}. \tag{6.34b}$$

In consequence, as $t \to \infty$, the particle speed v approaches a constant value $v_\infty \equiv g/\nu = W/c$, named *the terminal speed.* When the particle achieves its terminal speed, its weight is balanced by the drag force so that $cv_\infty = W$, and the particle continues to fall without further acceleration.

These facts are illustrated in Fig. 6.9. Equation (6.34b) shows that the rate at which $v(t)$ changes is governed by the coefficient of *dynamic viscosity* $\nu = c/m$, which has the physical dimensions $[\nu] = [F/MV] = [T^{-1}]$. Thus, at the instant $t = \nu^{-1}$, by (6.34b), $v(\nu^{-1}) = v_\infty(1 - e^{-1}) \approx 0.632 v_\infty$. Therefore, the speed reaches 63.2% of the terminal speed in the time $t = \nu^{-1}$, called the *retardation time*. The straight line of slope 1 in Fig. 6.9 shows that this also is the time at which the speed would reach the terminal value if it had continued to change at its initial constant rate $a(0) = g$, without air resistance. As the particle's speed approaches the terminal speed of its ultimate uniform motion shown by the horizontal asymptote, the weight $\mathbf{W} = W\mathbf{t}$ is balanced by the drag force $\mathbf{F}_D \to \mathbf{F}_\infty = -cv_\infty\mathbf{t}$.

Finally, with $v = ds/dt$ and the initial condition $s(0) = 0$, (6.34b) yields the distance through which the particle falls in time t:

$$s(t) = v_\infty \int_0^t (1 - e^{-\nu t})dt = v_\infty t - \frac{v_\infty}{\nu}(1 - e^{-\nu t}). \tag{6.34c}$$

Hence, the distance traveled in the retardation time interval is $s(1/\nu) = v_\infty/(\nu e) \approx 0.368 v_\infty/\nu$. The result (6.34c) also may be read from (6.33).

The reader may verify that in the absence of air resistance when $\nu \to 0$ the limit solutions of (6.34b) and (6.34c) are the elementary solutions (6.24). Now consider the case when the viscosity ν is small. First, recall the power series expansion of e^z about $z = 0$:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots. \tag{6.34d}$$

Then use of (6.34d) in (6.34b) and (6.34c) yields, to the first order in ν, an approximate solution for the case of small air resistance:

$$v(t) = gt\left(1 - \frac{\nu}{2}t\right), \qquad s(t) = \frac{g}{2}t^2\left(1 - \frac{\nu}{3}t\right). \tag{6.34e}$$

When $\nu \to 0$, we again recover (6.24) for which air resistance is absent. □

6.5.2.2. *Plane Motion with Resistance*

Now, let us consider the plane motion of a particle in frame $\varphi = \{O; \mathbf{i}, \mathbf{j}\}$, as shown in Fig. 6.8. With $\mathbf{t} = \mathbf{v}/v = \dot{x}/v\mathbf{i} + \dot{y}/v\mathbf{j}$ and $\mathbf{g} = -g\mathbf{j}$ in (6.30), the component equation (6.2) yields

$$\ddot{x} = -\frac{R(v)}{mv}\dot{x}, \qquad \ddot{y} = -g - \frac{R(v)}{mv}\dot{y}. \tag{6.35}$$

These equations are difficult to handle in this general form. For resistance governed by Stokes's law (6.29), however, the ratio $R(v)/mv = c/m$ is constant; and (6.35) simplifies to

$$\ddot{x} = -\nu\dot{x}, \qquad \ddot{y} = -g - \nu\dot{y} \qquad \text{with} \quad \nu \equiv \frac{c}{m}. \tag{6.36}$$

Example 6.12. *Projectile motion with air resistance*. A projectile S of mass m is fired from a gun with muzzle speed v_0 at an angle β with the horizontal plane. Neglect the Earth's motion and wind effects and assume that air resistance is governed by Stokes's law. Determine the projectile's motion as a function of time.

Solution. The equations of motion with air resistance governed by Stokes's law are given in (6.36). To find the motion $\mathbf{x}(S, t)$, we first integrate the system (6.36) to obtain $\mathbf{v}(S, t)$. Use of the initial condition $\mathbf{v}_0 = v_0(\cos\beta\mathbf{i} + \sin\beta\mathbf{j})$ yields

$$\int_{v_0\cos\beta}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -\nu t, \qquad \int_{v_0\sin\beta}^{\dot{y}} \frac{d\dot{y}}{g + \nu\dot{y}} = -t.$$

These deliver the projectile's velocity components as functions of time:

$$\dot{x} = (v_0\cos\beta)e^{-\nu t}, \qquad \dot{y} = -\frac{g}{\nu} + \left(v_0\sin\beta + \frac{g}{\nu}\right)e^{-\nu t}. \tag{6.37a}$$

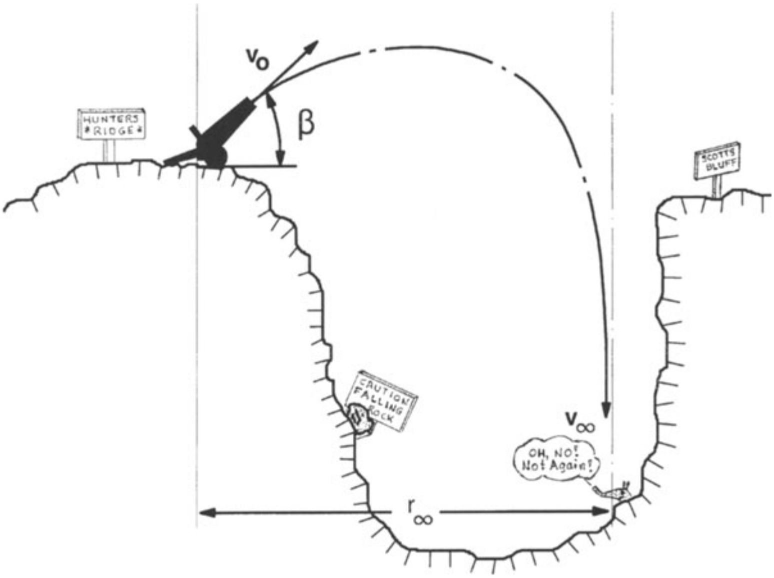


Figure 6.10. Projectile motion with air resistance.

Then integration of (6.37a) with the initial condition $\mathbf{x}_0 = \mathbf{0}$ yields the motion of the projectile as a function of time:

$$x(t) = \frac{v_0 \cos\beta}{\nu}(1 - e^{-\nu t}), \tag{6.37b}$$

$$y(t) = -\frac{g}{\nu}t + \frac{1}{\nu}\left(v_0 \sin\beta + \frac{g}{\nu}\right)(1 - e^{-\nu t}). \tag{6.37c}$$

Let us imagine that the projectile is fired from a hilltop into a wide ravine, as shown in Fig. 6.10. Then, as $t \to \infty$, in the absence of impact, (6.37a) gives $\dot{x} \to 0$ and $\dot{y} \to -g/\nu$. Hence, the projectile attains the terminal speed $v_\infty = g/\nu$ at which its weight is balanced by air resistance; and (6.37b) and (6.37c) show that the projectile approaches asymptotically, the vertical range line at $r_\infty \equiv \lim_{t\to\infty} x(t) = (v_0 \cos\beta)/\nu$ in Fig. 6.10. In the absence of air resistance, the range for the same situation would grow indefinitely with the width of the ravine. The simple Stokes model thus provides a more realistic picture of projectile motion with air resistance that limits its range. □

6.5.2.3. *The Millikan Oil Drop Experiment*

When oil is sprayed in fine droplets from an atomizer, the droplets become electrostatically charged, presumably due to frictional effects. The charge is usually negative, which means that the drops have acquired one or more excess electrons.

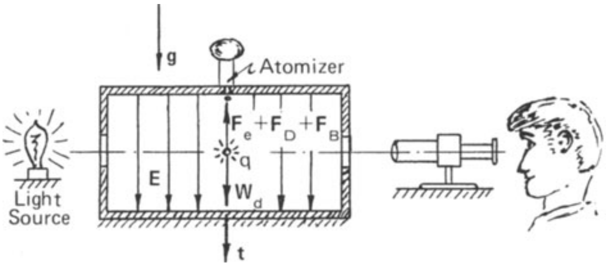


Figure 6.11. Schematic of the Millikan oil drop experiment.

This fact was exploited in 1909 by the famous American physicist Robert A. Millikan in a classic experiment designed to measure accurately the charge of an individual electron. Millikan's experimental method, its relation to our study of air resistance, and his remarkable result§ are discussed next.

A schematic of the oil drop test is shown in Fig. 6.11. Charged oil droplets, about a thousandth of a millimeter in diameter, are ejected from an atomizer at the top of the apparatus. A few drops escape through a small hole into an illuminated electric field $\mathbf{E}$ directed as shown. A lighted drop is seen in a telescope as a tiny, bright particle of mass m and negative electric charge $-q$ falling slowly under the influence of its weight $\mathbf{W}_d$, the electric force $\mathbf{F}_e$, the drag force $\mathbf{F}_D$, and the buoyant force $\mathbf{F}_B$ of the air, as shown in the free body drawing in Fig. 6.11; so, the total force on the droplet is $\mathbf{F}(P, t) = \mathbf{F}_e + \mathbf{F}_D + \mathbf{F}_B + \mathbf{W}_d$. The use of oil eliminates effects due to fluid evaporation, so only the drag force varies with time. Independent tests confirmed that the charge on the drops does not affect the air resistance to its motion, and because the particle's rate of fall is small, Stokes's law of resistance is applicable.

The intensity of the electric field, hence the electric force $\mathbf{F}_e = -q\mathbf{E}$ on a negatively charged drop, is adjusted until the droplet becomes stationary, spatially suspended in equilibrium between the field plates. In this case, $\mathbf{F}_D = \mathbf{0}$ and the equilibrium equation yields

$$\mathbf{F}_e + \mathbf{W}_d + \mathbf{F}_B = -q\mathbf{E} + \mathbf{W} = \mathbf{0}. \tag{6.38a}$$

The effective weight $\mathbf{W} \equiv \mathbf{W}_d + \mathbf{F}_B$ of a droplet in air depends on the mass density

§ In 1923, Robert A. Millikan was awarded the Nobel Prize for physics, principally for his work identifying precisely the unit of electric charge. Nearly 25 years subsequent to his death in 1953, however, he was strongly criticized for his treatment of students and others, and for his mishandling of the data. See the balanced account by D. Goodstein (among the references under Millikan) for the rest of the story. The importance that Millikan placed on his amended form of Stokes's law is underscored in this article. Also, it should be mentioned that besides frictional effects that induce negative charges on the droplets, the electric arc lamp is a source of ionization radiation of the space between the horizontal capacitor plates that also induces positive charges on an atom of an oil droplet, so the droplets are sometimes referred to as ions.

of the oil and of the sealed air as well as on the size of the drop. The mass densities are known, but the diameters of the drops are too small to be accurately measured directly with the telescope. Millikan devised an ingenious, indirect method for finding the size of the drops.

When the electric field is turned off, the drop accelerates in its fall until its terminal speed is reached. This happens when the drag force given by Stokes's rule balances the effective weight of the drop so that the particle is in a state of relative equilibrium at its constant terminal speed. Thus, the equation of the uniform motion yields

$$\mathbf{F}_\infty + \mathbf{W} = \mathbf{0}, \tag{6.38b}$$

in which $\mathbf{F}_\infty = -cv_\infty \mathbf{t}$ is the air resistance at the terminal speed v_∞. By timing the distance traveled at the constant slow rate of fall of the drop, Millikan measured the terminal speed and applied the result (6.38b) to compute the droplet size. Then the drag coefficient c, which depends on the size of the drop and the known viscosity of air, could be evaluated by a separate formula derived by Stokes from hydrodynamic theory. But Millikan found that Stokes's formula, due to the small size of the drops compared with the mean free path of a gas molecule, was inaccurate, and he provided an empirical correction to account for the discrepancy. With this adjustment in mind, c and v_∞ may be considered known. Thus, in effect, the charge on the drop is determined by eliminating $\mathbf{W}$ between (6.38a) and (6.38b) to obtain $-q\mathbf{E} = \mathbf{F}_\infty$. Clearly, the error in Stokes's formula for the calculation of c does not affect the basic linear nature of the rule (6.29), and hence the droplet charge is determined by

$$q = \frac{cv_\infty}{E}. \tag{6.38c}$$

Millikan and his co-workers found in many measurements the remarkable result that *every droplet had a charge q equal to an integral multiple of a number $e = 1.6019 \times 10^{-19}$ coulomb, the basic amount of negative charge of one electron. Thus, Millikan's conclusive experimental result that*

$$q = ne, \qquad n = 1, 2, 3, \ldots, \tag{6.38d}$$

showed that electric charge exists in nature only in integral units of magnitude e.

The procedure to obtain the data on one particular droplet sometimes took hours. At times, when interrupted while working on a drop, Millikan would put it into balance with the field and leave it. On one occasion he went home to dinner and returned after more than an hour to find the droplet only slightly displaced from where he had left it. At another time, Millikan realized he would not finish his experiment in time to attend dinner at home with invited guests, so he phoned Mrs. Millikan to explain that "I have watched an ion for an hour and a half and have to finish the job," but insisted that she and their guests go ahead with dinner. He learned later that Mrs. Millikan advised their guests that Robert would be delayed because he had "washed and ironed for an hour and a half and had to finish the job."

Measurement of e had been done earlier, but never with the accuracy achieved by Millikan's suspended oil drop test. He was studying the fundamental building block out of which, it is now believed, all electrical charges in the universe are composed, always in integral multiples of the basic unit electron charge e. The entire basis for the measurement of its magnitude rested on application of Stokes's law to the terminal speed of spherical droplets of oil in air. The apparatus was a device for catching and essentially seeing an individual electron riding on a drop of oil. Millikan recalled later in his autobiography this exciting observational experience: "He who has seen that experiment has in effect seen the electron."

Additional examples of particle motion with air resistance are provided in Problems 6.23 through 6.27. We continue with a new topic.

6.6. An Important Differential Equation

Many physical systems are governed by the second order differential equation

$$\ddot{u}(t) + r^2 u(t) = h(t), \tag{6.39}$$

for a scalar function $u(t)$. Herein r is a real or complex constant and $h(t)$ is a specified function of the independent variable t. We are going to encounter lots of applications in which one or more of the scalar equations of motion are of the type (6.39); so it is most helpful to understand the physical nature of its solution in general terms.

The solution of (6.39) when $r = 0$ describes a motion under a time varying force. This case was studied in Section 6.4.1, page 109; therefore, we shall assume that $r \neq 0$. In the general case, we recall from the theory of differential equations that the solution of (6.39) is given by the sum

$$u(t) = u_H(t) + u_P(t), \tag{6.40}$$

in which $u_H(t)$, called the *homogeneous solution*, is the general solution of the *related homogeneous equation*

$$\ddot{u}_H + r^2 u_H = 0, \tag{6.41}$$

and $u_P(t)$ is a *particular solution* that satisfies (6.39):

$$\ddot{u}_P + r^2 u_P = h(t). \tag{6.42}$$

6.6.1. General Solution of the Homogeneous Equation

The general solution of the homogeneous equation is obtained by consideration of a trial function $u_T = Ce^{\lambda t}$ in which λ and C are constants. This function satisfies (6.41) for each root of the *characteristic equation* $\lambda^2 + r^2 = 0$, namely, $\lambda = \pm ir$ in which $i = \sqrt{-1}$, so both $u_T(t) = C_1 e^{irt}$ and $u_T(t) = C_2 e^{-irt}$ are solutions of (6.41). Hence, the general solution of the homogeneous equation that contains two arbitrary integration constants C_1 and C_2 is given by the sum of these independent solutions:

$$u_H(t) = C_1 e^{irt} + C_2 e^{-irt}. \tag{6.43}$$

The homogeneous solution (6.43) is also known as the *complementary function*.

6.6.2. Particular Solution of the General Equation

The hardest part of our problem is to find a particular solution of (6.42) for a given function $h(t)$. Standard methods are available that may be applied to find one. The *method of variation of parameters*, for example, is a powerful procedure applicable to equations with variable or constant coefficients, but the complementary function must be known in advance. This presents no difficulty in the present problem for which it can be shown that this method leads to the following general relation for a particular solution of (6.42):

$$u_P(t) = \int^t \frac{h(\tau)}{\lambda_1 - \lambda_2} \left[e^{\lambda_1 (t-\tau)} - e^{\lambda_2 (t-\tau)} \right] d\tau, \tag{6.44}$$

wherein $\lambda = (\lambda_1, \lambda_2)$ are distinct roots of the characteristic equation. In the present case $\lambda = \pm ir$ yields $\lambda_2 = -\lambda_1 = -ir$. In evaluation of the indefinite integral in (6.44) arbitrary constants are omitted; they have no importance in the particular solution. The solution (6.44) also may be verified by its substitution into (6.42). (See Problems 6.28 and 6.29.) In many problems of physical interest, use of the formal relation (6.44) to compute the particular solution may be avoided. For the kinds of problems we shall encounter ahead, it is much easier to generate a particular solution on an ad hoc basis.

Example 6.13. Let us consider a particular solution for the case when $h(t)$ is a linear function of t, namely,

$$h(t) = c + bt, \tag{6.45a}$$

for constants b and c. Then because $\ddot{h}(t) = 0$, we see that a particular solution that satisfies (6.42) is

$$u_P(t) = \frac{h(t)}{r^2} = r^{-2}(c + bt). \tag{6.45b}$$

In this instance $\ddot{u}_P = 0$. Indeed, a particular solution of (6.42) has the property $\ddot{u}_P(t) = 0$ if and only if $u_P(t)$ is a linear function like (6.45b), and hence when and only when $h(t)$ is the linear function (6.45a). Therefore, in accordance with (6.40), the general solution of (6.39) for this case is given by the sum of

(6.43) and (6.45b):

$$u(t) = C_1 e^{irt} + C_2 e^{-irt} + r^{-2}(c + bt). \tag{6.45c}$$

The solution for other special functions $h(t)$ will be considered as the need arises.

6.6.3. Summary of the General Solution

In summary, we find with (6.43), (6.44), and (6.40) that the general solution of the differential equation (6.39) may be written as

$$u(t) = C_1 e^{irt} + C_2 e^{-irt} + u_P(t), \tag{6.46}$$

where the particular solution is defined formally by

$$u_P(t) = \int^t \frac{h(\tau)}{2ir} \left[e^{ir(t-\tau)} - e^{-ir(t-\tau)} \right] d\tau. \tag{6.47}$$

This is a convenient means of representing a particular solution of (6.39) for an arbitrary smooth function $h(t)$. Remember, however, that in many cases of practical interest, depending on the nature of $h(t)$, a particular solution of (6.39) may be obtained by simpler ad hoc methods.

6.6.4. Physical Character of the Solution

Now let us consider two important cases of physical interest. In the first instance we suppose that $r = p$ is a real constant so that $r^2 = p^2 > 0$. This leads to a trigonometric type solution. In the second case, we take $r = iq$, a pure complex constant, so that $r^2 = -q^2 < 0$. This leads to an exponential type solution which is then expressed in terms of hyperbolic functions. As a consequence, the physical nature of these two classes of solutions of (6.39) is quite different. (See Problem 6.33.)

6.6.4.1. Trigonometric Solution: $r = p$, a real constant

Equation (6.39) for this case becomes

$$\ddot{u}(t) + p^2 u(t) = h(t), \ p \text{ real}. \tag{6.48}$$

Of course, the general solution of this equation has precisely the form (6.46) with r replaced by p. But the complex exponential solution, convenient in some problems, suffers the undesirable disadvantage that the constants C_1 and C_2 are complex quantities. It proves more convenient, therefore, to transform this solution to its trigonometric form by use of *Euler's identity*

$$e^{\pm ipt} = \cos pt \pm i \sin pt. \tag{6.49}$$

Then, with $r = p$, the general solution (6.43) of the homogeneous equation (6.41), namely,

$$\ddot{u}_H + p^2 u_H = 0,\ p \text{ real}, \tag{6.50}$$

may be written as

$$u_H(t) = A \sin pt + B \cos pt, \tag{6.51}$$

wherein A and B are two real constants of integration. Thus, the complementary function has an oscillatory character typical of trigonometric functions.

Substitution of (6.49) into (6.47), with $r = p$, leads to the following expression for the particular solution of (6.48):

$$u_P(t) = \int^t \frac{h(\tau)}{p} \sin p(t - \tau) d\tau. \tag{6.52}$$

The general solution of (6.48) is the sum of (6.51) and (6.52). Formally,

$$u(t) = A \sin pt + B \cos pt + u_P(t). \tag{6.53}$$

The trigonometric functions in (6.51) and (6.53) have well-known periodic behavior whose physical relevance is discussed further in applications ahead.

Exercise 6.2. Let $C_1 = a_1 + ib_1$, $C_2 = a_2 + ib_2$, and set $r = p$ in (6.43). Use Euler's identity and show that the homogeneous solution (6.43) is real-valued when and only when $b_1 + b_2 = 0$ and $a_1 - a_2 = 0$. Determine in these terms the real constants in (6.51). □

6.6.4.2. Hyperbolic Solution: $r = iq$, *a complex constant*

The equation (6.39) for this case becomes

$$\ddot{u}(t) - q^2 u(t) = h(t),\ q \text{ real}. \tag{6.54}$$

It is important to recognize that *the principal difference between* (6.54) *and* (6.48) *is merely the sign of the second term.* This results in significantly different kinds of solutions. The general solution of (6.54) is given by (6.46) with r replaced by iq. We thus obtain

$$u(t) = C_1 e^{-qt} + C_2 e^{qt} + \int^t \frac{h(\tau)}{2q} \left(e^{q(t-\tau)} - e^{-q(t-\tau)}\right) d\tau. \tag{6.55}$$

Notice that the homogeneous solution, the first two terms in (6.55), has an exponential character. Unlike the oscillatory solution (6.51), this exponential solution grows increasingly large with t. Hence, plainly, equations (6.48) and (6.54) will describe totally distinct kinds of physical effects.

It is useful to observe that hyperbolic functions may be introduced to express the solution by formulas analogous to those used in the trigonometric case. For comparison, the results are presented in order parallel to the trigonometric formulas (6.49)–(6.53).

Representation in terms of hyperbolic functions. The hyperbolic sine and cosine functions are defined by

$$\sinh z = \frac{1}{2}(e^z - e^{-z}), \qquad \cosh z = \frac{1}{2}(e^z + e^{-z}). \tag{6.56}$$

These equations may be solved to obtain the exponential functions e^z and e^{-z}:

$$e^{\pm z} = \cosh z \pm \sinh z. \tag{6.57}$$

This is similar to (6.49). Then, with $r = iq$, the general solution (6.43) of the homogeneous equation associated with (6.54), namely,

$$\ddot{u}_H - q^2 u_H = 0, q \text{ real}, \tag{6.58}$$

may be written as

$$u_H(t) = A \sinh qt + B \cosh qt, \tag{6.59}$$

wherein A and B are two real constants of integration. Use of the first of (6.56) in (6.47) when $r = iq$ yields the following formula for a particular solution of (6.54):

$$u_P(t) = \int^t \frac{h(\tau)}{q} \sinh q(t-\tau)\, d\tau. \tag{6.60}$$

The general solution of (6.54), given by (6.55), is the sum of (6.59) and (6.60). Formally,

$$u(t) = A \sinh qt + B \cosh qt + u_P(t). \tag{6.61}$$

This completes the parallel representation of results (6.58)–(6.61) which are to be compared with the corresponding equations (6.50)–(6.53) for the trigonometric solution. Although the forms of solutions (6.53) and (6.61) are similar, it is evident that their physical nature is quite different. The trigonometric functions in (6.53) are periodic, they recur over and over again. But, as seen by (6.56), the hyperbolic functions in (6.61) grow indefinitely with the variable t. The graphs and some additional basic properties of the hyperbolic functions follow.

Further properties of the hyperbolic functions. Graphs of the functions (6.56) and some basic properties of the hyperbolic functions provide a helpful picture of their growth behavior. To start with, differentiation of (6.56) shows that

$$\frac{d}{dz}(\sinh z) = \cosh z, \qquad \frac{d}{dz}(\cosh z) = \sinh z. \tag{6.62}$$

We thus see an important difference in the derivatives of the hyperbolic functions

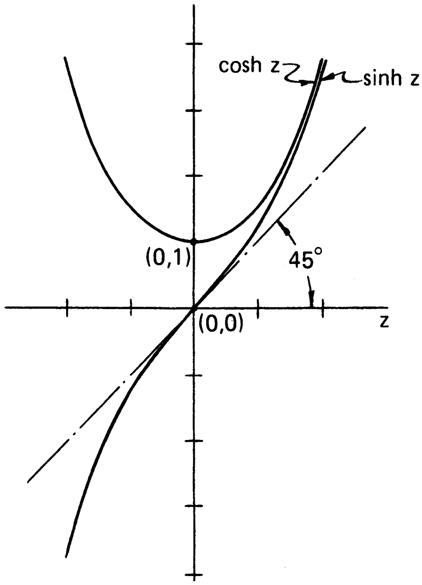


Figure 6.12. Plots of the hyperbolic functions sinh z and cosh z.

compared with their trigonometric counterparts. It also follows easily from (6.56) or (6.57) that

$$\cosh^2 z - \sinh^2 z = 1. \tag{6.63}$$

This identity reveals a simple geometrical property that accounts for the name of these functions. Indeed, with $x = \cosh z$ and $y = \sinh z$, (6.63) yields $x^2 - y^2 = 1$, the equation of equilateral hyperbolas with asymptotes along the bisectors of the coordinate lines. Hence, the functions (6.56) are named *hyperbolic functions*. The trigonometric functions $x = \cos z$ and $y = \sin z$, on the other hand, yield $x^2 + y^2 = 1$, the equation of a unit circle. And we recall that the trigonometric functions are also known as *circular functions*.

The identity (6.63) shows that, unlike their trigonometric cousins, $\cosh z > \sinh z$ for all values of z. This means that their graphs never intersect; the graph of $\cosh z$ lies always above the graph of $\sinh z$. Moreover, (6.56) and (6.62) show that $\sinh z$ vanishes at $z = 0$ where its slope, $\cosh z$, has value 1. Since $d^2(\sinh z)/dz^2 = 0$, the graph of $\sinh z$ has an inflection at $z = 0$. Equation (6.56) shows that $\sinh(-z) = -\sinh z$ is an odd function of z. The graph of $\sinh z$ thus has the form shown in Fig. 6.12. The graph of $\cosh z$, also shown there, has a minimum at $z = 0$ where its value is 1, and, by (6.56), $\cosh(-z) = \cosh z$ shows that $\cosh z$ is an even function of z. Clearly, as z grows indefinitely large, (6.56) indicates that both functions grow indefinitely, as shown in Fig. 6.12. It can be proved from statics that the graph of the hyperbolic cosine function, also called the *catenary*, is the shape assumed by a uniform, heavy cord supported at its ends and hanging under its own weight, an easy experiment for the reader.

6.7. The Simple Harmonic Oscillator

The differential equations (6.48) and (6.54) occur in a wide variety of dynamical problems, the simplest kind being those for which $h(t) = 0$. These equations reduce in this case to the respective homogeneous equations (6.50) and (6.58). In particular, the oscillations of a mass attached to an ideal linear spring and the small amplitude oscillations of a pendulum are motions of physical systems that are governed by the same homogeneous equation (6.50)—the equation of a so-called *simple harmonic oscillator.* An example in which (6.58) occurs will follow shortly. We begin with the linear spring/mass system.

6.7.1. Hooke's Law of Linear Elasticity

We usually think of a spring as a helically wound wire device. But all solid bodies, like a solid rubber block or cord, behave in the same springy way, except that the deformation of most solid bodies is usually very small. So all solid bodies whatsoever, whether metal, wood, glass, or stone; hair, silk, tissue, or bone; and so on, are springs too. In general, to characterize the uniform, uniaxial elastic behavior of a deformable solid body under tensile or compressive end loads, we adopt an ideal spring model described by *Hooke's law*: *The uniaxial force F_H required to stretch or to compress an ideal spring is proportional to the uniaxial change of length δ of the spring from its natural, undeformed state*:

$$F_H = k\delta. \tag{6.64}$$

The constant k is called the *spring constant*. Sometimes the terms *elasticity, modulus*, or *stiffness* are also used. Clearly, $[k] = [F/L]$. An ideal spring for which (6.64) holds is known as a *linear spring*.

The linear force–deformation law (6.64) was proposed by Robert Hooke in 1675. To protect his discovery from use by others while he exploited its applications, he claimed priority for the law and published its substance in a Latin anagram, "*ceiiinosssttuu*." Three years later, and 18 years since his first knowledge of it, Hooke unscrambled the puzzle to read:¶ "*ut tensio sic vis*;" that is, the

¶ Notice that the anagram has a double "u," contrary to its Latin decipherment by Hooke. See R. Hooke, *De Potentia Restitutiva or of Spring*, 1678; reproduced in R. T. Gunther, *Early Science in Oxford*, Volume VIII, The Cutler Lectures of Robert Hooke, pp. 331–56, Oxford University Press, Oxford, 1931. This is not an error. In early Latin manuscripts *v* often appears in print as *u*. In fact, M. Espinasse in *Robert Hooke,* University of California Press, Berkeley, 1962, p. 78, writes literally, "*ut tensio sic uis*." Hooke's own decipherment, however, is commonly adopted in books on elasticity, its history, and Hooke's life. See Volume 1, p. 5, of I. Todhunter and K. Pearson, *A History of the Theory of Elasticity and of the Strength of Materials*, Dover, New York, 1960; L. Jardin, *Ingenious Pursuits: Building the Scientific Revolution*, pp. 322–3, Doubleday, New York, 1999; and the remarkable treatise by J. F. Bell, "The Experimental Foundations of Solid Mechanics," *Flügge's Handbuch der Physik*, Volume **VIa/1**, pp. 156–60, Springer-Verlag, New York, 1973.

extension of any spring increases in proportion to the tension. Hooke demonstrated by experiments that in addition to solid bodies, the rule also holds for helical wire springs for which the deformation δ may be large. Because Hooke's law is linear, it follows that the extra force F^* required to stretch (or compress) the spring an additional amount η, say, is proportional to η; i.e., $F^* = k\eta$. This simple superposition rule does not apply to any nonlinear spring. Any potential confusion about the effect of initial deformation of a spring in the formulation of a problem may be avoided by use of a deformation variable defined with respect to the natural state.

Hooke's rule is not a fully accurate characterization of a springy body for all cases of practical interest. It ignores, for example, the possibly large twisting effect induced by uniaxial loading of a helical spring, whose torsional stiffness and mass usually are neglected in applications of (6.64). And it does not hold for large deformations possible in nonlinear rubberlike materials or biological tissues. On the other hand, Hooke's law provides a mathematically simple and useful description of the physical nature of phenomena in a great variety of practical cases where the elastic response of a solid may be reasonably modeled by a linear spring.

6.7.2. The Linear Spring-Mass System

Let us consider a linear spring fixed at one end and having a mass m (sometimes called the load) attached to its other end, and either suspended vertically or supported by a smooth plane surface. The mass of the spring is generally considered negligible in comparison with the mass m; so, henceforward, its mass is ignored. When m is displaced a distance δ from the natural, unstretched spring configuration, it exerts on the spring a uniaxial force F_H given by (6.64). In response, the spring exerts an equal but oppositely directed restoring force $F_S = -F_H = -k\delta$, called *the spring force,* that acts always to return the mass toward the natural state of the spring. Hence, if released, the mass will vibrate under the alternating extension and compression reactions of the spring itself. Let us first study the oscillations of the mass on a smooth horizontal surface, as shown in Fig. 6.13.

6.7.2.1. Horizontal Vibrations of a Mass on a Linear Spring

To characterize the horizontal oscillatory motion of the mass, we suppose that m is given an initial uniaxial velocity $\mathbf{v}_0 = v_0\mathbf{i}$ from its natural equilibrium configuration in $\Phi = \{F; \mathbf{i}, \mathbf{j}\}$ shown in Fig. 6.13. The free body diagram of m is shown in Fig. 6.13a. The weight $\mathbf{W}$ is balanced by the normal reaction $\mathbf{N}$ of the smooth surface, so the only force that affects the horizontal, uniaxial motion of m is the spring force $\mathbf{F}_S = -kx\mathbf{i}$, in which $x = \delta$ denotes the displacement of m, the change of length of the spring from its natural state. Therefore, the equation

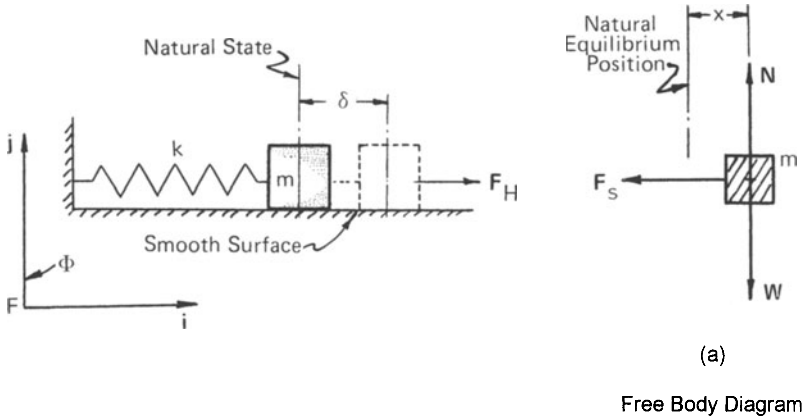


Figure 6.13. An ideal spring-mass system.

of motion of m, namely, $\mathbf{F}_S = m\ddot{x}\mathbf{i}$, becomes

$$\ddot{x} + p^2 x = 0 \qquad \text{with} \quad p = \sqrt{\frac{k}{m}}. \tag{6.65a}$$

This equation has the form (6.50) whose general solution is given by (6.51):

$$x(t) = A \sin pt + B \cos pt. \tag{6.65b}$$

An oscillatory motion described by (6.65b) is called a *simple harmonic motion.*

The integration constants A and B are determined by specified initial conditions; presently, $x(0) = 0$, $\dot{x}(0) = v_0$. Since $x(0) = B = 0$, (6.65b) reduces to $x(t) = A \sin pt$, and with $\dot{x}(0) = Ap = v_0$, we have the general solution

$$x(t) = \frac{v_0}{p} \sin pt. \tag{6.65c}$$

The maximum displacement of m from its equilibrium state is called the *amplitude* of the oscillation. The amplitude of the motion (6.65c) is given by $x_A \equiv v_0/p$. The graph of the motion (6.65c) and the corresponding velocity $\dot{x} = v_0 \cos pt$ are shown in Fig. 6.14. The motion of m varies from x_A to $-x_A$ over and over again. Also, the displacement $x(t)$, and similarly the velocity $v(t)$, has the same value at times $t + 2n\pi/p = t + n\tau$ for $n = 0, 1, 2, \ldots$; that is, $\sin(pt + 2n\pi) = \sin p(t + n\tau) = \sin pt$. Hence, the motion (6.65c) is said to be *periodic*, and the least nonzero time $\tau = 2\pi/p$ for which $x(t) = x(t + \tau)$ is called the *period* of the motion—it is the time required to complete one oscillation. (See Fig. 6.14.) The number of periods that occur in a unit of time is the number of oscillations of the mass per unit time. This number, denoted by

$$f = \frac{1}{\tau} = \frac{p}{2\pi}, \tag{6.65d}$$

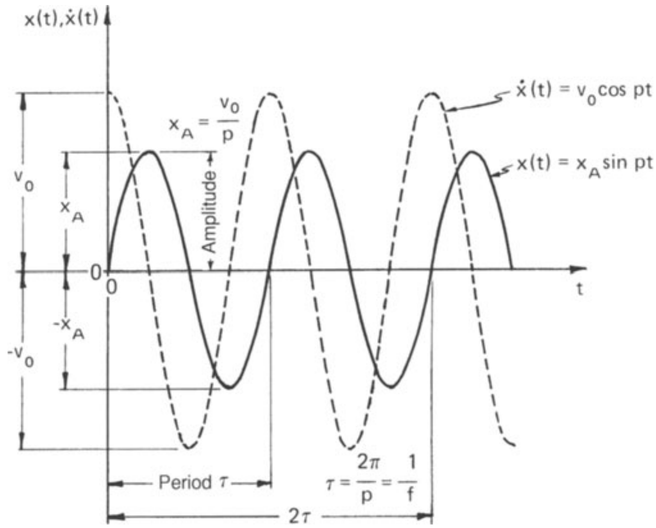


Figure 6.14. Graphs illustrating the periodic nature of the simple harmonic motion and the simultaneous velocity of the load in a linear spring-mass system.

is called the *frequency* of the oscillations. The measure units of f are expressed as cycles per unit time. When the time is in seconds, the measure of f commonly is stated in cycles per second or Hertz, abbreviated 1 cps$\equiv$ 1 Hz. Since there are 2π radians in one cycle, $p = 2\pi f$ is called the *circular frequency*; its measure units are radians per unit time. Clearly, $[p] = [f] = [T^{-1}]$ and $[\tau] = [T]$.

The relation for the circular frequency of a simple harmonic motion may be read immediately from the coefficient of the differential equation of motion (6.65a). Consequently, the period and the frequency of the motion of the mass of a linear spring-mass system may be obtained at once from (6.65d). We thus find

$$\tau = 2\pi\sqrt{\frac{m}{k}}, \qquad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{6.65e}$$

The graph of the uniaxial velocity $\dot{x} = v = v_0 \cos pt$ versus the uniaxial position $x = x_A \sin pt$, called a *phase plane diagram*, is an ellipse centered at the origin and having semi-axes determined by x_A and v_0:

$$\left[\frac{x}{x_A}\right]^2 + \left[\frac{\dot{x}}{v_0}\right]^2 = 1. \tag{6.65f}$$

We may suppose that p in (6.65a) is known. Then for each choice of initial velocity v_0, the pair $(x_A = v_0/p,\, v_0)$ determines a different ellipse, and hence (6.65f) describes a family of concentric ellipses. Moreover, the normalized plot of $\dot{x}/v_0$ versus x/x_A reduces every member of the family (6.65f) to a single unit circle.

The periodic nature of the motion is exhibited by these closed phase plane paths, all of which are traversed in the same time $\tau = 2\pi/p$, the period of the motion. Equation (6.65f) has exactly two solutions $x = \pm x_A$ for which $\dot{x} = 0$, and hence *the amplitudes* $\pm x_A$ *are the extreme points in the motion at which the mass comes momentarily to rest*. The $\pm$ sign reflects the symmetry of the motion about $x = 0$, the equilibrium state of rest—the unique time independent solution of (6.65a). And, by (6.65f), *the greatest velocity* $\dot{x} = v_0$ *also occurs at* $x = 0$. These results are evident in Fig. 6.14. We shall find in Chapter 7 that the phase plane curves are related to the energy of the system.

For other initial conditions, the form of the solution (6.65c), hence also the relations describing the amplitude and the phase plane trajectory, will be somewhat different. General formulas for the amplitude and the phase plane graph for arbitrary initial data assigned in any simple harmonic motion (6.65b) are presented later.

6.7.2.2. *Vertical Vibrations of a Mass on a Linear Spring*

Now let us consider the effect of gravity on the oscillatory motion of a load m supported vertically by a linear spring. The weight produces a static deflection δ_E of the spring from its natural state, and the mass is then set into vertical oscillatory motion about this equilibrium state. We shall see that the motion of m relative to the unstretched state is described by an equation that may be transformed to another having the same form as (6.65a) relative to the static equilibrium state.

Let us fix the origin at the natural state of the spring so that $\mathbf{i}$ is in the downward direction of $\mathbf{g} = g\mathbf{i}$. Construction of the free body diagram of m is left for the reader. The weight $\mathbf{W} = mg\mathbf{i}$ produces a static deflection δ_E such that the spring force exerted on m is $\mathbf{F}_S = -k\delta_E\mathbf{i}$; hence, the static equilibrium equation $\mathbf{W} + \mathbf{F}_S = (mg - k\delta_E)\mathbf{i} = \mathbf{0}$ yields

$$k\delta_E = mg. \tag{6.66a}$$

When the mass is set into vertical motion, the weight $\mathbf{W}$ is unchanged but the spring force becomes $\mathbf{F}_S = -kx\mathbf{i}$, where x denotes the stretch of the spring from its natural state. Hence, the equation of motion $\mathbf{W} + \mathbf{F}_S = (mg - kx)\mathbf{i} \doteq m\ddot{x}\mathbf{i}$ yields

$$\ddot{x} + p^2 x = g, \qquad \text{with} \quad p^2 = \frac{k}{m} = \frac{g}{\delta_E}, \tag{6.66b}$$

wherein (6.66a) is introduced. This equation has the form of (6.48) in which $h(t) \equiv g$ is constant. Therefore, recalling the method leading to (6.45b) and (6.53), we see that the general solution of (6.66b) is

$$x(t) = C\cos pt + D\sin pt + \frac{g}{p^2}, \tag{6.66c}$$

in which C and D are integration constants to be fixed by the initial data.

This shows that the motion of m is simple harmonic, but the center of the oscillations is displaced to the position at $x_E = g/p^2 = mg/k = \delta_E$, the static equilibrium position of m. Hence, introducing the new variable $z \equiv x - \delta_E$ to describe the displacement of m from its static equilibrium position, the equation of motion in (6.66b) transforms to

$$\ddot{z} + p^2 z = 0. \tag{6.66d}$$

This has the same form as our earlier equation (6.65a) for the horizontal motion; so, the solution (6.66c) may be cast in the form

$$z(t) = x(t) - \delta_E = C\cos pt + D\sin pt. \tag{6.66e}$$

Hence, both linear spring-mass systems are governed by the same kind of equation. When the displacement is measured from the vertical static equilibrium position of a *linear* spring-mass system, in effect, the static deflection and the weight of the load may be ignored in view of the balance equation (6.66a). Therefore, the static equilibrium state is a convenient reference state from which to study the motion of a linear system, because we need only consider the additional spring force $F^* = -kz = m\ddot{z}$ for the displacement from that state. This superposition procedure, however, cannot be used to study the motion of a load on a nonlinear spring; in this case, the undeformed reference state must be used.

The frequency $f = p/2\pi = (\sqrt{k/m})/2\pi$ of the vibration of m is independent of the initial data. In view of (6.66a), this may be rewritten in terms of the static deflection alone, namely,

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_E}}. \tag{6.66f}$$

Thus, regardless of the spring stiffness and independently of the amplitude, any vertical loading that produces the same static deflection in different linear springs will oscillate with the same frequency. Of course, for springs of different moduli, the loads needed to produce the same static deflection differ; nevertheless, the measured frequency of their oscillations is identical for all amplitudes, and hence, in this sense, formula (6.66f) is *universal*.

We now study the small amplitude oscillations of a pendulum. Although this physical system is quite different from the spring-mass system, both are governed by the same basic equation of motion characteristic of a simple harmonic oscillator.

6.7.3. The Simple Pendulum

A *simple pendulum*, shown in Fig. 6.15, consists of a small heavy body of mass m, called the *bob*, attached to one end of a thin rigid rod or string of length ℓ, negligible mass, and suspended from a smooth pin or hinge at the point O. The pendulum is displaced to swing about its vertical equilibrium position. Air resistance and the mass of the rod are ignored. We wish to determine the

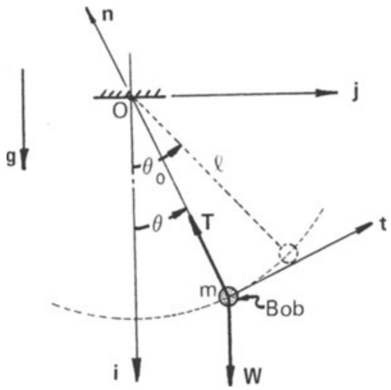


Figure 6.15. A simple pendulum and its free body diagram.

frequency and period of its small amplitude oscillations in the vertical plane, when the pendulum is released from rest at a small angle θ_0. The problem, however, is first formulated exactly for large amplitude oscillations.

The free body diagram in Fig. 6.15 shows the gravitational force $\mathbf{W} = m\mathbf{g}$ and the rod tension $\mathbf{T}$ acting on the bob. The equation of motion for the angular placement $\theta(t)$ of the bob from its vertical equilibrium position is readily described in terms of intrinsic variables. In accordance with (6.3), $\mathbf{F} = \mathbf{W} + \mathbf{T} = m(\ddot{s}\mathbf{t} + \kappa \dot{s}^2\mathbf{n})$, in which $\dot{s} = \ell\dot{\theta}$, $\ddot{s} = \ell\ddot{\theta}$, and $\kappa = 1/\ell$. Therefore,

$$m(\ell\ddot{\theta}\mathbf{t} + \ell\dot{\theta}^2\mathbf{n}) = T\mathbf{n} - W(\sin\theta\,\mathbf{t} + \cos\theta\,\mathbf{n}). \tag{6.67a}$$

This yields the two scalar equations of motion:

$$\ddot{\theta} + p^2\sin\theta = 0, \qquad T = m\ell(\dot{\theta}^2 + p^2\cos\theta), \tag{6.67b}$$

where

$$p \equiv \sqrt{\frac{g}{\ell}}. \tag{6.67c}$$

Let the reader confirm these equations by use of (6.4).

The first equation in (6.67b) is an ordinary nonlinear differential equation for the angular motion $\theta(t)$ and the second gives the rod tension $T(\theta)$ as a function of θ. The exact solution of these equations for finite amplitude oscillations of a pendulum will be studied in Chapter 7. Presently, however, we consider only small values of θ so that all squared and higher order terms in θ and its derivative $\dot{\theta}$ may be neglected. Then use of the series functions (2.17) in (6.67b) leads to

$$\ddot{\theta} + p^2\theta = 0, \qquad T = mg = W. \tag{6.67d}$$

Certainly, for sufficiently small placements $\theta(t)$, it is expected that the rod tension does not vary significantly from its static value, the weight of the bob, as shown in (6.67d). The first equation in (6.67d) has the same form as (6.65a); so, the small amplitude pendulum motion is simple harmonic with circular frequency p defined by (6.67c). Therefore,

$$\theta(t) = A \sin pt + B \cos pt. \tag{6.67e}$$

The constants A and B are determined by the given initial conditions that the pendulum is released from rest at a small angle θ_0, so that $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$. From (6.67e), the angular speed is $\dot{\theta} = Ap \cos pt - Bp \sin pt$. We thus find $A = 0$ and $B = \theta_0$, and (6.67e) yields the solution

$$\theta(t) = \theta_0 \cos pt. \tag{6.67f}$$

The angle θ_0 is the amplitude, the maximum angular placement from the vertical equilibrium position in the motion (6.67f). Recalling the relations (6.65d), we find the small amplitude frequency and period of a simple pendulum with circular frequency (6.67c):

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}, \qquad \tau = 2\pi\sqrt{\frac{\ell}{g}}. \tag{6.67g}$$

The period is the time required for the pendulum to swing from θ_0 to $-\theta_0$ and back again to θ_0. The period of the small amplitude, simple harmonic motion of a pendulum is independent of this amplitude. The finite amplitude motion of a pendulum described by the first equation in (6.67b), though still periodic, is not simple harmonic. It is shown in Chapter 7 that the periodic time in the finite motion varies with the amplitude.

6.7.4. The Common Mathematical Model in Review

The linear spring-mass system and the simple pendulum (for small amplitude oscillations) are merely two examples of a great many physical systems that are characterized by the same mathematical model. Their common model, called the *simple harmonic oscillator*, is described by the homogeneous differential equation

$$\ddot{u} + p^2 u = 0, \tag{6.68}$$

whose solution

$$u = A \sin pt + B \cos pt, \tag{6.69}$$

is simple harmonic. The constant circular frequency p and period $\tau = 2\pi/p$, or the frequency $f = 1/\tau$, may be read immediately from the positive coefficient in (6.68).

The amplitude of the motion of a harmonic oscillator may be obtained by introduction of two other constants U and α defined by

$$A = U \cos\alpha, \qquad B = U \sin\alpha. \tag{6.70}$$

Therefore, the new constants are related to the former by

$$U = \sqrt{A^2 + B^2}, \qquad \tan\alpha = \frac{B}{A}. \tag{6.71}$$

We lose no generality in taking U positive, and α may be either positive or negative valued.

Use of (6.70) in (6.69) yields the following alternative form for the motion $u(t)$ of the simple harmonic oscillator:

$$u(t) = U\sin(pt + \alpha). \tag{6.72}$$

The reader may show that with $A = U\sin\beta,\ B = U\cos\beta$, where $\beta = \pi/2 - \alpha$, an alternative form of the solution (6.72) is given by $u(t) = U\cos(pt - \beta)$. Either solution shows that U, the maximum value of $u(t)$, is the *amplitude* of the motion. The angle $pt + \alpha$ (or $pt - \beta$) is called the *phase angle* or simply the *phase* of the motion; it characterizes the state of the oscillation at a specific time. The *phase constant* α (or β) defines the *initial phase* of the motion. From (6.72), the velocity may be written as

$$\dot{u}(t) = Up\cos(pt + \alpha). \tag{6.73}$$

Thus, if initially we are given $u(0) \equiv u_0$ and $\dot{u}(0) \equiv v_0$, then (6.72) and (6.73) yield $u_0 = U\sin\alpha,\ v_0 = Up\cos\alpha$. In consequence, by (6.71), the amplitude and initial phase may be expressed in terms of the initial data:

$$U = \sqrt{u_0^2 + \left(\frac{v_0}{p}\right)^2}, \qquad \alpha = \tan^{-1}\left(\frac{u_0 p}{v_0}\right).$$

A graphical description of the motion is obtained from (6.72) and (6.73). *For arbitrary initial data and for each fixed frequency p, the graph of $\dot{u}(t)$ versus $u(t)$ for the simple harmonic oscillator motion is a family of concentric ellipses having semi-axes determined by U and Up*:

$$\left(\frac{u}{U}\right)^2 + \left(\frac{\dot{u}}{Up}\right)^2 = 1. \tag{6.74}$$

In general, the plane graph of $\dot{u}$ versus u for any single degree of freedom system is called the *phase plane graph*. Thus, for each choice of initial data, the phase plane graph for the simple harmonic oscillator is an ellipse defined by (6.74). However, it is seen further that the normalized plot of $\dot{u}/Up$ versus u/U reduces every member of the family (6.74) to a single unit circle. The periodic nature of the simple harmonic motion is exhibited by these closed curves. Since p is fixed and the period does not depend on the initial data, all trajectories in the phase plane are traversed in the same time τ. Equation (6.74) has exactly two solutions $u = \pm U$ for which $\dot{u} = 0$ and two solutions $\dot{u} = \pm Up$ at the equilibrium position $u = 0$, the unique time independent solution of (6.68). *Hence, the amplitudes $\pm U$ mark the extreme positions in a simple harmonic motion at which the velocity* *momentarily vanishes; and the greatest velocity* $\pm Up$ *occurs at the equilibrium state.*

This concludes our study of the simple harmonic oscillator. The effect of viscous damping on mechanical vibrations, and some effects of inertial forces induced by rotating bodies, including effects of the Earth's rotation, and other kinds of forces are explored later in this chapter. It is important to recognize that not every vibration need be periodic, and not every periodic motion need be vibratory. Random vibrations, for example, are not periodic, and steady orbital motions are periodic but not vibratory. The following example of a particle moving in an electromagnetic field exhibits a motion that is periodic but not oscillatory. The solution procedure, however, is the same.

6.8. Motion of a Charged Particle in an Electromagnetic Field

A particle of charge q and mass m is ejected from an electronic device, with initial velocity $\mathbf{v}_0 = v\mathbf{j}$ at the place $\mathbf{x}_0 = R\mathbf{i}$ in an inertial frame $\Phi = \{F; \mathbf{i}_k\}$. The charge moves under the influence of constant and oppositely directed electric and magnetic fields that are parallel to the axis of the gravitational field, as shown in Fig. 6.16. The total body force acting on q is $\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m + \mathbf{W}$; hence, with (6.18), the equation of motion may be written as

$$\ddot{\mathbf{x}} - c\dot{\mathbf{x}} \times \mathbf{B} = \frac{d}{dt}(\dot{\mathbf{x}} - c\mathbf{x} \times \mathbf{B}) = c\mathbf{E} + \mathbf{g}, \tag{6.75a}$$

where in $c \equiv q/m$. This vector equation is readily integrated to obtain the velocity

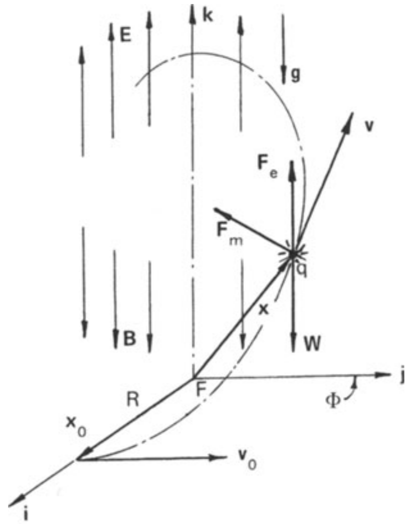


Figure 6.16. Motion of a charged particle in uniform and oppositely directed electric and magnetic fields.

$\dot{\mathbf{x}}$ as a function of $\mathbf{x}$ and t; thus,

$$\dot{\mathbf{x}} = c\mathbf{x} \times \mathbf{B} + (c\mathbf{E} + \mathbf{g})t + \mathbf{C}_0. \tag{6.75b}$$

The constant vector of integration is fixed by the initial data $\mathbf{x}(0) = \mathbf{x_0}$, $\dot{\mathbf{x}}(0) = \mathbf{v}_0$, so that

$$\mathbf{C}_0 = \mathbf{v}_0 - c\mathbf{x}_0 \times \mathbf{B}. \tag{6.75c}$$

Although (6.75b) cannot be integrated further, its use in (6.75a) leads to another integrable result. Bearing in mind that the vectors **B**, **E**, and **g** are parallel, we obtain

$$\ddot{\mathbf{x}} - c^2(\mathbf{x} \times \mathbf{B}) \times \mathbf{B} = c\mathbf{C}_0 \times \mathbf{B} + c\mathbf{E} + \mathbf{g}. \tag{6.75d}$$

Finally, substitution of (6.75c) into (6.75d), expansion of the triple products in the result, use of the orthogonality condition $\mathbf{B} \cdot \mathbf{x}_0 = 0$, and $\omega \equiv cB$ yields the vector differential equation

$$\ddot{\mathbf{x}} + \omega^2\mathbf{x} - c^2(\mathbf{x} \cdot \mathbf{B})\mathbf{B} = \omega^2\mathbf{x}_0 + c\mathbf{v}_0 \times \mathbf{B} + c\mathbf{E} + \mathbf{g}. \tag{6.75e}$$

Exercise 6.3. Show that (6.75e) may be written as $\ddot{\mathbf{x}} + \mathbf{P}^2\mathbf{x} = \boldsymbol{\gamma}$, in which $\mathbf{P}^2 \equiv \omega^2(\mathbf{i} \otimes \mathbf{i} + \mathbf{j} \otimes \mathbf{j})$ and $\boldsymbol{\gamma}$ is a constant vector. □

We now introduce $\mathbf{B} = -B\mathbf{k}$, $\mathbf{E} = E\mathbf{k}$, $\mathbf{g} = -g\mathbf{k}$, $\mathbf{x}_0 = R\mathbf{i}$, $\mathbf{v}_0 = \nu\mathbf{j}$, and $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ into (6.75e) and equate the corresponding vector components to obtain the following three scalar equations of motion:

$$\ddot{x} + \omega^2 x = \omega^2\left(R - \frac{\nu}{\omega}\right), \qquad \ddot{y} + \omega^2 y = 0, \qquad \ddot{z} = 2A, \tag{6.75f}$$

in which $2A \equiv cE - g$. The first pair of these equations shows that both x and y are simple harmonic functions, and hence the general solution of the system (6.75f) is given by

$$x(t) = a + K\cos\omega t + L\sin\omega t, \qquad \text{with} \quad a \equiv R - \frac{\nu}{\omega}, \tag{6.75g}$$

$$y(t) = M\cos\omega t + N\sin\omega t \quad \text{and} \quad z(t) = At^2 + Pt + Q. \tag{6.75h}$$

The initial data $\mathbf{x}(0) = \mathbf{x}_0 = R\mathbf{i}$ and $\dot{\mathbf{x}}(0) = \mathbf{v}_0 = \nu\mathbf{j}$ determine the integration constants $L = M = P = Q = 0$ and $K = N = \nu/\omega$. Hence, the foregoing system has the solution

$$x(t) = a + \frac{\nu}{\omega}\cos\omega t, \qquad y(t) = \frac{\nu}{\omega}\sin\omega t, \qquad z(t) = At^2. \tag{6.75i}$$

It is seen that $(x - a)^2 + y^2 = (\nu/\omega)^2$ is the equation of a circle centered at $(a, 0)$, hence (6.75i) suggests that the trajectory of q looks a bit like a cylindrical helix of radius $\rho \equiv \nu/\omega$. By taking $R = \rho$ in (6.75g), we have $a = 0$, and the cylinder axis is shifted to the origin of Φ. The first two equations in (6.75f) have essentially the same form as equation (6.68) for the harmonic oscillator, but the motion of q is not oscillatory. On the other hand, the motion in the xy-plane is periodic; the circular frequency $\omega = cB$, evident from (6.75f), describes the constant rate of rotation of q about the cylinder axis, and the periodic time $\tau = 2\pi/\omega$ is the time required for the particle to make one full swing around that axis as it advances along z. Notice that the tangent to the path does not make a fixed angle with the cylinder axis; rather, $\mathbf{t} \cdot \mathbf{k} = 2At(v^2 + 4A^2t^2)^{-1/2}$. The pitch increases with the square of the number of turns: $p_n \equiv z(n\tau) = n^2 z(\tau)$, and $z(\tau) = A/4\pi\omega^2 = p_1$. So, the path is not a true cylindrical helix.

6.9. Motion of a Slider Block in a Rotating Reference Frame

We now turn to a different class of problems whose solutions involve the hyperbolic functions. Two problems concerning the motion of a slider block in a slot milled in a rotating table are studied. The first concerns the free sliding motion of the block due to inertial forces induced by the table's rotation. The second problem is similar, but more interesting. An additional controlling spring is introduced, and depending on the nature of two physical parameters, one due to the rotation and the other due to the spring, the governing equation of motion may have a solution of either trigonometric or hyperbolic type, or neither.

6.9.1. Uncontrolled Motion of a Slider Block

A block S of mass m shown in Fig. 6.17 is constrained initially by a cord fastened at the end point A of a smooth slot milled in a table that turns in the horizontal plane with a constant angular speed ω. When the string is cut suddenly, the block slides freely in the slot. We wish to determine the motion $\mathbf{x}(S, t)$ of the slider block relative to the spinning table, and the behavior of the force that acts on the block as a function of its position in the slot and as a function of time.

The free body diagram of the sliding block is shown in Fig. 6.17a. Of course, the string force $\mathbf{F}_S = \mathbf{0}$, and the weight of the block is $\mathbf{W} = -W\mathbf{k}$. Because the slot is smooth, it exerts on S only the normal contact forces $\mathbf{N} = -N\mathbf{j}$ in the plane of the table and $\mathbf{R} = R\mathbf{k}$ perpendicular to it. The total force acting on S is $\mathbf{F} = \mathbf{N} + \mathbf{R} + \mathbf{W}$, and hence the equation of motion for S in the inertial frame $\Phi = \{F; \mathbf{I}_k\}$ fixed in the laboratory is given by

$$\mathbf{F} = -N\mathbf{j} + (R - W)\mathbf{k} = m\mathbf{a}_S. \tag{6.76a}$$

The absolute acceleration $\mathbf{a}_S$ of S in Φ may be obtained from (4.48). With $\mathbf{a}_O = \mathbf{0}$, $\boldsymbol{\omega}_f = \omega\mathbf{k}$, $\dot{\boldsymbol{\omega}}_f = \mathbf{0}$, and $\mathbf{x}(S, t) = x\mathbf{i} + a\mathbf{j}$ in the reference frame $\varphi = \{O; \mathbf{i}_k\}$ fixed in the table, as shown in Fig. 6.17, the total acceleration of S referred

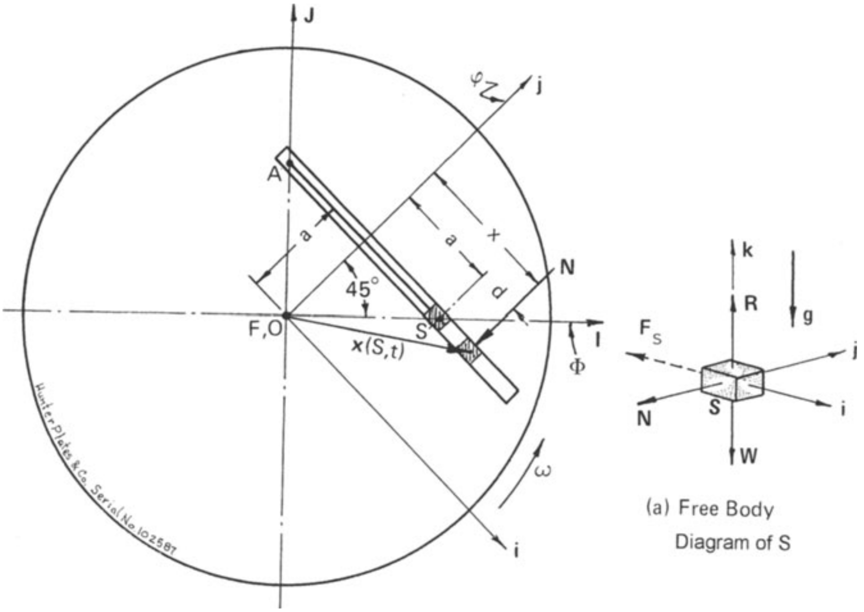


Figure 6.17. Relative motion of a slider block on a rotating table.

to φ is

$$\mathbf{a}_S = (\ddot{x} - \omega^2 x)\mathbf{i} + (2\omega\dot{x} - a\omega^2)\mathbf{j}. \tag{6.76b}$$

Substitution of (6.76b) into (6.76a) yields the scalar equations

$$\ddot{x} - \omega^2 x = 0, \qquad N = m(a\omega^2 - 2\omega\dot{x}), \qquad R = W. \tag{6.76c}$$

The first of these equations determines the motion $x(t)$ of S relative to the table, and the next one determines the normal contact force N either as a function of x or of t. The last relation confirms that since there is no motion of S normal to the table, the slot reaction force $\mathbf{R}$ balances the weight $\mathbf{W}$, so that $\mathbf{R} + \mathbf{W} = \mathbf{0}$ in (6.76a). Therefore, in future problems where the motion is constrained to a smooth horizontal plane, for simplicity, the trivial normal equilibrated forces may be ignored.

The first equation in (6.76c) has the same form as the homogeneous equation (6.58) whose solution is given by (6.59). Therefore, the slider's motion is given by

$$x(t) = A \sinh \omega t + B \cosh \omega t. \tag{6.76d}$$

The slider is initially at rest at $x(0) = a$ in frame φ, as shown in Fig. 6.17, and hence $\dot{x}(0) = 0$. Thus, with $\dot{x}(t) = A\omega \cosh \omega t + B\omega \sinh \omega t$, it follows that

$A = 0$, $B = a$; hence

$$x(t) = a \cosh \omega t, \qquad \dot{x}(t) = a\omega \sinh \omega t. \tag{6.76e}$$

Therefore, the motion of S relative to the table frame may be written as

$$\mathbf{x}(S, t) = a(\cosh \omega t \mathbf{i} + \mathbf{j}). \tag{6.76f}$$

Use of (6.76e) in the second equation in (6.76c) gives the slot reaction force $\mathbf{N} = -N\mathbf{j}$ as a function of time;

$$\mathbf{N} = \tilde{\mathbf{N}}(t) = -ma\omega^2(1 - 2\sinh \omega t)\mathbf{j}. \tag{6.76g}$$

Alternatively, use of the identity (6.63) yields the slot reaction force as a function of the slider's position along the slot:

$$\mathbf{N} = \hat{\mathbf{N}}(x) = -m\omega^2(a - 2\sqrt{x^2 - a^2})\mathbf{j}. \tag{6.76h}$$

Let the reader show that the same result may be derived directly by integration of the first equation in (6.76c) to find $\dot{x}(x)$.

These results show from (6.76g) that initially $\tilde{\mathbf{N}}(0) = -ma\omega^2\mathbf{j}$, and, as time advances, the normal force $\tilde{\mathbf{N}}(t)$ decreases to zero in the time t^* for which $\omega t^* = \sinh^{-1}(1/2) \approx 0.481$, the instant when the slider is at the place $x^* = a\sqrt{5}/2 \approx 1.118a$ in the slot. Afterwards, the normal, slot reaction force reverses its sense of application to the opposite side of the block and grows again, indefinitely for as long as the block is able to move outward. Suppose, for example, that $\mathbf{x}(S, 0) = a(\mathbf{i} + \mathbf{j}) = 25\mathbf{I}$ cm and $\omega = 20\pi/3$ rad/sec (200 rpm). Then $a = 25/\sqrt{2}$ cm, and the previous formulas show that $\mathbf{N}$ vanishes, and then reverses its sense of application, after $t^* \approx 0.023$ sec when S has moved a distance $d^* = x^* - a \approx 2.086$ cm from its initial position.

When the string was cut, the motion of the block along the slot was no longer controlled, and the inertial effect of the table's rotation drove the slider increasingly farther from its rest state toward the end of the slot. The controlling effect of an additional spring force is illustrated next.

6.9.2. Controlled Motion and Instability of a Slider Block

Suppose that the string shown in Fig. 6.17 is replaced by a linear spring of stiffness k fastened at A and to the block S, initially at rest at the natural state of the spring at $x = a$ but otherwise free to slide in the smooth slot. We wish to investigate the motion $\mathbf{x}(S, t)$ of the block relative to the rotating table.

The free body diagram of the sliding block is shown in Fig. 6.17a in the table frame φ. The forces are the same as before with the addition of the spring force $\mathbf{F}_S = -k(x - a)\mathbf{i}$. Since there is no motion normal to the horizontal plane, $\mathbf{R} + \mathbf{W} = \mathbf{0}$, as noted before. Therefore, the equation of motion for S in the inertial frame Φ, but referred to the table frame φ, is given by

$$\mathbf{F} = \mathbf{N} + \mathbf{F}_S = -N\mathbf{j} - k(x-a)\mathbf{i} = m\mathbf{a}_S. \tag{6.77a}$$

Here we recall (6.76b) to obtain the scalar equations

$$\ddot{x} + p^2(1-\eta^2)x = ap^2, \tag{6.77b}$$

$$N = m(a\omega^2 - 2\omega\dot{x}), \tag{6.77c}$$

wherein, by definition,

$$p \equiv \sqrt{\frac{k}{m}}, \qquad \eta \equiv \frac{\omega}{p}. \tag{6.77d}$$

The physical nature of the motion determined by (6.77b) depends on the coefficient $p^2(1-\eta^2)$. There are three cases to explore: (i) $\eta < 1$, (ii) $\eta = 1$, and (iii) $\eta > 1$. Each case is studied in turn for the assigned initial data

$$x(0) = a, \qquad \dot{x}(0) = 0. \tag{6.77e}$$

Case (*i*): $\eta < 1$; i.e. the angular speed $\omega < p$. Then the equation of motion in (6.77b) has the form of (6.48) in which p^2 is replaced by $\Omega^2 \equiv p^2(1-\eta^2)$ and $h(t) \equiv ap^2$ is constant. Recalling (6.45b) and (6.53), we see that the general solution of (6.77b) is

$$x(t) = A\sin\Omega t + B\cos\Omega t + \frac{a}{1-\eta^2}, \qquad \text{with} \quad \Omega = p\sqrt{1-\eta^2}. \tag{6.77f}$$

The relative motion of S is simple harmonic, but the center of the oscillation is displaced to the relative equilibrium position at

$$x_E = \frac{a}{1-\eta^2}, \tag{6.77g}$$

defined by the unique time independent solution of the equation of motion (6.77b). Notice that $x_E > a$. Hence, introducing the new variable $z \equiv x - x_E$ to describe the displacement of S from its relative equilibrium position, we may write

$$z(t) \equiv x(t) - \frac{a}{1-\eta^2} = A\sin\Omega t + B\cos\Omega t. \tag{6.77h}$$

Consequently, the equation of motion in (6.77b) transforms to the familiar equation

$$\ddot{z} + \Omega^2 z = 0, \tag{6.77i}$$

the differential equation for the simple harmonic oscillator.

The initial values (6.77e) yield $B = a - x_E = -a\eta^2/(1-\eta^2)$ and $A = 0$; so, $z(t) = B\cos\Omega t$. The oscillations occur symmetrically about the relative equilibrium position x_E with the amplitude $z_{\max} = |B| = a\eta^2/(1-\eta^2)$ and circular frequency Ω given in (6.77f). We thus find that the motion of the slider in the case when $\omega < p$ is given by $\mathbf{x}(S, t) = x(t)\mathbf{i} + a\mathbf{j} \overset{\text{or}}{=} (z(t) + x_E)\mathbf{i} + a\mathbf{j}$, in which

$$x(t) = \frac{a}{1-\eta^2}(1 - \eta^2 \cos \Omega t), \qquad z(t) = -\frac{a\eta^2}{1-\eta^2} \cos \Omega t. \tag{6.77j}$$

Case (*ii*): $\eta = 1$; i.e. the angular speed $\omega = p$. The general solution of the differential equation of motion (6.77b), for the initial data (6.77e), is given by

$$x(t) = \frac{1}{2} a p^2 t^2 + a. \tag{6.77k}$$

This result suggests that $\omega = p$ is the critical angular speed of the table at which the motion of S about its relative equilibrium position x_E ceases to be oscillatory and now tends to grow indefinitely with time. The previously stable relative equilibrium position (6.77g) of the slider block about which it oscillates when $\omega < p$, no longer exists, a fact evident from (6.77b) for which no time independent solution exists when $\eta = 1$. And hence, the relative equilibrium position x_E of the slider block is said to be unstable at $\omega = p$. In our study of infinitesimal stability defined later, it is proved that the relative equilibrium state is stable if and only if $\omega < p$. Investigation of the physical nature of the slot reaction force (6.77c), both here and below, is left for the reader.

Case (*iii*): $\eta > 1$; i.e. the angular speed $\omega > p$. The equation of motion (6.77b), in which the coefficient is now negative, has the form of (6.54) in which $q^2 \equiv p^2(\eta^2 - 1)$ and $h(t) \equiv ap^2$. Therefore, with (6.45b) and (6.61) in mind, the general solution of (6.77b) is given by

$$x(t) = A \sinh qt + B \cosh qt - \frac{a}{\eta^2 - 1}, \tag{6.77l}$$

where $q \equiv p(\eta^2 - 1)^{1/2}$. Alternatively, the change of variable $\xi(t) \equiv x(t) + a/(\eta^2 - 1)$ transforms the equation of motion (6.77b) to $\ddot{\xi} - q^2\xi = 0$, an equation of the type (6.58) whose solution is given by (6.59).

The initial data (6.77e) yields $B = a\eta^2/(\eta^2 - 1)$ and $A = 0$. We thus obtain from (6.77l) the relative motion $\mathbf{x}(S, t) = x(t)\mathbf{i} + a\mathbf{j} \overset{\text{or}}{=} (\xi(t) - a/(\eta^2 - 1))\mathbf{i} + a\mathbf{j}$ in which

$$x(t) = \frac{a}{\eta^2 - 1}(\eta^2 \cosh qt - 1), \qquad \xi(t) = \frac{a\eta^2}{\eta^2 - 1} \cosh qt. \tag{6.77m}$$

The motion $x(t)$ relative to the table frame when $\omega > p$ and the slider block is released from rest at $x(0) = a$ thus tends to grow increasingly large with time. At some point, of course, Hooke's law fails, the limiting extensibility of the spring restricts the extent of the motion, and (6.77m) is no longer valid. Notice that the time independent solution of (6.77b) in this case is not a physically meaningful relative equilibrium state.

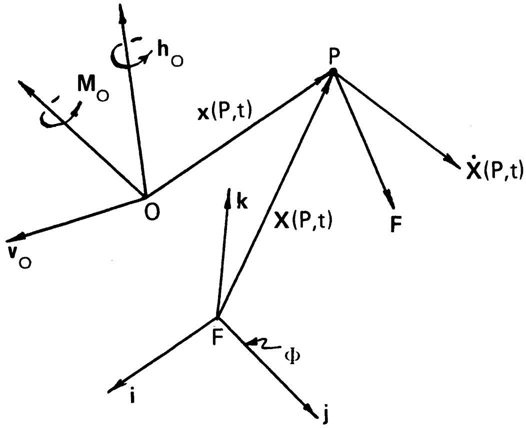


Figure 6.18. Schema for the moment of momentum principle.

6.10. The Moment of Momentum Principle

In this section the Newton–Euler law is applied to derive an additional principle of motion that relates torque and the moment of momentum of a particle. First, however, we recall the definition (5.20) to write *the moment* $\mathbf{M}_O$ *of a force* $\mathbf{F}$ *about a point* O*, either fixed or in motion relative to an assigned frame* Φ:

$$\mathbf{M}_O = \mathbf{x} \times \mathbf{F}, \tag{6.78}$$

in which $\mathbf{x} \equiv \mathbf{x}_O$ is the position vector from O to the particle P on which the total force $\mathbf{F}$ acts, as shown in Fig. 6.18. Let O be a *fixed point* in the inertial frame $\Phi = \{F; \mathbf{i}_k\}$ in Fig. 6.18, so that $\dot{\mathbf{x}} = \dot{\mathbf{X}} = \mathbf{v}$, the velocity of P in Φ. Now recall the definition (5.31) of the moment of momentum of a particle P, differentiate it with respect to time, and use (5.34) to obtain

$$\frac{d\mathbf{h}_O}{dt} = \mathbf{x} \times \frac{d\mathbf{p}}{dt} + \mathbf{v} \times m\mathbf{v} = \mathbf{x} \times \mathbf{F}.$$

In view of (6.78), this yields our additional principle of motion.

The moment of momentum principle: *The moment about a fixed point* O *of the total force acting on a particle* P *in an inertial frame* Φ *is equal to the time rate of change of the moment about* O *of the momentum of* P *in* Φ:

$$\mathbf{M}_O = \frac{d\mathbf{h}_O}{dt}. \tag{6.79}$$

6.10.1. Application to the Simple Pendulum Problem

The moment of momentum principle (6.79) provides an alternative and often simpler means to derive the appropriate equation of motion for a particle without our having to address details concerning certain forces of constraint; otherwise, it delivers no more information on the motion than may be obtained from the Newton–Euler law. This is demonstrated in our review of the equation of motion for a simple pendulum.

The forces that act on the bob are shown in Fig. 6.15, page 138. To apply (6.79), we first determine the moment of these forces about the fixed point O. The central directed string tension has no moment about O, while the weight exerts a torque about O given by

$$\mathbf{M}_O = \mathbf{x} \times \mathbf{W} = -mg\ell \sin\theta \mathbf{b},$$

where $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ is a constant unit vector perpendicular to the plane of motion. The moment of momentum of the bob about the fixed point O in Fig. 6.15 is given by $\mathbf{h}_O = \mathbf{x} \times m\mathbf{v} = -\ell\mathbf{n} \times m\ell\dot{\theta}\mathbf{t} = m\ell^2\dot{\theta}\mathbf{b}$, and hence

$$d\mathbf{h}_O/dt = m\ell^2\ddot{\theta}\mathbf{b}.$$

Collecting this data in (6.79), equating the components, and writing $p^2 = g/\ell$, we obtain the equation $\ddot{\theta} + p^2 \sin\theta = 0$ for the angular motion $\theta(t)$ of the pendulum bob, which is the same as the first equation in (6.67b). Because the cord tension has no moment about O, the moment of momentum principle eliminates the need to consider it further in the discussion of the motion of the bob.

6.10.2. The Moment of Momentum Principle for a Moving Point

The moment of momentum principle (6.79) holds only for an arbitrary point O fixed in the inertial frame Φ. We now determine the form of this principle when O is an arbitrary moving point in Φ.

The moment about O of the momentum $\mathbf{p}(P, t) = m(P)\dot{\mathbf{X}}(P, t)$ in the inertial frame Φ is defined by (5.31) in which point O may be either a fixed or a moving moment center. Hence, when O has an arbitrary velocity $\mathbf{v}_O$ in Φ, the derivative of (5.31) with respect to time in Φ is given by

$$\dot{\mathbf{h}}_O = \dot{\mathbf{x}} \times \mathbf{p} + \mathbf{x} \times \mathbf{F},$$

wherein $\dot{\mathbf{x}} = \dot{\mathbf{X}} - \mathbf{v}_O$. Hence, use of (6.78) now yields the *moment of momentum principle for an arbitrary moving reference point* O:

$$\mathbf{M}_O = \dot{\mathbf{h}}_O + \mathbf{v}_O \times \mathbf{p}. \tag{6.80}$$

Therefore, the moment of momentum principle (6.79) *may hold with respect to a moving point O if and only if $\mathbf{v}_O \times \mathbf{p} = \mathbf{0}$, i.e. when and only when the velocity of O is parallel to the velocity of the particle P; otherwise, O must be a fixed point.*

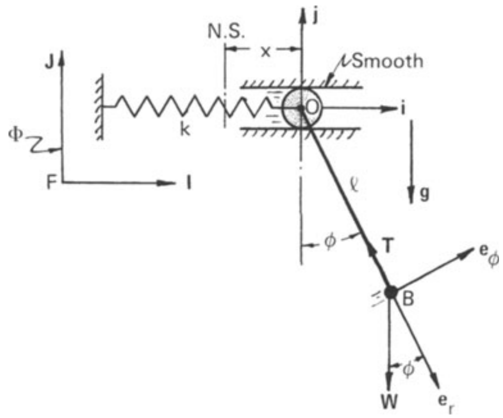


Figure 6.19. Motion of a pendulum having a moving support.

In general, then, the modified principle (6.80) must be used when O is a moving reference point. An application of this rule follows.

Example 6.14. A pendulum bob B attached to a rigid rod of negligible mass and length ℓ is suspended from a smooth movable support at O that oscillates about the natural undeformed state of the spring so that $x(t) = x_O \sin \Omega t$ in Fig. 6.19. Apply equation (6.80) to derive the equation of motion for the bob.

Solution. The forces that act on the pendulum bob B are shown in the free body diagram in Fig. 6.19. Notice that the tension $\mathbf{T}$ in the rod at B is directed through the moving point O. Moreover, the spring force and normal reaction force of the smooth supporting surface also are directed through O; but these forces do not act on B, so they hold no direct importance in its equation of motion. Consequently, the moment about the point O of the forces that act on B at $\mathbf{x}_B = \ell \mathbf{e}_r$ in the cylindrical system shown in Fig. 6.19 is given by

$$\mathbf{M}_O = \mathbf{x}_B \times \mathbf{W} = -\ell W \sin\phi \mathbf{k}. \tag{6.81a}$$

The absolute velocity of B is determined by $\mathbf{v}_B = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{x}_B$, in which $\omega = \dot{\phi}\mathbf{k}$ and $\mathbf{v}_O = \dot{x}\mathbf{i} = x_O \Omega \cos \Omega t \mathbf{i} = v_O \mathbf{i}$. Thus,

$$\mathbf{v}_B = v_O \mathbf{i} + \ell \dot{\phi} \mathbf{e}_\phi, \qquad \text{with} \quad v_O = x_O \Omega \cos \Omega t. \tag{6.81b}$$

With the linear momentum $\mathbf{p} = m\mathbf{v}_B$ and use of (6.81b), we find

$$\mathbf{v}_O \times \mathbf{p} = v_O \mathbf{i} \times m\ell\dot{\phi}\mathbf{e}_\phi = m v_O \dot{\phi} \ell \sin\phi \mathbf{k}. \tag{6.81c}$$

The moment of momentum about O is given by $\mathbf{h}_O = \mathbf{x}_B \times \mathbf{p} = m\ell(v_O \cos\phi + \ell\dot{\phi})\mathbf{k}$, and its time rate of change is

$$\dot{\mathbf{h}}_O = m\ell(a_O \cos\phi - v_O\dot{\phi}\sin\phi + \ell\ddot{\phi})\mathbf{k}, \tag{6.81d}$$

in which $a_O = \dot{v}_O = -x_O\Omega^2 \sin \Omega t$. Substituting (6.81a), (6.81c), and (6.81d) into (6.80), we find $-\ell W \sin\phi \mathbf{k} = m\ell(-x_O\Omega^2 \sin \Omega t \cos\phi + \ell\ddot{\phi})\mathbf{k}$. Hence, with $W = mg$, the equation of motion for the bob may be written as

$$\ddot{\phi} + p^2 \sin\phi = \frac{x_O\Omega^2}{\ell} \cos\phi \sin \Omega t, \tag{6.81e}$$

where $p^2 = g/\ell$. The solution of (6.81e) for small $\phi(t)$ is discussed later in our study of mechanical vibrations. (See Example 6.15, page 161.) □

6.11. Free Vibrations with Viscous Damping

The simple harmonic oscillator is the fundamental model of the theory of mechanical vibrations. Its motion is a perpetual sinusoidal oscillation; once set into motion, the oscillation continues indefinitely. In real situations, however, there usually is a dissipative or viscous drag force, called a *damping force*, that causes the vibration eventually to die out. If the damping force is very small, the simple harmonic oscillator often is a useful model. On the other hand, when friction devices or shock absorbers are used in mechanical systems, it is the intent of the design that their damping effect be considerable. The suspension system of an automobile, for example, is designed to dampen smoothly and quickly the vibrations induced by the irregular motion of the vehicle over a rough road. The viscous damper used to ease the automatic closing of a door and prevent its slamming is another example of the useful effects of damping. Other cases where damping effects are sometimes desirable and sometimes not arise in instrument design. Damping of the potentially violent needle motion of a galvanometer can prevent damage to the instrument when the current is measured, whereas dissipative effects in a gravitometer may seriously affect the accuracy of gravity measurements.

The analysis of induced motion, damped or not, is also important. The motion of a structure induced by an earthquake or by aerodynamic effects of wind, the sudden wing vibration of an aircraft exposed to high winds or turbulence, and the vibration of a vehicle induced by a bumpy road obviously are undesirable but unavoidable environmental effects. On the other hand, magnification of induced motions is essential in the design of seismographs and certain flight instruments.

The analysis of the kinds of problems described above generally is quite complex, especially when vibrational effects are nonlinear; however, a great variety of problems that involve damping and induced motions can be adequately modeled by a simplified damped spring-mass system that consists of a load of mass m, a linear spring of constant stiffness k, and a linear *viscous damper* or *dashpot*. A typical model of a damped spring-mass system is shown in Fig. 6.20.

A dashpot consists of a piston that moves within a cylinder containing a fluid, usually oil. When the piston is moved by the load, it exerts a viscous retarding force on the load. For simplicity, we model this viscous force by Stokes's law (6.29) and

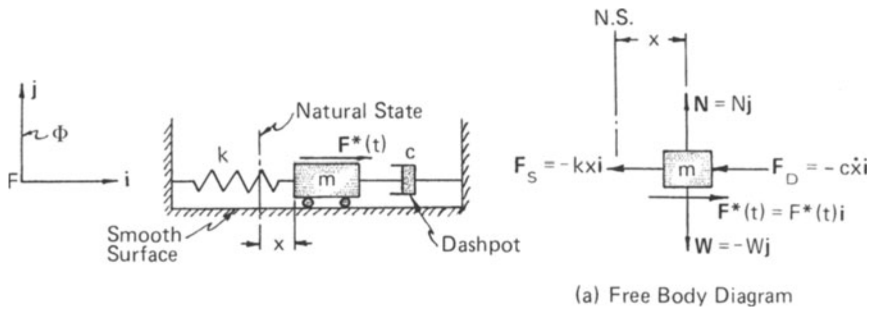


Figure 6.20. Model of a damped spring-mass system.

write $\mathbf{F}_D = -c\dot{x}\mathbf{i}$, in which c is a constant damping coefficient. The spring force is a restoring force given by $\mathbf{F}_S = -kx\mathbf{i}$, where $x(t)$ denotes the displacement of the load from the natural state of the system. The other applied forces in Fig. 6.20 include a disturbing force $\mathbf{F}^*(t) = F^*(t)\mathbf{i}$, attributed to certain environmental effects of the sort mentioned above. The free body diagram in Fig. 6.20a shows that the weight $\mathbf{W}$ is balanced by the normal reaction force $\mathbf{N}$ of the smooth surface, and hence the motion $x(t)$ is determined by the differential equation

$$m\ddot{x} + c\dot{x} + kx = F^*(t). \tag{6.82}$$

If the disturbing force $F^*(t) = F_0$ is constant, the motion is called a *free vibration*; otherwise, it is called a *forced vibration*. When c is zero or may be considered negligible, the motion is said to be *undamped*. The undamped, free vibrational motion is just the simple harmonic motion (6.65a) studied earlier. We next consider the problem of damped, free vibrations of the load.

6.11.1. The Equation of Motion for Damped, Free Vibrations

In a free vibration, the only effect of a constant disturbing force $F^* = F_0$, such as gravity, is to shift the origin to the new position $z \equiv x - x_E$, where $x_E = F_0/k$ is the unique time independent, relative equilibrium solution of (6.82). Therefore, by this simple transformation, all damped, free vibrations of the system in Fig. 6.20 are characterized by *the differential equation for the damped, free vibrational motion of the load m about its relative equilibrium position*:

$$\ddot{z} + 2\nu\dot{z} + p^2 z = 0, \tag{6.83}$$

wherein the coefficients are constants defined by

$$2\nu \equiv \frac{c}{m}, \qquad p \equiv \sqrt{\frac{k}{m}}, \tag{6.84}$$

in which p is the circular frequency of the familiar undamped spring-mass system. The coefficient ν is named the *damping exponent*. The damping coefficient has the physical dimensions $[c] = [FV^{-1}] = [MT^{-1}]$, and hence $[\nu] = [p] = [T^{-1}]$. The dimensionless ratio

$$\zeta \equiv \frac{\nu}{p} = \frac{c}{2mp}, \tag{6.85}$$

is known as the *viscous damping ratio.*

6.11.2. Analysis of the Damped, Free Vibrational Motion

The general solution of (6.83) may be obtained by several methods. One familiar approach is described at the end of this section in an exercise for the reader. Another useful method that simplifies the presentation and emphasizes the physical nature of the damping adopts a trial solution of the form

$$z(t) = e^{-\beta t}u(t). \tag{6.86a}$$

The constant β and the function $u(t)$ are then chosen to eliminate the damping term from the transformed equation for $u(t)$. Substitution of (6.86a) into (6.83) yields

$$\ddot{u} + 2(\nu - \beta)\dot{u} + (\beta^2 - 2\nu\beta + p^2)u = 0.$$

We thus choose $\beta = \nu$ to remove the damping term; then $u(t)$ is given by the general solution of the homogeneous equation

$$\ddot{u} + r^2 u = 0, \tag{6.86b}$$

wherein, with the aid of (6.85),

$$r^2 \equiv p^2 - \nu^2 = p^2(1 - \zeta^2). \tag{6.86c}$$

Equation (6.86b) has the structure of equation (6.41) whose general solution for $r \neq 0$ is given in (6.43) in which r may be either real or complex. We use this result in (6.86a) to obtain the solution of (6.83) in the general form

$$z(t) = e^{-\nu t}(C_1 e^{irt} + C_2 e^{-irt}), \tag{6.86d}$$

in which $C_1,\ C_2$ are arbitrary constants. The role of the damping exponent ν is now clear. From (6.86c), there are three physical cases to consider: $\nu < p,\ \nu > p,\ \nu = p$. In the latter case, $r \equiv 0$ and we need only solve the equation $\ddot{u}(t) = 0$. We shall begin with the case for which $\nu < p$.

Case (i): Lightly damped motion. If $\zeta = \nu/p < 1$, then $r^2 > 0$ in (6.86c); hence (6.86b), with $r \equiv \omega > 0$, has the general solution $u(t) = A\cos\omega t + B\sin\omega t$, wherein

$$\omega \equiv p\sqrt{1 - \zeta^2} < p. \tag{6.86e}$$

Therefore, the general solution of (6.83) provided by (6.86a) is

$$z(t) = e^{-\nu t}(A\cos\omega t + B\sin\omega t), \tag{6.86f}$$

wherein A and B are real constants determined by the initial data.

The solution (6.86f) is oscillatory but not periodic. Because of the *damping factor* $e^{-\nu t}$, the oscillations decay in time so that $z \to 0$ as $t \to \infty$; but in its oscillatory motion the load returns again and again to the relative equilibrium state at $z = 0$. In fact, by (6.86f), if the mass passes through $z = 0$ in a given direction at time t_o, then at time $t = t_o + 2\pi/\omega$ it will pass $z = 0$ again in the same direction. The time $\tau = 2\pi/\omega$, therefore, is called the *period of the lightly damped motion*, and the constant ω defined in (6.86e) is named the *damped circular frequency*. Hence,

$$f_d \equiv \frac{1}{\tau} = \frac{\omega}{2\pi} \tag{6.86g}$$

defines the *frequency of the damped, free vibration*. Notice, however, that the motion itself in (6.86f) is not periodic, because $z(t+\tau) \neq z(t)$.

The lightly damped motion (6.86f) may also be visualized from its equivalent form

$$z(t) = z_0 e^{-\nu t}\cos(\omega t + \lambda) \overset{\text{or}}{=} z_0 e^{-\nu t}\sin(\omega t + \psi), \tag{6.86h}$$

in which z_0 and λ (or ψ) are integration constants. The graph of the first equation is illustrated in Fig. 6.21. The initial displacement is $z_0 \cos\lambda$. The initial phase

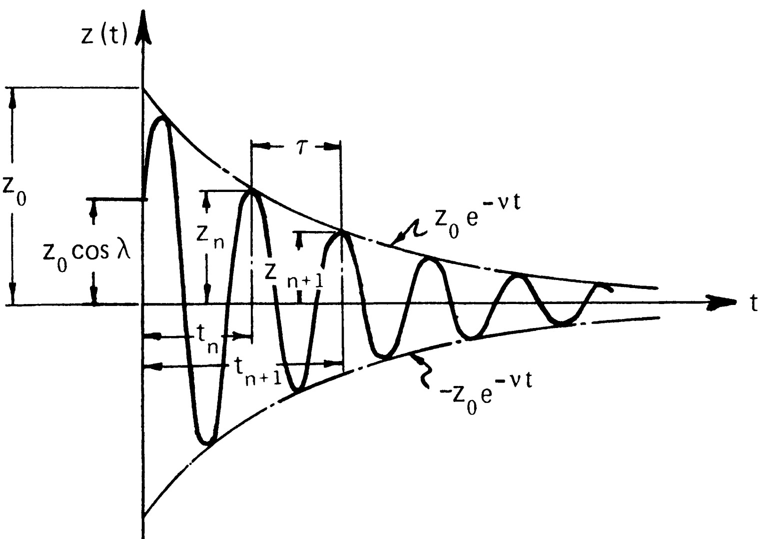


Figure 6.21. Graph of the motion of a lightly damped harmonic oscillator.

λ, however, may be chosen to adjust the time origin so that z_0 is the initial displacement. The damping factor $e^{-\nu t}$ reduces in time the amplitudes of successive oscillations; these occur in time τ. We see from (6.86e) that the *damped circular frequency ω is smaller than the circular frequency p for the undamped, simple harmonic case. Therefore, the effect of damping is to decrease the frequency of the oscillations compared with those of the undamped case.* However, if $\nu \ll p$, so that the damping is very slight, the term $e^{-\nu t}$ stays close to unity for large values of t, and (6.86f) models more precisely the actual physical behavior of the ideal simple harmonic oscillator.

An oscillographic recording of the motion in Fig. 6.21 may be obtained by experiment, and this graph can be used to determine the damping parameters from measurements of any two successive amplitudes at times t_n and $t_{n+1} = t_n + \tau$. Although the peak values of $z(t)$ do not quite touch the exponential envelope lines, they often are sufficiently close for practical experimental purposes. With (6.86h) and $z_n = z(t_n)$, we find $z_n/z_{n+1} = e^{\nu\tau}$. Thus, the natural logarithm of this ratio, called the *logarithmic decrement* Δ, determines ν and hence c in terms of measurable quantities:

$$\Delta \equiv \log \frac{z_n}{z_{n+1}} = \nu\tau. \tag{6.86i}$$

Therefore, with (6.86g), the damping exponent is determined by $\nu = f_d\Delta$, and (6.84) yields the damping coefficient $c = 2mf_d\Delta = 2m\Delta/\tau$. Alternatively, with the aid of (6.85) and (6.86e) in (6.86i), Δ may be written in terms of the viscous damping ratio ζ; we find $\Delta = 2\pi\nu/\omega = 2\pi\zeta/(1-\zeta^2)^{1/2}$. Then ζ may be expressed in terms of the frequency ratio $\omega/p = f_d/f$ or the logarithmic decrement Δ, which are measurable quantities, to obtain $\zeta = (1-(f_d/f)^2)^{1/2} = \Delta/(4\pi^2+\Delta^2)^{1/2}$.

It is useful to observe for the experimental situation that the damping parameters can be evaluated by use of data for any number of complete cycles in the oscillograph record in Fig. 6.21. Let z_1 and z_{n+1} denote the measured amplitudes at times t_1 and $t_1 + n\tau$, for integers $n = 1, 2, \ldots$. Then, in view of (6.86i) applied in turn to each n in the set just indicated,

$$\log\left(\frac{z_1}{z_{n+1}}\right) = \log\left(\frac{z_1}{z_2}\cdot\frac{z_2}{z_3}\cdot\frac{z_3}{z_4}\cdots\frac{z_n}{z_{n+1}}\right) = n\log\left(\frac{z_n}{z_{n+1}}\right) = n\Delta.$$

Therefore,

$$\Delta = \frac{1}{n}\log\left(\frac{z_1}{z_{n+1}}\right), \tag{6.86j}$$

which may be used to determine the damping parameters ν, c, and ζ, as shown above. This rule is particularly helpful in reducing experimental measurement error when recorded successive amplitudes are so close together that even small measurement errors in the amplitude and period will generate significant errors in data used to compute the damping parameters.

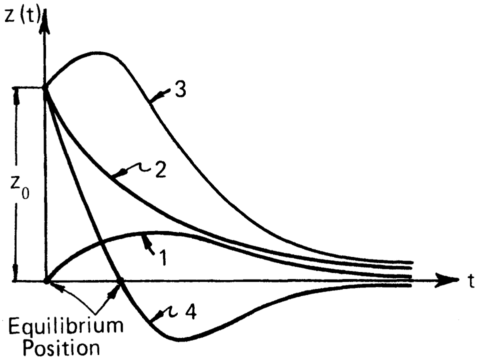


Figure 6.22. Graph of four typical motions of a heavily damped system.

Case (ii): Heavily damped motion. If $\zeta = \nu/p > 1$, then $r^2 = -q^2 < 0$ in (6.86c), where

$$q \equiv \sqrt{\nu^2 - p^2} = p\sqrt{\zeta^2 - 1} < \nu. \tag{6.86k}$$

Hence, with $r = \pm iq$ in (6.86d), the general solution of (6.83) in this case is

$$z(t) = e^{-\nu t}(Ae^{qt} + Be^{-qt}). \tag{6.86l}$$

The constants A and B are determined by the initial data. Equation (6.86l) may also be expressed in terms of hyperbolic functions.

This motion is not oscillatory. Since $q < \nu$, the damping factor $e^{-\nu t}$ is dominant; so, whatever initial conditions may be assigned, once the particle passes through its relative equilibrium position, if at all, it will never do so again. The unique null solution of (6.86l) is obtained in the time

$$t_o = \frac{\log(-B/A)}{2q}. \tag{6.86m}$$

The viscosity in a heavily damped system is so great that the load cannot vibrate about its relative equilibrium position; rather, it must creep slowly back to it as $t \to \infty$.

Some typical cases are shown in Fig. 6.22. Curve 1 occurs for the initial conditions $z(0) = 0,\ \dot{z}(0) = v_0$, from which $-B/A = 1$ and hence (6.86m) has only the trivial solution $t_o = 0$. This motion begins with a push away from the equilibrium position and the mass can never cross it again; for, $z \to 0$ again only as $t \to \infty$. Curve 2 in Fig. 6.22 illustrates the case $z(0) = z_0,\ \dot{z}(0) = 0$. For the general case $z(0) = z_0,\ \dot{z}(0) = v_0$, the motion may resemble either curve 2, 3, or 4. In the last instance, the load passes through its equilibrium position only once and then creeps gradually back to it from below. See Problem 6.62.

Case (iii): Critically damped motion. If $\zeta = \nu/p = 1$, the general solution of (6.86b) for which $r^2 = 0$ is $u = A + Bt$, where A and B are integration constants.

Thus, by (6.86a), the general solution of (6.83) for the critically damped motion is

$$z(t) = (A + Bt)e^{-\nu t}. \tag{6.86n}$$

As $t \to \infty$, the motion $z(t) \to 0$. The critically damped motion, therefore, is similar to that for the heavily damped model illustrated in Fig. 6.22. Discussion of the motion graphs is left for the reader in Problem 6.63.

From (6.85), the damping coefficient for this case has the value

$$c^* = 2mp = 2\sqrt{mk}, \tag{6.86o}$$

which is named the *critical damping coefficient*. This is the value of the damping coefficient at which the motion loses its oscillatory, lightly damped character in transition to a nonvibratory, heavily damped decaying motion. In view of (6.86o), the damping ratio (6.85) in the general case is the ratio of the damping coefficient to its critical value:

$$\zeta = \frac{c}{c^*} = \frac{\nu}{p}. \tag{6.86p}$$

In both the oscillatory lightly damped case $\zeta < 1$ and the nonoscillatory heavily damped case $\zeta > 1$, the load takes a longer time to come to rest than it does in the critically damped case $\zeta = 1$. This effect is illustrated by the familiar automatic storm-door closer. If the closer mechanism is adjusted to have light damping, the door will want to swing through its closed equilibrium position in an effort to oscillate, so the door will slam. If the closer is adjusted to have too much damping, the heavily damped door will close too slowly, perhaps not at all. The optimum case is when the closer is critically adjusted so that the door will close as quickly as possible, without slamming. Thus, the critical damping case $\zeta = 1$ describes the most efficient damping condition, because the motion is damped in the least time.

6.11.3. Summary of Solutions for the Damped, Free Vibrational Motion

For the damped, free vibrational motions, $z(t) \to 0$ as $t \to \infty$, so all of these motions eventually die out. To summarize, equation (6.83) for the damped, free vibrational motion of the load about its relative equilibrium position is characterized by three physical situations depending on the value of the viscous damping ratio $\zeta = \nu/p = c/2mp$:

- Lightly damped motion, $\zeta < 1$:

$$z(t) = e^{-\nu t}(A\cos\omega t + B\sin\omega t), \qquad \omega = p\sqrt{1-\zeta^2}. \tag{cf. 6.86f}$$

- Heavily damped motion, $\zeta > 1$:

$$z(t) = e^{-\nu t}(Ae^{qt} + Be^{-qt}), \qquad q = p\sqrt{\zeta^2 - 1}. \tag{cf. 6.86l}$$

- Critically damped motion, $\zeta = 1$:

$$z(t) = e^{-\nu t}(A + Bt). \qquad \text{(cf. 6.86n)}$$

The reader may explore the following additional elements.

Exercise 6.4. The usual solution method for linear equations with constant coefficients adopts a trial solution $z_T = Ae^{\lambda t}$. Find the characteristic equation for λ in order that (6.83) may be satisfied. Determine its roots, and thus show that the solution of (6.83) is given by (6.86d). □

Exercise 6.5. The method based on (6.86a) may be applied more generally in problems where the coefficients ν and p^2 in (6.83) are functions of time. Let $z(t) = u(t)e^{h(t)}$ and find $h(t)$ and $r^2(t)$ in order that (6.83) may be transformed to an equation of the form (6.86b) for the function $u(t)$. The solution $u(t)$ will now depend on the nature of the function $r(t)$; so, in general, $u(t)$ need not be a periodic function. □

6.12. Steady, Forced Vibrations with and without Damping

The oscillatory motion of a mechanical system subjected to a time varying external disturbing force is called a *forced vibration*. In this section, we investigate the forced vibration of the system in Fig. 6.20 due to a steady, sinusoidally varying disturbing force

$$F^*(t) = F_0 \sin \Omega t. \tag{6.87}$$

The constant F_0 is the *force amplitude* and the constant circular frequency Ω is called the *forcing* or *driving frequency*.

The motion of a load induced by a time varying driving force of the kind (6.87) is known as a *steady, forced vibration;* otherwise, the response is called *unsteady* or *transient*. In general, a vibratory motion consists of identifiable steady and transient parts. The transient part of the motion eventually dies out, and the subsequent remaining part of the motion is called the *steady-state vibration*. A disturbing force that changes suddenly by a constant value, called a step function, and an impulsive exciting force which is suddenly applied for only a very short time, are examples of forces for which the response is transient. Some other examples are described in the problems. In the text, however, we shall explore only the steady, forced vibration problem for which the equation of motion (6.82) has the form

$$\ddot{x} + 2\nu\dot{x} + p^2 x = Q \sin \Omega t, \tag{6.88}$$

in which ν and p are defined in (6.84) and

$$Q \equiv \frac{F_0}{m}. \tag{6.89}$$

We recall that p is the free vibrational circular frequency of the undamped oscillator; it is the intrinsic frequency of the system. Therefore, for future clarity and brevity, p is called the *natural (circular) frequency*.

The general solution x_H of the homogeneous equation associated with (6.88) when $Q = 0$ is given by (6.65b) when $\nu = 0$, and by (6.86f), (6.86l), or (6.86n), according as $0 < \zeta = \nu/p < 1, \zeta > 1$, or $\zeta = 1$, respectively, as summarized earlier (page 157). Consequently, the general solution of (6.88) is obtained by adding to the appropriate homogeneous solution $x_H = e^{-\nu t}u(t)$ a particular solution x_P of (6.88) that gives the effect of the external force.

A particular solution of (6.88) may be obtained by the method of undetermined coefficients. Accordingly, we take $x_P = C_1 \sin \Omega t + C_2 \cos \Omega t$, where $\Omega \neq p$ is the forcing frequency and the constants C_1, C_2 are chosen to satisfy (6.88) identically. Substitution of x_P into (6.88) yields

$$\left[(p^2 - \Omega^2)C_1 - 2\nu\Omega C_2 - Q\right] \sin \Omega t + \left[2\nu\Omega C_1 + (p^2 - \Omega^2)C_2\right] \cos \Omega t = 0,$$

which holds identically for all t if and only if the coefficients vanish. This provides two equations for the constants C_1 and C_2, which yield

$$C_1 = \frac{X_S(1-\xi^2)}{(1-\xi^2)^2 + (2\xi\zeta)^2}, \qquad C_2 = \frac{-2X_S\xi\zeta}{(1-\xi^2)^2 + (2\xi\zeta)^2}, \tag{6.90a}$$

wherein, by definition,

$$X_S \equiv \frac{Q}{p^2} = \frac{F_0}{k}, \qquad \xi \equiv \frac{\Omega}{p}, \tag{6.90b}$$

and ζ is the viscous damping ratio defined in (6.86p). Notice that X_S is the static deflection of the spring due to F_0, and ξ is the ratio of the forcing frequency to the natural frequency.

The general solution of (6.88) is the sum $x(t) = x_H + x_P$. This gives the forced vibrational motion

$$x(t) = e^{-\nu t}u(t) + C_1 \sin \Omega t + C_2 \cos \Omega t, \tag{6.90c}$$

provided that $\Omega \neq p$. The first term in (6.90c) is the transient part of the motion. It describes the damped, free vibrational part of the motion for which $u(t)$ is identified in (6.86f) for the lightly damped problem, in (6.86l) for the heavily damped case and in (6.86n) for the critically damped problem. In any event, the transient, damped part of the motion (6.90c) vanishes as $t \to \infty$, and the motion attains the *steady-state* simple harmonic form described by the last two terms. When $\nu = 0$, however, $u(t)$ is the simple harmonic solution of (6.68); and this part of the undamped, forced vibrational motion (6.90c) does not die out, it is not a transient motion. Nevertheless, the part of the undamped, forced vibrational motion described by the last two terms in (6.90c) is still named the steady-state part. Thus, in every case the effect of the sinusoidal driving force is to superimpose on the free, damped or undamped vibrational motion a simple harmonic motion whose frequency Ω equals that of the driving force (6.87) and whose steady-state amplitude, in accordance with (6.90a), is defined by

$$H \equiv \sqrt{C_1^2 + C_2^2} = \frac{X_S}{\sqrt{(1-\xi^2)^2 + (2\xi\zeta)^2}}. \tag{6.90d}$$

See (6.71). The steady-state amplitude is constant for a fixed value of Ω, hence ξ; but it grows larger as $\xi \to 1$, that is, as the forcing frequency Ω approaches the natural frequency p.

The foregoing results are used to study the ideal undamped, and the lightly, heavily, and critically damped vibration problems. We begin with the undamped case.

6.12.1. Undamped Forced Vibrational Motion

The equation for the undamped forced vibrational motion of the load is obtained from (6.88) with $\nu = p\zeta = 0$:

$$\ddot{x} + p^2 x = Q \sin \Omega t. \tag{6.91}$$

We recall (6.90a) and (6.90d) to obtain $(C_1, C_2) = (H, 0)$; then (6.90c), in which $u(t)$ is the simple harmonic solution of (6.68), yields the general solution of (6.91):

$$x(t) = A \cos pt + B \sin pt + H \sin \Omega t, \tag{6.92a}$$

where, with (6.90b),

$$H = \frac{Q}{p^2(1-\xi^2)} = \frac{F_0/k}{1-\xi^2} = \frac{X_S}{1-\xi^2}, \qquad \xi = \frac{\Omega}{p} \neq 1. \tag{6.92b}$$

The motion (6.92a) is the superposition of two distinct simple harmonic motions. The first two terms, which contain the two integration constants, represent an undamped, free vibration of circular frequency p. The third term is the steady-state, forced vibrational contribution; it depends on the driving force amplitude in (6.92b) but is independent of the initial data and has the same circular frequency Ω as the disturbing force. In general, the two motions have different amplitudes, frequencies, and phase. Therefore, their composition, and hence the motion, is not periodic unless the ratio $\xi = \Omega/p$ is a rational number, or unless A and B are zero. Thus, the undamped, forced vibrational motion (6.92a) usually is a complicated aperiodic motion.

Suppose, for example, that the system is given an initial displacement $x(0) = x_0$ and velocity $\dot{x}(0) = v_0$. Then (6.92a) yields $A = x_0$ and $B = (v_0 - H\Omega)/p$, and the undamped, forced vibrational motion is described by

$$x(t) = x_0 \cos pt + \frac{v_0}{p} \sin pt + H(\sin \Omega t - \xi \sin pt). \tag{6.92c}$$

Even if the system were started from its natural rest state so that $x_0 = v_0 = 0$, the solution $x(t) = H(\sin \Omega t - \xi \sin pt)$ still contains both free and forced vibration terms. This motion generally is not periodic. Suppose, however, that the initial data may be chosen so that $x_0 = 0$ and $v_0 = H\Omega$ for a fixed forcing frequency. Then $A = B = 0$ and the motion (6.92a) reduces to the steady-state, periodic motion $x(t) = H \sin \Omega t$.

The effects of damping and the critical case when $\xi = 1$ will be discussed momentarily. First, we consider an example that illustrates the application of these results to a mechanical system.

Example 6.15. The equation for the undamped, forced vibration of the pendulum device described in Fig. 6.19, page 150, is given in (6.81e). Solve this equation for the case when both the motion of the hinge support and the angular motion of the pendulum are small. Assume that the pendulum is released from rest at a small angle ϕ_0.

Solution. The differential equation (6.81e) describes a complicated nonlinear, undamped, forced vibrational motion of the pendulum. To simplify matters, we consider the case when the angular placement is sufficiently small that terms greater than first order in ϕ may be ignored. Then (6.81e) simplifies to

$$\ddot{\phi} + p^2 \phi = \frac{x_O \Omega^2}{\ell} \sin \Omega t, \tag{6.93a}$$

where $p^2 = g/\ell$. This equation has the same form as (6.91); it describes the small, undamped, steady forced vibrational motion of the pendulum. For consistency with the small motion assumption, however, we consider only the case for which the motion of the hinge support O also is small, so that $x_O/\ell \ll 1$. Because the amplitude of the disturbing force in (6.93a) varies with its frequency, for small motions $\phi(t)$, the range of operating frequencies also is limited.

The general solution of (6.93a), with $Q \equiv x_O \Omega^2/\ell$, may be read from (6.92a):

$$\phi(t) = A \cos pt + B \sin pt + H \sin \Omega t, \tag{6.93b}$$

in which A and B are constants and the steady-state amplitude, by (6.92b), is

$$H = \frac{x_O \xi^2}{\ell(1 - \xi^2)}, \qquad \xi \equiv \frac{\Omega}{p} \neq 1. \tag{6.93c}$$

The assigned initial data determine the constants in (6.93b),

$$\phi(0) = A = \phi_0, \qquad \dot{\phi}(0) = Bp + H\Omega = 0, \tag{6.93d}$$

which then yields the solution for the small angular motion of the pendulum:

$$\phi(t) = \phi_0 \cos pt + \frac{x_O \xi^2}{\ell(1-\xi^2)} (\sin \Omega t - \xi \sin pt). \tag{6.93e}$$

It is evident that this small motion solution is meaningful only for sufficiently small values of the driving frequency ratio ξ; otherwise, the smallness of $\phi(t)$ is violated. □

6.12.1.1. *The Resonance Phenomenon*

As $\xi = \Omega/p \to 1$, the motion (6.92c) in response to the driving force grows increasingly larger, and at $\xi = 1$, its amplitude (6.92b) is infinite. The condition $\xi = \Omega/p = 1$ when the forcing frequency is tuned to the natural frequency of the system is known as *resonance*. It is useful to examine the solution for the undamped motion at the resonant frequency.

Let $x^*(t)$ denote the motion at the resonant frequency $\Omega = p$, and recall (6.92b). Then from (6.92c), we evaluate $x^*(t) = \lim_{\Omega \to p} x(t)$ to obtain

$$x^*(t) = (x_0 - Kt)\cos pt + \frac{1}{p}(v_0 + K)\sin pt, \qquad K \equiv \frac{p}{2} X_S = \frac{F_0}{2mp}.$$

This is not a steady-state motion; its amplitude increases continuously with time, so the vibrations grow increasingly larger. Although the condition of resonance does not occur instantaneously, the motion of the load may grow excessively and exceedingly large in a short time.

6.12.1.2. *Steady-State Amplitude Factors*

Two kinds of dimensionless *amplitude factors* arise often in forced vibration problems, both characterize the steady-state response of the system in terms of the frequency ratio. One of these amplitude factors, defined by

$$\alpha_0 \equiv \frac{1}{1-\xi^2}, \tag{6.94a}$$

called the *magnification factor,* appears in the steady-state amplitude relation (6.92b). The magnification factor is the ratio of the steady-state dynamic response amplitude H to the static amplitude X_s of the system, hence $\alpha_0 = H/X_s$ is a measure of the dynamic displacement compared to the static displacement of the load.

A different dimensionless amplitude factor, defined by

$$\alpha_1 \equiv \frac{\xi^2}{1-\xi^2}, \tag{6.94b}$$

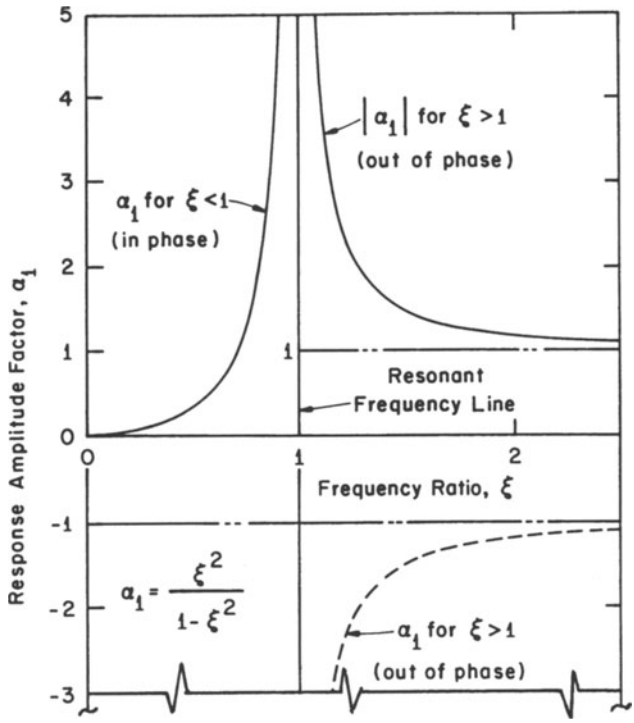


Figure 6.23. Response amplitude factor $\alpha_1(\xi)$ for steady-state forced vibrations without damping as a function of the system frequency ratio $\xi = \Omega/p$.

appears in the steady-state amplitude relation (6.93c) for the forced vibration of the pendulum. Since $\phi\ell$ describes the small horizontal motion of the pendulum bob, we see that $H\ell$ is its maximum value in the steady-state motion $\phi_\sigma \equiv H \sin \Omega t$. Thus, the amplitude factor in this case, according to (6.93c), is the ratio of the dynamical amplitude $H\ell$ of the bob to the amplitude x_O of the support; hence α_1 is a measure of the dynamical response of the system.

Graphs of the amplitude factors (6.94b) and (6.94a) are shown in Figs. 6.23 and 6.24, respectively. These *response graphs* are independent of the particular physical problems in which these amplitude factors may arise. The general physical relevance of (6.94b), however, is readily illustrated in connection with the driven pendulum example.

The map of (6.94b) is shown in Fig. 6.23. Accordingly, at small operating frequencies ξ, the amplitude factor α_1 also is small, both near zero. Thus, the influence of the vibrating support on the small amplitude oscillations of the pendulum is insignificant, and the motion in (6.93b) is essentially a simple harmonic motion of natural frequency p. Moreover, for $\xi < 1, \alpha_1 > 0$ and $H = x_O\alpha_1/\ell > 0$.

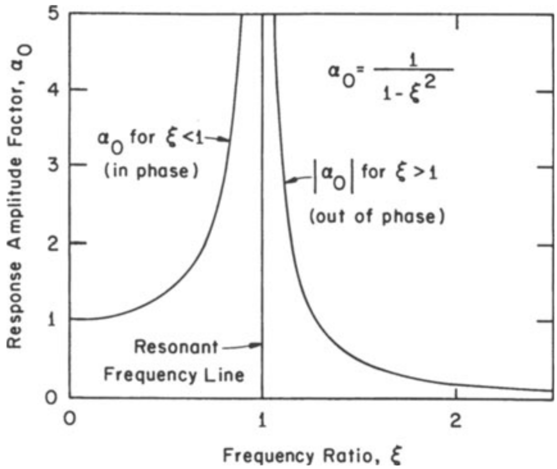


Figure 6.24. Response amplitude factor $\alpha_0(\xi)$ for steady-state forced vibrations without damping as a function of the system frequency ratio $\xi = \Omega/p$.

Therefore, the steady-state motion $\phi_\sigma \equiv H \sin \Omega t$ of the bob is in phase with the driving force (6.87); that is, the bob's motion is in the direction in which the support is moving. This is characterized by the solid left-hand curve in Fig. 6.23. At resonance, the forcing frequency is tuned to the natural frequency at $\xi = 1$, and therefore the amplitude factor (6.94b), and hence the amplitude of the pendulum motion, becomes infinite, as indicated by the vertical line in the response graph. But this is not an instantaneous effect, rather it indicates a growth in the amplitude in time, growth which eventually violates the small amplitude motion assumption used in the solution. When $\xi > 1$, the amplitude factor $\alpha_1 < 0$, and hence $H = x_O\alpha_1/\ell < 0$ also. Thus, the steady-state response of the pendulum, the part $\phi_\sigma = H \sin \Omega t = |H| \sin(\Omega t \pm \pi)$, is simple harmonic and 180° out of phase with the driving force (6.87); that is, the bob's motion is opposite to the direction in which the support is moving. This case is represented by the dotted response curve in Fig. 6.23. At high operating frequencies for which $\xi \gg 1, \alpha_1 \to -1$; that is, $H \to -x_O/\ell$. Because $x_O \ll \ell$, the high frequency, steady-state dynamical amplitude of the pendulum swing will be small, and the steady-state pendulum motion (6.93b) is a high frequency, simple harmonic vibration, but 180° out of phase with the motion of the support. For graphical convenience, it is customary to plot the absolute value of the amplitude factor. When this is done for α_1, the dotted curve in Fig. 6.23 is transformed into its mirror reflection shown as the solid right-hand curve above it.

Interpretation of the general physical relevance of the magnification factor (6.94a) in its relation to the response graph shown in Fig. 6.24 is a bit different. In accordance with (6.92b), for a small operating frequency the magnification factor

$\alpha_0 \approx 1$, as shown in Fig. 6.24. This means that the steady-state motion of the mass shown in Fig. 6.20 has an amplitude equal to the static displacement of the spring due to a force F_0. The motion is in phase with the driving force, so the mass moves in the direction of this force. As $\xi \to 1$ at resonance, the amplitude grows indefinitely great, as described earlier. Beyond resonance $\xi > 1$; so, the steady-state motion in (6.92a) is out of phase with the driving force, and hence the mass in Fig. 6.20 moves in a direction opposite to the disturbing force. Under a high frequency driving force for which $\xi \to \infty$ in Fig. 6.24, the steady-state amplitude response $\alpha_0(\xi) \to 0$, and hence the steady-state amplitude in (6.92b) approaches zero. Therefore, the high frequency vibration of the supporting structure has virtually no effect on the motion of the system, and the mass in Fig. 6.20 remains essentially stationary.

Of course, some sort of damping or friction is always present in real mechanical systems. Damping effects in the forced vibration of a load are studied next.

6.12.2. Steady-State Vibrational Response of a Damped System

When damping is present, the free vibrational part of the motion, the first term in (6.90c) called the *transient state*, eventually dies out, and the vibrational motion thus converges toward a harmonic motion having the same frequency as the disturbing force, the steady-state heartbeat of the system. In consequence, only the steady-state part of the motion (6.90c) of a damped system need be considered.

Let x_σ denote the steady-state solution. Then by (6.90c)

$$x_\sigma = H \sin(\Omega t - \lambda), \tag{6.95a}$$

where H is defined in (6.90d) and, from (6.90a), the initial phase λ is given by

$$\tan \lambda = -\frac{C_2}{C_1} = \frac{2\xi\zeta}{1-\xi^2}. \tag{6.95b}$$

Clearly, for $\xi = 1$, $\lambda = 90^\circ$ at resonance; and in this case, when $\Omega t = \pi/2$, $F^* = F_0$ in (6.87) and $x_\sigma = 0$ in (6.95a). Hence, at resonance, the vibrating body in Fig. 6.20 is moving through its mid position in its steady-state motion at the same instant when the driving force is at its greatest value. Notice that the *response amplitude* H in (6.90d) *does not depend on any initial data*. Thus, regardless of how the system may be set into motion initially, after a time, it settles down to the steady-state motion (6.95a) whose amplitude (6.90d) and phase (6.95b) depend upon the damping and frequency ratios.

The *amplitude factor* defined by

$$\alpha \equiv \frac{1}{\sqrt{(1-\xi^2)^2 + (2\xi\zeta)^2}} = \frac{H}{X_s} \tag{6.95c}$$

is a measure of the dynamic response; it is the ratio of the dynamic amplitude H to the static spring deflection X_s of the load due to the maximum disturbing force

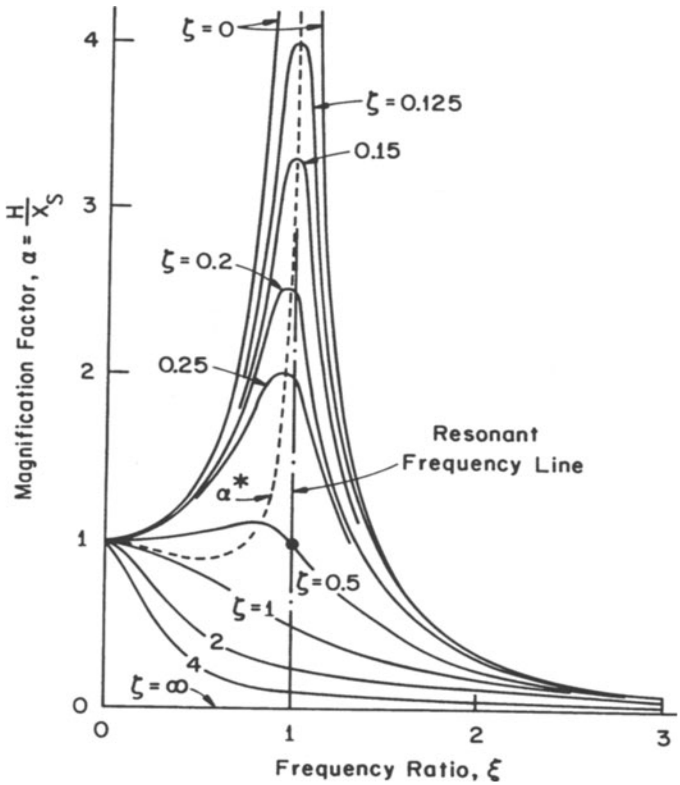


Figure 6.25. Magnification factor as a function of the frequency ratio ξ for various values of the damping parameter ζ in a forced vibration of a system.

F_0. Notice that when $\zeta = 0$, $\alpha = |\alpha_0|$ in (6.94a), and hence α is also known as the *magnification factor*. The *response curves* corresponding to (6.95c) for various values of the damping ratio are shown in Fig. 6.25. The curve for $\zeta = 0$ is the same as the plot of $|\alpha_0|$ in Fig. 6.24.

At low frequencies, $\xi = \Omega/p$ is very close to zero, and (6.95c) shows that α is very nearly equal to 1 in Fig. 6.25. In this case, the disturbing force has such a low frequency Ω in comparison with the undamped natural frequency p that it behaves very nearly as a static dead load; hence H is nearly the same as the static response to the disturbing force: $H = X_s = F_0/k$, very nearly. Notice by (6.95c) that for $\xi = 1$, the curve for $\zeta = \frac{1}{2}$ yields $\alpha = 1$. This is the emphasized point on the resonance line $\xi = 1$ in Fig. 6.25.

At high frequencies, $\xi \gg 1$, and (6.95c) shows that the dynamic response amplitude H becomes very small with α and approaches zero as $\xi \to \infty$ in Fig. 6.25. The frequency of the disturbing force in this instance changes so rapidly that the mass cannot respond but slightly, though at the same frequency, in accordance with (6.95a). Figure 6.25 thus shows that for very small or very large values of ξ, the effect of any sort of damping is insignificant.

At the resonant frequency, the forcing frequency Ω is tuned to the natural frequency p so that $\xi = 1$. Then (6.95c) gives $H = X_s/2\zeta = F_0/2\zeta k$, (6.95b) yields $\lambda = \pi/2$ for the angle by which the disturbing force F^* in (6.87) leads the steady-state motion x_σ in (6.95a), which becomes

$$x_\sigma = -\frac{F_0}{2\zeta k}\cos pt. \tag{6.95d}$$

Hence, if the damping ratio is small, the amplitude of the steady motion may become seriously large when Ω is close to p. Resonance in the undamped system corresponds to $\zeta = 0$ in Fig. 6.25. *The effect of damping is to reduce the response amplitude, and at the resonant frequency ratio* $\xi = 1$ *the reduction may be especially significant.* Thus, the intensity of the resonant motion may be substantially reduced by the introduction of damping in the system.

The peak magnification in the damped motion, however, does not occur at $\xi = 1$. For fixed values of ζ and p, the maximum magnification occurs when ξ has the value

$$\xi^* \equiv \sqrt{1 - 2\zeta^2}. \tag{6.95e}$$

This is known as the *damped resonant frequency ratio* and $\Omega^* = p\xi^*$ is called the *damped resonant forcing frequency*. From (6.95e), *the peak frequency* Ω^* *occurs at a ratio* ξ^* *which is somewhat smaller than* 1, *depending upon the degree of damping.*

At the damped resonant frequency ratio ξ^*, the maximum dynamic amplitude is H^* and the magnification factor (6.95c) has the maximum value

$$\alpha^* = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \equiv \frac{H^*}{X_S}, \tag{6.95f}$$

which depends on the damping ratio. The locus of these maxima, indicated by the dotted curve in Fig. 6.25, shows that the peak value α^* increases as the damping ratio ζ decreases. Since ζ usually is much less than 1, (6.95e) shows that $\xi^* = 1$; that is, the value of the lightly damped resonant forcing frequency Ω^* differs very little from the undamped, free vibrational frequency p of the system. In this case, from (6.95f), the maximum dynamic amplitude at the damped resonant frequency is $H^* \equiv X_s\alpha^* = X_s/2\zeta$, very nearly. For small damping the amplitude is greatest near the resonant frequency ratio $\xi = 1$. As ζ increases, α^* decreases and shifts toward the left until it reaches $\alpha^* = 1$ at $\xi^* = 0$ for $\zeta = \sqrt{2}/2$. Afterwards, the peak $\alpha^* = 1$ is a relative maximum value for all $\zeta > \sqrt{2}/2$, and (6.95f) is no longer applicable.

6.12.3. Force Transmissibility in a Damped System

The vibrating load in its steady-state obviously transmits force to the supporting structure of the system. Therefore, it is important to have a measure of the intensity of this force. In this section, a certain transmissibility factor is introduced, and effects due to variation in the damping and in the operating frequency are discussed.

In the steady-state motion (6.95a), the spring and damping forces for the mechanical system in Fig. 6.20 are given by

$$F_S = kx_\sigma = kH\sin(\Omega t - \lambda), \qquad F_D = c\dot{x}_\sigma = c\Omega H\cos(\Omega t - \lambda), \tag{6.96a}$$

whose amplitudes are $\hat{F}_S = kH$ and $\hat{F}_D = cH\Omega$. Each force in (6.96a) contributes to the *total force transmitted to the support*: $F_S + F_D = F_T\sin(\Omega t - \lambda + \psi)$ where $\tan\psi \equiv c\Omega/k$ and the maximum impressed force, denoted by F_T, is defined by

$$F_T \equiv \sqrt{\hat{F}_S^2 + \hat{F}_D^2} = H\sqrt{k^2 + c^2\Omega^2}. \tag{6.96b}$$

Then the ratio of the total impressed force to the maximum value of the disturbing force $F_0 = kX_s$ defines the *transmission ratio* T_R, also known as the *transmission factor* or the *transmissibility*. Thus, with (6.95c), we find the *transmission ratio*

$$T_R = \frac{F_T}{F_0} = \sqrt{\frac{1 + (2\xi\zeta)^2}{(1 - \xi^2)^2 + (2\xi\zeta)^2}}. \tag{6.96c}$$

The graph of the transmission ratio as a function of the frequency ratio $\xi = \Omega/p$ for various values of the damping ratio $\zeta = c/2mp$ is shown in Fig. 6.26. The greatest transmission to the supporting structure for small damping occurs at resonance, and the effect of increased damping is to decrease the amplitude of the transmission and shift it toward the left of the resonant frequency line $\xi = 1$. Notice, however, that a transmission factor $T_R = 1$ occurs at a universal frequency ratio $\xi = \sqrt{2}$ (shown as the small circle in Fig. 6.26), regardless of the amount of damping. For $\xi > \sqrt{2}$, the transmission ratio $T_R < 1$, and hence the transmitted force is smaller than the applied disturbing force. Moreover, the transmission ratio actually is made smaller by decreasing the amount of damping at high operating frequencies. Therefore, less vibrational force is transmitted to the supporting structure. As a result, smoother operation may be expected. At very low operating frequencies, the transmissibility is again close to 1 for all values of the damping. Otherwise, Fig. 6.26 shows that increasing the amount of damping ζ when $0 < \xi < \sqrt{2}$ decreases the maximum transmitted force. In summary, *if* $\xi < \sqrt{2}$, $T_R > 1$ *and greater damping is recommended for smoother operation of the system; however, when* $\xi > \sqrt{2}$, $T_R < 1$ *and decreased damping will result in smoother operation, that is, the effect of the transmitted force intensity is reduced.*

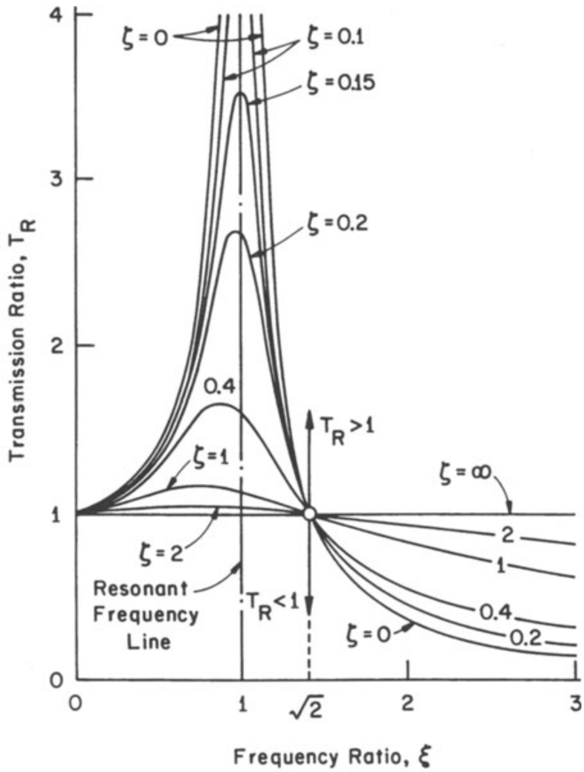


Figure 6.26. Transmissibility as a function of the frequency ratio ξ for various values of the damping ratio ζ in the forced vibration of a system.

For $\xi = \sqrt{2}$, $T_R = 1$ *for every damped (linear) mechanical system.* Mechanical design with these ideas in mind is known as *vibration isolation.*

6.13. Motion under a General Nonlinear Force $f(x, \dot{x})$

So far, we have considered free and forced vibrations of damped and undamped systems subjected to forces that are linear in x and $\dot{x}$. Here we study the motion $x(t)$ of a particle under a general nonlinear force $f = f(x, \dot{x})$ per unit mass. This total force may include inertial forces as well as other sorts of linear and nonlinear contact and body forces. The equation of motion is

$$\ddot{x} = f(x, \dot{x}). \tag{6.97}$$

Although exact solutions of such equations can be obtained, this is not always possible, and the analysis of (6.97) often is difficult. Some readily integrable situations arise when $f(x, \dot{x})$ has special properties. The reader will see easily, for example, that for a nonlinear force of the form $f(x, \dot{x}) = g(x)h(\dot{x})$ for smooth functions $g(x)$ and $h(\dot{x})$, the equation of motion (6.97) has the first integral $\int h^{-1}(\dot{x})d\dot{x}^2 = 2\int g(x)dx + C$, where C is a constant. Another example follows.

6.13.1. Special Class of Nonlinear Equations of Motion

A variety of dynamical systems are characterized by an integrable nonlinear equation of motion (6.97) of the form

$$q(x)\ddot{x} + \frac{1}{2}\dot{x}^2\frac{dq(x)}{dx} = g(x), \tag{6.98a}$$

for any smooth functions $q(x)$ and $g(x)$. This equation may be written as

$$\frac{d}{dx}\left[\frac{1}{2}\dot{x}^2 q(x)\right] = g(x), \tag{6.98b}$$

which is twice integrable. We first derive

$$\dot{x}^2 q(x) = 2\int g(x)dx + C \equiv p(x), \tag{6.98c}$$

where C is a constant, and thus obtain the velocity function

$$v(x) = \dot{x}(x) = \pm\sqrt{\frac{p(x)}{q(x)}}. \tag{6.98d}$$

A second integration yields the travel time in the motion:

$$t = \pm\int\sqrt{\frac{q(x)}{p(x)}}dx + t_0, \tag{6.98e}$$

t_0 denoting the initial instant. In principle, this determines the nonlinear motion $x(t)$; then $v(t)$ can be found from (6.98d). The inversion of (6.98e), however, may require numerical integration. Two explicit examples are provided in the following exercises.

Exercise 6.6. The motion of a particle free to slide on a smooth parabolic wire $y = \frac{1}{2}kx^2$ that rotates about its vertical y-axis with a constant angular speed is described by the nonlinear equation

$$(1 + k^2x^2)\ddot{x} + \Omega x + k^2x\dot{x}^2 = 0,$$

where k and Ω are constants. Derive a first integral for $\dot{x}(x)$. □

Exercise 6.7. The motion of a dynamical system is governed by the equation

$$(h^2 + r^2\theta^2)\ddot{\theta} + r^2\theta\dot{\theta} + kr\theta\cos\theta = 0,$$

where h, k, and r are constants. The system is initially at rest at $\theta = 0$. Derive an integral giving the travel time in the motion. □

6.13.2. Radial Oscillations of an Incompressible Rubber Tube

Nonlinear equations of the type (6.98a) arise often in physical problems. An important example in nonlinear elasticity theory, discovered by J. K. Knowles in 1960, concerns the finite amplitude, free radial oscillations of a very long cylindrical tube made of an incompressible, rubberlike material. The tube has an inner radius r_1 and outer radius r_2 in its undeformed state and is initially inflated uniformly by an internal pressure. A purely radial motion of the tube is induced by its sudden deflation, so that the radial motion of any concentric cylindrical material surface of radius R in the deformed state at time t is described by $R = R(r, t)$, where r is the radius of the corresponding undeformed cylindrical material surface. Let R_1, R_2 respectively denote the inner and outer radii of the deformed tube surfaces at time t. Because of the incompressibility of the material, these radii are related by $R^2 - R_1^2 = r^2 - r_1^2$. Hence, the motion is determined completely if $R_1(t)$ is known. It proves convenient to introduce the dimensionless ratios

$$x(t) \equiv \frac{R_1(t)}{r_1}, \qquad \mu \equiv \frac{r_2^2}{r_1^2} - 1. \tag{6.99a}$$

Knowles found for arbitrary rubberlike materials that the free radial motion of the tube is described by the nonlinear differential equation

$$x \log\left(1 + \frac{\mu}{x^2}\right)\ddot{x} + \left(\log\left(1 + \frac{\mu}{x^2}\right) - \frac{\mu}{\mu + x^2}\right)\dot{x}^2 + h(x, \mu) = 0, \tag{6.99b}$$

where $h(x, \mu)$ is a known function that depends on the constitutive character of the rubberlike material. Notice that while this problem concerns the motion of a highly deformable body, the equation of motion actually involves only the motion of a particle on the inner surface of the tube. All other particles on the inner surface have the same radial motion.

At first glance, equation (6.99b) certainly appears formidable. Upon multiplication by x, however, it is seen that (6.99b) assumes the form (6.98a) and may be written as

$$\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2 x^2 \log\left(1 + \frac{\mu}{x^2}\right)\right) + xh(x, \mu) = 0. \tag{6.99c}$$

This yields the first integral

$$\dot{x}^2 x^2 \log\left(1 + \frac{\mu}{x^2}\right) = -2\int xh(x, \mu)dx + C \equiv p(x). \tag{6.99d}$$

The integration constant C depends on the specified initial data $x(0) = x_0$, $\dot{x}(0) = v_0$. Equation (6.99d) thus determines the radial "velocity" function

$$v(x) = \dot{x}(x) = \pm \sqrt{\frac{p(x)}{x^2 \log\left(1 + \frac{\mu}{x^2}\right)}}, \tag{6.99e}$$

in which $\dot{x} = \dot{R}_1/r_1$, hence $[v(x)] = [T^{-1}]$. The analytical properties of the function $p(x)$ show that the phase plane curves described by (6.99e) are closed and that (6.99e) yields exactly two values $x = a$, $x = b > a$ for which $v(a) = v(b) = 0$; so the motion is periodic. See the referenced paper by Knowles for details.

Integration of (6.99e), with the appropriate sign chosen to render $t > 0$, yields the travel time

$$t = \int_{x_0}^{x} \frac{dx}{v(x)}. \tag{6.99f}$$

The finite periodic time τ of the purely radial oscillations of the tube, the time required for the tube to pulsate from $x = a$ to $x = b$ and back again, is thus determined by the formula

$$\tau = 2\int_{a}^{b} \frac{dx}{v(x)}. \tag{6.99g}$$

It turns out that the exact solution of (6.99g) may be obtained for special kinds of rubberlike materials. Without getting into these matters, however, we see that these general results are useful because they provide physical insight into what is otherwise a very difficult dynamical problem. Some additional simpler examples may be found in the problems at the end of this and subsequent chapters. (See Problems 6.68 and 6.69.) Similar ideas are applied in Chapter 7 to determine exactly the motion and period of the finite amplitude oscillations of a pendulum.

6.14. Infinitesimal Stability of the Relative Equilibrium States of a System

In other problems for which the exact solution of (6.97) is not possible, a variety of analytical and graphical methods described in other works may be used to construct an approximate solution or to study various physical aspects of the motion of the dynamical system. An important physical attribute of particular interest is the infinitesimal stability of the relative equilibrium states of a dynamical system governed by (6.97).

Relative equilibrium solutions of (6.97), if any exist, are the time independent solutions x_E of the equation

$$f(x_E, 0) = 0. \tag{6.100}$$

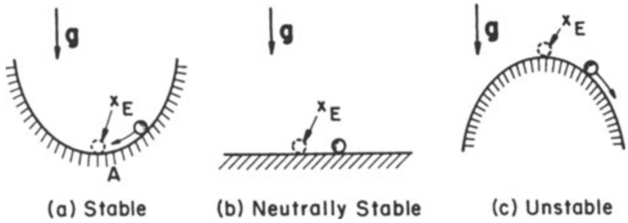


Figure 6.27. Schematic illustrating the concepts of (a) infinitesimal stability, (b) neutral stability, and (c) instability.

This provides the positions x_E at which the mass is at relative rest. In the special case when f is linear in x, there is only one equilibrium solution of (6.100), but for nonlinear systems there may be many relative equilibrium positions. In particular, if $f(x_E, 0)$ is a polynomial in x_E, there are as many equilibrium positions as there are real roots of (6.100); but some of these may not be stable.

The question of how the system behaves if disturbed only slightly from a relative equilibrium position is of special interest. If the body either returns eventually to the relative equilibrium position x_E, or oscillates about x_E so that its motion always remains in a small neighborhood of x_E, the relative equilibrium position is said to be *infinitesimally stable*, or briefly, *stable*. For greater clarity, the term *asymptotically stable* is also used to characterize the relative equilibrium position in the case when the body returns eventually to this state. If the body, following its arbitrary small disturbance from an equilibrium position, remains at a fixed small distance from the relative equilibrium position, the equilibrium state is called *neutrally stable*. On the other hand, if the body moves away indefinitely from x_E, the relative equilibrium state is called *unstable*. These three situations are illustrated in Fig. 6.27 for the small disturbance of a heavy particle from its equilibrium position x_E. The particle will perform small oscillations indefinitely about the equilibrium state at the lowest point of the bowl in Fig. 6.27a, and hence this state is infinitesimally stable. Now suppose the bowl contains water, then the oscillations eventually will die out as the heavy particle settles down to x_E; in this instance x_E is asymptotically stable. If the particle is given a small displacement from x_E on the horizontal plane surface in Fig. 6.27b and released from rest, it will remain there; therefore, the equilibrium state x_E is neutrally stable. Finally, in Fig. 6.27c, if the particle is disturbed only very slightly from its equilibrium position at the vertex of the inverted bowl, it will move away indefinitely from x_E, so this position is unstable.

To investigate the motion in the neighborhood of a relative equilibrium position x_E, we write

$$x(t) = x_E + \chi(t), \tag{6.101}$$

where $\chi(t)$ is a small disturbance from x_E, compatible with any constraints on x, so that $\dot{x} = \dot{\chi}$ also is a small quantity of the same order. The function $f(x, \dot{x})$ is then expanded in a Taylor series about x_E to obtain to the second order in χ and $\dot{\chi}$,

$$f(x, \dot{x}) = f(x_E, 0) + \left.\frac{\partial f(x, \dot{x})}{\partial x}\right|_{x_E} \chi + \left.\frac{\partial f(x, \dot{x})}{\partial \dot{x}}\right|_{x_E} \dot{\chi} + O(\chi^2, \dot{\chi}^2).$$

Thus, recalling (6.100) and (6.101), introducing

$$\alpha \equiv -\left.\frac{\partial f(x, \dot{x})}{\partial \dot{x}}\right|_{x_E}, \qquad \beta \equiv -\left.\frac{\partial f(x, \dot{x})}{\partial x}\right|_{x_E}, \tag{6.102}$$

and neglecting all terms of order greater than the first in χ and $\dot{\chi}$, we thus obtain from (6.97) the *linearized differential equation of motion of the body about the relative equilibrium position* x_E:

$$\ddot{\chi} + \alpha\dot{\chi} + \beta\chi = 0. \tag{6.103}$$

The relative equilibrium position will be stable if and only if the solution $\chi(t)$ of this equation remains bounded for all time t or approaches zero as $t \to \infty$. Otherwise, the initial infinitesimal displacement grows with time and eventually violates the smallness assumptions leading to (6.103); so, the position x_E is unstable.

We recognize that (6.103) is similar to (6.83) for the damped, free vibrations of a body about its relative equilibrium state. Here, however, the constant coefficients obtained from (6.102) are arbitrary; they may be positive, negative, or zero, so all possible solutions of (6.103) must be examined. The usual trial solution $\chi_T = Ae^{\lambda t}$ of (6.103) yields the characteristic equation

$$\lambda^2 + \alpha\lambda + \beta = 0, \tag{6.104a}$$

which has the two solutions

$$\lambda_1 = -\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}, \qquad \lambda_2 = -\frac{\alpha}{2} - \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}. \tag{6.104b}$$

Therefore, the general solution of (6.103) is

$$\chi(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}, \tag{6.104c}$$

in which $A_1,\ A_2$ are arbitrary constants. The physical nature of the solution, and hence the stability of the relative equilibrium positions, is characterized by the signs of α and β, which determine the roots λ_1 and λ_2. There are several cases to explore.

1. Roots λ_1, λ_2 are real and negative. Then (6.104c) shows that $\chi(t) \to 0$ as $t \to \infty$. Therefore, the equilibrium position is asymptotically stable. For real roots (6.104b), $(\alpha/2)^2 > \beta$ must hold. Moreover, $\alpha > 0$ is necessary for a negative root λ_1. If $\beta = 0$ or $\beta < 0$, λ_1 will be non-negative, contrary to the initial requirement. Consequently, it is necessary and sufficient

that $\alpha > 0$, $(\alpha/2)^2 > \beta > 0$ hold. Hence, $\alpha > 0,\ \beta > 0$ in (6.104a) imply asymptotic stability.

2. Roots λ_1, λ_2 are real and positive. Then $\chi(t) \to \infty$ as $t \to \infty$ in (6.104c), and hence the equilibrium position is unstable. Real roots require $(\alpha/2)^2 > \beta$. For $\lambda_2 > 0$, $\alpha < 0$ is necessary, and hence $\beta = 0$ or $\beta < 0$ cannot hold. Therefore, it is necessary and sufficient that $\alpha < 0$, $(\alpha/2)^2 > \beta > 0$ hold. Thus, $\alpha < 0,\ \beta > 0$ in (6.104a) imply instability.
3. Roots $\lambda_1 > 0, \lambda_2 < 0$, or conversely. The second term in (6.104c) $\to 0$ and the first $\to \infty$, or conversely; so the relative equilibrium position is unstable. Real roots require $(\alpha/2)^2 > \beta$; and $\beta \neq 0$, otherwise $\lambda_1 = 0$. Case 1 and Case 2 show that $\beta > 0$, $\alpha > 0$ and $\beta > 0$, $\alpha < 0$ cannot satisfy the assigned conditions. Therefore, $\beta < 0$ must hold, and the conditions on λ_1, λ_2 are then satisfied for all real α. So, $\beta < 0$, α arbitrary imply instability.
4. Roots λ_1, λ_2 are complex conjugates. Now $\beta > (\alpha/2)^2 > 0$ must hold and (6.104c) may be written as

$$\chi(t) = e^{-\alpha t/2}\left(A_1 e^{irt} + A_2 e^{-irt}\right), \tag{6.104d}$$

where $r = (\beta - (\alpha/2)^2)^{1/2}$ is real and positive. If $\alpha > 0$, we have Case 1: $\alpha > 0,\ \beta > 0$, and hence the equilibrium position is asymptotically stable. Notice that $\chi(t) \to 0$ as $t \to \infty$. If $\alpha = 0$, the motion (6.104d) is simple harmonic, and hence the relative equilibrium position is infinitesimally stable. Finally, when $\alpha < 0$, we have Case 2: $\alpha < 0$, $\beta > 0$, and $\chi(t) \to \infty$ with t. The equilibrium state is unstable.
5. For $\beta = 0$, (6.104b) yields $\lambda_1 = 0,\ \lambda_2 = -\alpha$, and hence the motion is given by

$$\chi(t) = A_1 + A_2 e^{-\alpha t}. \tag{6.104e}$$

When $\alpha > 0$, $\chi \to A_1$ as $t \to \infty$; the equilibrium position is neutrally stable. When $\alpha < 0$, $\chi(t) \to \infty$ with t, and the equilibrium position is unstable. The degenerate case when $\alpha = 0$ also yields $\ddot{\chi} = 0$ in (6.103); so $\chi(t) = A_1 + A_2 t$. The equilibrium state is again unstable.

In summary, for all real or complex characteristic roots (6.104b), the infinitesimal stability of the relative equilibrium states is characterized by the following four circumstances expressed in terms of the *infinitesimal stability parameters* α and β, the coefficients (6.102) of the linearized equation of motion (6.103). *The relative equilibrium position is*

$$\left.\begin{array}{l} \text{(a) \textit{infinitesimally stable when} } \alpha = 0,\ \beta > 0, \\ \text{(b) \textit{asymptotically stable for} } \alpha > 0,\ \beta > 0, \\ \text{(c) \textit{neutrally stable for} } \alpha > 0,\ \beta = 0, \\ \text{(d) \textit{unstable for all remaining cases.}} \end{array}\right\} \tag{6.105}$$

These results also may be conveniently arranged in a matrix shown below.

$\alpha \downarrow \mid \beta \rightarrow$	>0	$=0$	<0
>0	A	N	U
$=0$	S	U	U
<0	U	U	U

Notice that the system is always unstable when either α or β is negative.

6.14.1. Stability of the Equilibrium Positions of a Pendulum

We now investigate the infinitesimal stability of the (relative) equilibrium positions of a simple pendulum whose finite angular motion is described by (6.67b):

$$\ddot{\theta} = -p^2 \sin\theta. \tag{6.106a}$$

This has the form of (6.97) in which $f(\theta, \dot{\theta}) = -p^2 \sin\theta$ is independent of $\dot{\theta}$. Hence, by (6.102), $\alpha \equiv 0$ and $\beta = p^2 \cos\theta_E$ at an equilibrium position θ_E. Thus, from (a) in (6.105), θ_E is a stable equilibrium position if and only if $\beta > 0$.

The (relative) equilibrium states, by (6.100), are given by

$$f(\theta_E) = -p^2 \sin\theta_E = 0, \tag{6.106b}$$

also evident from (6.106a). This yields infinitely many equilibrium positions $\theta_E = \pm n\pi$, $n = 0, 1, 2, \ldots$. But only two, $\theta_E = 0, \pi$, are physically distinct positions. For $\theta_E = 0$, $\beta = p^2 > 0$, and for $\theta_E = \pi$, $\beta = -p^2 < 0$. Hence, $\theta_E = 0$ is an infinitesimally stable equilibrium position, whereas $\theta_E = \pi$ is unstable.

To see this somewhat differently, recall (6.101), write $\theta(t) = \theta_E + \chi(t)$, and then linearize equation (6.106a) to obtain

$$\ddot{\chi} + (p^2 \cos\theta_E)\chi = 0. \tag{6.106c}$$

This corresponds to the linearized equation (6.103). Specifically, then

$$\ddot{\chi} + p^2\chi = 0 \text{ for } \theta_E = 0, \qquad \ddot{\chi} - p^2\chi = 0 \text{ for } \theta_E = \pi. \tag{6.106d}$$

We know that the first of (6.106d) yields a stable simple harmonic solution for any given initial data, whereas the second yields a solution that grows exponentially with time. Hence, we again conclude that $\theta_E = 0$ is an infinitesimally stable equilibrium position, while $\theta_E = \pi$ is unstable.

The physical nature of the results is evident. Any small disturbance of the pendulum bob from its lowest point at $\theta_E = 0$ results in a small oscillation about this equilibrium position. Any infinitesimally small disturbance from its extreme vertical position $\theta_E = \pi$, on the other hand, grows increasingly larger and quickly violates the smallness assumption leading to (6.106c).

6.14.2. Application to Linear Oscillators

The foregoing discussion has focused on infinitesimal stability for nonlinear problems in the class defined by (6.97), but the same infinitesimal perturbation procedure can be applied to all sorts of dynamical systems, including problems in which $f(x, \dot{x})$ is linear in either one or both variables x and $\dot{x}$. For illustration, let us reexamine the stability of the equilibrium positions of the rotating spring-mass system studied in Section 6.9.2, page 145. Equation (6.77b) gives the equation of motion in the form (6.97): $\ddot{x} = f(x, \dot{x}) = ap^2 - p^2(1 - \eta^2)x$, independent of $\dot{x}$ and linear in x. Equation (6.100) yields the evident equilibrium position $x_E = a/(1 - \eta^2)$, the same as (6.77g). The infinitesimal stability parameters in (6.102) are $\alpha \equiv 0$, $\beta = -\left. df(x)/dx \right|_{x_E} = p^2(1 - \eta^2)$. Therefore, the equilibrium position x_E is infinitesimally stable if and only if $\beta > 0$, that is, when and only when $\eta \equiv \omega/p < 1$. This is precisely the result derived earlier for arbitrary amplitude oscillations consistent with obvious constraints but based on the familiar nature of equation (6.77i).

The free vibrational motion of the general linear damped oscillator is described in (6.83), and this equation is not restricted to infinitesimal motions $z(t)$ from the equilibrium position $z_E = 0$. In view of the physical nature of the damping and spring coefficients, the infinitesimal stability parameters in (6.103) are positive; we identify $\alpha \equiv 2\nu > 0$ and $\beta \equiv p^2 > 0$. Therefore, we know from infinitesimal stability analysis that the equilibrium position $z_E = 0$ is asymptotically stable. In fact, it is physically clear that the system, when disturbed by any amount consistent with design constraints, will return eventually to its equilibrium position. If $\alpha \equiv 0$, the motion about the equilibrium state will be stable for $\beta \equiv p^2 > 0$, as learned earlier.

It is not necessary to recall the details of the formal infinitesimal stability analysis of the equilibrium states of a dynamical system. In special problems, it is straightforward to simply determine the equilibrium states from the equation of motion, introduce a disturbance like (6.101) for an infinitesimal perturbation from these states, and then carry out a linearized analysis of the equation of motion. This process leads to an incremental equation of motion similar to (6.103) from which the stability may be determined in accordance with (6.105). For further study of vibration problems and stability analysis see the referenced text by Meirovitch.

6.15. Equations of Motion Relative to the Earth

To investigate effects of the Earth's rotation on the motion of a particle, we recall the *equation of motion of a particle relative to the Earth*:

$$m\mathbf{a}_\varphi = \mathbf{F} - 2m\mathbf{\Omega} \times \mathbf{v}_\varphi. \qquad \text{(cf. 5.102)}$$

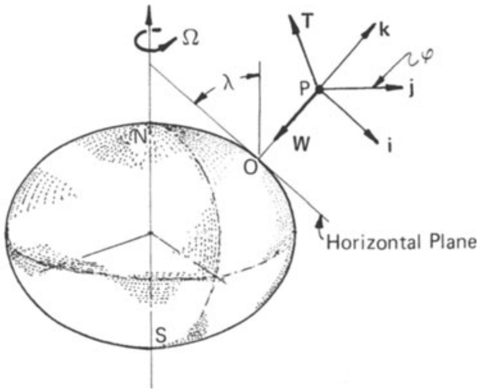


Figure 6.28. Motion of a particle relative to the Earth.

Let the Earth frame $\varphi = \{O; \mathbf{i}_k\}$ be oriented so that $\mathbf{i}$ is directed southward and $\mathbf{j}$ is eastward in the horizontal plane tangent to the Earth's surface at the latitude λ, the angle of elevation of the Earth's axis above the horizontal plane, as shown in Fig. 6.28. Then $\mathbf{k}$ is normal to the surface, directed skyward. Referred to φ, the angular velocity of the Earth frame is

$$\boldsymbol{\Omega} = \Omega(-\cos\lambda\mathbf{i} + \sin\lambda\mathbf{k}). \tag{6.107}$$

Hence, the Coriolis acceleration is given by

$$2\boldsymbol{\Omega} \times \mathbf{v}_\varphi = -2\Omega\dot{y}\sin\lambda\mathbf{i} + 2\Omega(\dot{x}\sin\lambda + \dot{z}\cos\lambda)\mathbf{j} - 2\Omega\dot{y}\cos\lambda\mathbf{k}, \tag{6.108}$$

wherein $\mathbf{v}_\varphi \equiv \delta\mathbf{x}/\delta t$ and $\mathbf{x}(P,t) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the relative position vector. Finally, the total force acting on the particle P is $\mathbf{F} = \mathbf{T} + \mathbf{W}$, where $\mathbf{W} = -mg\mathbf{k}$ is its apparent weight and $\mathbf{T} = Q\mathbf{i} + R\mathbf{j} + S\mathbf{k}$ is the total of all other contact and body forces that act on P. Then, use of (6.108) in (5.102) yields *the scalar equations for the particle's motion relative to the Earth*:

$$m\ddot{x} = Q + 2m\Omega\dot{y}\sin\lambda, \tag{6.109}$$

$$m\ddot{y} = R - 2m\Omega(\dot{x}\sin\lambda + \dot{z}\cos\lambda), \tag{6.110}$$

$$m\ddot{z} = S - mg + 2m\Omega\dot{y}\cos\lambda. \tag{6.111}$$

Some interesting Coriolis effects of the Earth's rotation may be read from these equations, or more directly from (6.108). When a particle is traveling eastward so that $\mathbf{v}_\varphi = \dot{y}\mathbf{j}$, for example, the Coriolis force $-2m\boldsymbol{\Omega} \times \mathbf{v}_\varphi = 2m\Omega\dot{y}(\sin\lambda\mathbf{i} + \cos\lambda\mathbf{k})$ for $\lambda > 0$ in the northern hemisphere drives the particle toward the right, southward and upward; and at the same latitude in the southern hemisphere for which $\lambda < 0$, it drives the particle toward the left, northward and upward. Therefore, in the moving Earth frame over a period of time, a ship or plane in its eastward directed motion must make a small course correction northward in the northern hemisphere and southward in the southern hemisphere, to counter the Coriolis force effect in (5.102). At the equator $\lambda = 0$, only the vertical component is active: $-2m\boldsymbol{\Omega} \times \mathbf{v}_\varphi = 2m\Omega\dot{y}\mathbf{k}$, so no course adjustment is needed. Other subtle

Coriolis effects on the motion of a particle relative to the Earth are demonstrated in some applications that follow.

6.16. Free Fall Relative to the Earth—An Exact Solution

The elementary result (6.24) for the motion of a particle that falls from rest relative to the Earth shows that the particle falls on a straight line—the plumb line, a result that ignores the Coriolis effect of the Earth's spin. Due to the Earth's rotation, however, the particle in its free fall from rest is deflected horizontally from the vertical plumb line. This Coriolis deflection effect is determined, and afterwards the theoretical result is compared with experimental data. For simplicity, however, effects due to air resistance, wind, and buoyancy are ignored.

The free fall problem is the simplest example for which an exact solution of the equations of motion of a particle relative to the Earth may be obtained. In this case, with $(Q, R, S) = 0$ and $\mathbf{v}_\varphi(P, 0) = \mathbf{0}$ initially, (6.109)–(6.111) may be readily integrated to obtain

$$\dot{x} = 2\Omega y \sin\lambda, \tag{6.112a}$$

$$\dot{y} = -2\Omega(x \sin\lambda + z\cos\lambda), \tag{6.112b}$$

$$\dot{z} = -gt + 2\Omega y \cos\lambda. \tag{6.112c}$$

The next step is less evident. We first substitute (6.112a) and (6.112c) into (6.110) and set $R = 0$ to obtain

$$\ddot{y} + 4\Omega^2 y = 2\Omega g t \cos\lambda. \tag{6.112d}$$

The general solution of (6.112d) is given by

$$y = \frac{gt\cos\lambda}{2\Omega} + A\cos 2\Omega t + B\sin 2\Omega t. \tag{6.112e}$$

Without loss of generality, the origin may be chosen so that $\mathbf{x}(P, 0) = \mathbf{0}$. Thus, with $y = 0$ and $\dot{y} = 0$ at $t = 0$, we find $A = 0$, $B = -(g\cos\lambda)/4\Omega^2$, and hence

$$y = \frac{g\cos\lambda}{4\Omega^2}(2\Omega t - \sin 2\Omega t). \tag{6.112f}$$

Now substitute this relation into (6.112a) and (6.112c), recall the initial data, and integrate the results to derive *the exact time-parametric equations for the particle path in its free fall relative to the rotating Earth frame*:

$$x = \frac{g\sin 2\lambda}{8\Omega^2}(2\Omega^2 t^2 - 1 + \cos 2\Omega t), \tag{6.112g}$$

$$y = \frac{g\cos\lambda}{4\Omega^2}(2\Omega t - \sin 2\Omega t), \tag{6.112h}$$

$$z = -\frac{gt^2}{2} + \frac{g\cos^2\lambda}{4\Omega^2}(2\Omega^2 t^2 - 1 + \cos 2\Omega t). \tag{6.112i}$$

Notice that both horizontal and vertical components of the motion are affected by the Earth's rotation, and that the results are independent of the particle's mass. When the Earth's rotational rate $\Omega \to 0$, these equations show that $x \to 0$, $y \to 0$, $z \to -\frac{1}{2}gt^2$. That is, $\mathbf{x}(P, t) = z\mathbf{k} = \frac{1}{2}\mathbf{g}t^2$, the elementary solution (6.24) for which the Earth's rotation is neglected.

6.16.1. Free Fall Deflection Analysis

Since Ω is small, but not zero, and the time of fall near the Earth's surface is of short duration, the path equations (6.112g)–(6.112i) may be simplified by series expansion of the trigonometric functions to retain only terms of $O(\Omega t)^2$. This yields

$$\mathbf{x}(P,t) = \frac{gt^2}{12}((\Omega t)^2 \sin 2\lambda \mathbf{i} + 4\Omega t \cos\lambda \mathbf{j} - 2(3 - 2(\Omega t)^2 \cos^2\lambda)\mathbf{k}). \qquad (6.112\text{j})$$

We thus find an eastward (**j**-directed) deflection of the first order and a north–south (**i**-directed) essentially negligible deflection of the second order in Ωt. To terms of the first order in Ωt, therefore, the motion is described by

$$\mathbf{x}(P,t) = \frac{1}{3}g\Omega t^3 \cos\lambda \mathbf{j} - \frac{1}{2}gt^2\mathbf{k}. \qquad (6.112\text{k})$$

The first term describes the Coriolis deflection, and the second is the elementary solution (6.24). *Therefore, a particle P in its free fall relative to the Earth experiences in either hemisphere an eastward directed deflection from the vertical axis*. The trajectory of P, shown in Fig. 6.29, to the first order in Ωt is a semicubical parabola in the east–west vertical plane:

$$y^2 = -\frac{8\Omega^2\cos^2\lambda}{9g}z^3. \qquad (6.112\text{l})$$

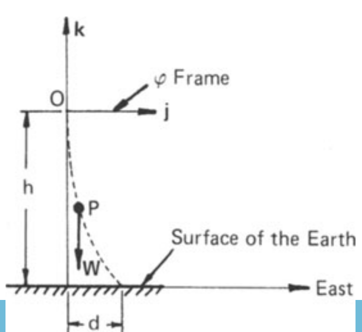


Figure 6.29. Free fall deflection of a particle relative to the Earth.

The deflection $y = d$ for fall through a height $z = -h$ is

$$d = \frac{2}{3} h\Omega \cos\lambda \sqrt{\frac{2h}{g}}. \tag{6.112m}$$

The deflection is greatest at the equator ($\lambda = 0$) and vanishes at the poles ($\lambda = \pm\pi/2$). For example, the greatest deflection of a raindrop falling freely through 10,000 ft (3049 m), without air resistance, wind, and buoyancy effects, according to (6.112m), is $d_{\max} = 12.1$ ft (3.69 m). Though only 0.12% of the height, the deflection in the ideal free fall case would be clearly observable. In fact, some experimental results on falling solid pellets have been reported.

6.16.2. Reich's Experiment

The free fall of pellets down a deep mine shaft at Freiberg, Germany was studied by F. Reich in 1831 and published a few years before Coriolis reported his formula for relative rotational effects in 1835. The depth of the mine was 158.5 m, and Reich observed an average deflection of 28.3 mm in 106 trials. The corresponding value estimated by (6.112m) for the data $\Omega = 7.29 \times 10^{-5}$ rad/sec, $g = 9.82$ m/sec^2, and $\lambda = 51°N$ is 27.5 mm. Our theoretical estimate thus demonstrates excellent agreement with Reich's experimental result on the eastward deflection. It is known, however, that the eastward deflection is slightly reduced by air resistance.

Long before the expression for the Coriolis acceleration was discovered, the eastward deflection due to the Earth's rotation was argued intuitively by natural philosophers, though usually incorrectly, and Reich knew about this. In addition, however, Reich found a small *southerly* deflection at Freiberg. This north–south deviation is determined exactly by (6.112g) and to terms of the order $(\Omega t)^2$ by (6.112j). If the time of fall τ from the height h is estimated by their omission in (6.112j) so that $\tau^2 = 2h/g$, the north–south deflection is approximated by $\delta = x(\tau) = (h^2\Omega^2/3g)\sin 2\lambda$. Hence, the southerly deflection predicted for Reich's experimental data is roughly 0.004 mm. Within the error of experiment, this would be zero and in fact negligible; so, it seems unlikely that such a minute free fall effect could be accurately measured. The fact that Reich and others have observed and reported the effect at all is surprising.

6.17. Foucault's Pendulum

In 1851, J. B. Léon Foucault** (1819–1868) discovered by experiment that the effect of the Earth's rotation on the motion of a carefully constructed pendulum

** The story of Léon Foucault's life, his pendulum experiments, his invention of the gyroscope, his numerous other accomplishments, and the illustrious period of French history during which he is to produce relative to the Earth an apparent rotation of its plane of oscillation at an angular rate $\omega = \Omega \sin \lambda$, clockwise in the northern hemisphere ($\lambda > 0$) and anticlockwise in the southern hemisphere ($\lambda < 0$). Foucault's first pendulum consisted of a 5 kg brass bob attached to a 2 m long steel wire suspended from the ceiling in the cellar of his house, its end held in a device that enabled the pendulum's unhindered rotation. To avoid disturbing extraneous vibrations from the thunderous clatter of passing carriages and other neighborhood noise, echoes of busy Paris streets that followed him to his cellar laboratory, he worked during the wee small hours of the night. His first test, 1–2 AM, Friday, January 3, 1851, ended quickly in failure when suddenly the wire broke. Several days later, modifications concluded, at two o'clock in the morning of Wednesday, January 8, 1851, he recorded in his journal the slow steady rotation of the plane of the pendulum's swing. Secluded from the rest of the world in the cellar of his house, without reference to heavenly bodies, he thus witnessed for the first time in history direct proof of the rotation of the Earth about its axis! (Incidentally, to relate the time of Foucault's pendulum experiments in France to American history, we may recall that Millard Fillmore was 13th President of the United States (1850–1853).)

Needless to say, Foucault was most anxious to demonstrate his important discovery to French scientists, but he needed a prominent public place to display his pendulum. Moreover, the effect could be enhanced by the use of a longer pendulum wire—remember, the period of oscillation for a simple pendulum is increased with its length; so, with a longer wire the pendulum swings more slowly, and the turning of the Earth is more easily seen.

Having no scientific credentials himself, he was generally not well-regarded by the members of the French Academy of Sciences. On the other hand, François Arago, a man of scientific prominence and a member of the Academy, the renowned and distinguished Director of the Paris Observatory, a large building with a high dome, was a somewhat friendly, admiring associate, who was certain to appreciate his discovery. Foucault convinced Arago to permit the presentation of his pendulum discovery in the Meridian Hall, the largest, longest, and highest room in the Observatory and, though unimportant to the experiment, perfectly aligned lengthwise with the Paris Meridian. (This is the very Meridian a certain specified length of which was proposed to define the length of the standard meter, but because of errors of its measurement, which is another story, actually it does

struggled for recognition by his colleagues in the Academy of Sciences, is told in the books by Aczel and by Tobin cited in the References and from which this summary narrative is adapted. There are, however, some ambiguities and discrepancies in their reports. For instance, it is not clear from their separate presentations that Foucault's pendulum demonstration and his paper presented by Arago at the Academy announcing the discovery occurred on the same day. Also, Tobin, page 141, sets the time for Foucault's Meridian Hall invitation at 2–3 PM, while Aczel, page 93, reports 3–5 PM; and they express a difference of opinion on other historical matters, including the date of Foucault's first successful test! Consequently, when I perceived a conflict, unable to check the original sources myself, though the difference might seem insignificant, I generally leaned toward Tobin's view.

not.) The high ceiling of Meridian Hall would allow use of a pendulum of 11 m length.

Foucault prepared invitations and sent them to all members of the Academy and some others—"You are invited to come to see the Earth turn, in the Meridian Hall of the Paris Observatory, tomorrow, from two to three."—an invitation clearly designed to stimulate curiosity and to drive attendance. On February 3, 1851, Foucault (see the References) announced his pendulum discovery in a paper presented to the Academy by then supportive Arago. Later that day, many of France's most famous scientists and mathematicians assembled in Meridian Hall to see the Earth turn. Word of Foucault's pendulum experiment success instantly excited the interest of science-minded Louis-Napoléon, President of the French Republic, who decreed straightaway that the experiment be repeated in the Panthéon, a grand temple and mausoleum for great Frenchmen, the highest domed building in all of Paris. A new pendulum 67 m long and weighing 28 kg, the then longest and heaviest in the world, was fabricated. At the end of March 1851, Foucault's dream was realized—the Panthéon pendulum exhibition was open for all visitors to witness. Later that year, a report of a pendulum experiment in Brazil confirmed the counterclockwise, southern hemisphere ($\lambda < 0$) rotation of the pendulum in agreement with Foucault's empirical sine relation $\omega = \Omega \sin \lambda$.

The dynamical equations of motion of a particle relative to a moving reference frame were widely known long before 1851. The earliest derivation appears to have come from A. Clairaut in 1742 (see Dugas in the References). The result, however, is commonly attributed to G. G. de Coriolis (1792–1843), a student of Siméon Denis Poisson (1781–1840), who presented the correct equations in a paper read to the Academy of Sciences in 1831 and published a year later. Moreover, it is known that probably around 1837, Poisson had analyzed the Coriolis effect on the motion of a pendulum; but failing to appreciate its cumulative effect, he rejected the result as too small to be noticeable and apparently never published it. Foucault's demonstration sparked new interest among mathematicians and scientists to explain by analysis Foucault's empirical sine rule. At a meeting of the Academy a few days after Arago's presentation of Foucault's memoir, J. P. M. Binet, an obscure professor of mechanics and astronomy, wrote down the equations of motion from the principles of dynamics and following some approximations and a lengthy analysis, there, for the first time, derived Foucault's equation for the rate of rotation of the pendulum. (See the References.)

Though widely acclaimed around the world for his work in science and engineering, the ultimate honor that Foucault desperately desired, his election as a member of the Academy of Sciences, was continually denied to him. A seat in the Academy opened only upon the death of a member and then, of course, the number of candidates seeking election was many, to say least about vote-rigging politics that sometimes raised its ugly head. Foucault had narrowly missed election several times. Finally, on January 23, 1865, 14 years after his famous demonstration of the Earth's rotation and 3 years before his death, his quest was finally realized when he was elected to the Academy of Sciences. Foucault described the long awaited approbation of his peers, his election to Academy membership, as one of the great joys of his life. (See Tobin in the References.)

Nowadays, one may find a Foucault pendulum in just about every major city around the world. In Lexington, Kentucky, for example, a Foucault pendulum swings in the Public Library on Main Street. Surprisingly, the pendulum has been exhibited at the Panthéon only since 1995. In St. Petersburg (formerly Leningrad), Russia, during the Soviet years from 1931, interrupted by the war of 1941–1945, and thereafter continuing until the late 1980s, the world's longest Foucault pendulum, nearly 100 m in length, was suspended from the dome of St. Isaac's Cathedral, one of the tallest churches in the world, built in 1818–1858. Students were taken regularly by their professors to see this remarkable display proving the rotation of the Earth. Soon after Soviet President Mikhail Gorbachev's initiation of perestroika and his rise to power in 1988, St. Isaac's was returned to the Church, and the phenomenal Foucault pendulum, the incongruous centerpiece of St. Isaac's swinging from its cupola, was promptly removed. Today, the image of a white dove in flight adorns the pinnacle of the incredibly beautiful and spectacular ceiling within the golden dome of this magnificent church. Though still principally a museum as decreed by the Soviet government in 1931, from time to time St. Isaac's nowadays holds religious services on special occasions, and a Foucault pendulum may be seen at the St. Petersburg Planetarium. Everyone who has observed the swing of a Foucault pendulum has, in effect, seen the rotation of the Earth!

6.17.1. General Formulation of the Problem

We now turn to the analysis of Foucault's pendulum phenomenon. Let us consider a pendulum bob of mass m attached by a long wire of length ℓ to a fixed point $(0, 0, \ell)$ along the vertical plumb line in the Earth frame $\varphi = \{O; \mathbf{i}_k\}$ in Fig. 6.30. The relative position vector of m in φ is $\mathbf{x}(m, t) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The total force $\mathbf{F}$ on the bob is its apparent weight $m\mathbf{g}$ and the wire tension $\mathbf{T} = T\mathbf{n} = Q\mathbf{i} + R\mathbf{j} + S\mathbf{k}$, where $\mathbf{n} = -x/\ell\mathbf{i} - y/\ell\mathbf{j} + (1 - z/\ell)\mathbf{k}$. Hence, the general equations (6.109)–(6.111) yield the following relations for the motion of the pendulum bob relative to the Earth:

$$m\ddot{x} = -\frac{Tx}{\ell} + 2m\Omega\dot{y}\sin\lambda, \tag{6.113a}$$

$$m\ddot{y} = -\frac{Ty}{\ell} - 2m\Omega(\dot{x}\sin\lambda + \dot{z}\cos\lambda), \tag{6.113b}$$

$$m\ddot{z} = \frac{T(\ell - z)}{\ell} - mg + 2m\Omega\dot{y}\cos\lambda. \tag{6.113c}$$

These equations cannot be integrated exactly for large amplitude oscillations. The manner in which the wire tension varies with the motion is unknown, and its elimination from these equations serves only to further complicate matters. It is possible, however, to derive an approximate solution for small amplitude oscillations.

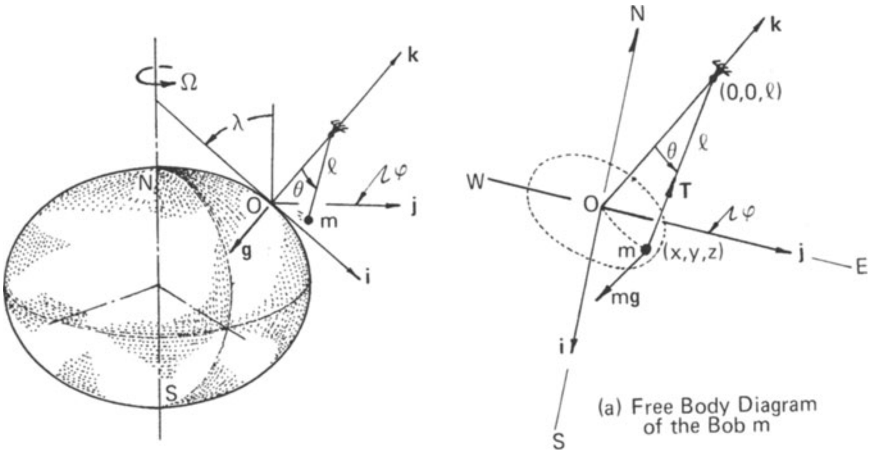


Figure 6.30. Foucault's pendulum and its motion relative to the Earth.

6.17.2. Equations for Small Amplitude Oscillations

Let us assume that the wire is long compared with the displacement so that x/ℓ, y/ℓ, and all of their time derivatives are small terms. Since $\ell - z = \ell[1 - (x^2 + y^2)/\ell^2]^{1/2}$, our smallness assumption shows that $z/\ell = (x^2 + y^2)/2\ell^2$, approximately. Hence, z/ℓ and its time derivatives are small quantities of the second order and may be discarded from (6.113a)–(6.113c). In particular, (6.113c) then yields an equation for the wire tension,

$$T = m(g - 2\Omega\dot{y}\cos\lambda). \tag{6.113d}$$

Since Ω is very small, (6.113d) shows that the tension, as expected, is very nearly equal to the apparent weight of the bob.

Using (6.113d) in (6.113a) and (6.113b) and neglecting terms of second order, we obtain the simpler, but coupled system of linear equations

$$\ddot{x} - 2\omega\dot{y} + p^2x = 0, \qquad \ddot{y} + 2\omega\dot{x} + p^2y = 0, \tag{6.113e}$$

in which

$$p \equiv \sqrt{\frac{g}{\ell}}, \qquad \omega \equiv \Omega\sin\lambda. \tag{6.113f}$$

The constant p is the familiar small amplitude, circular frequency of the simple pendulum when the Earth's rotation is ignored. It is evident from (6.113e), however, that *the motion of Foucault's pendulum is not simple harmonic.*

6.17.3. Solution of the Small Amplitude Equations

The solution of the coupled system (6.113e) may be obtained following an unusual change of variable. We multiply the second of (6.113e) by $i = \sqrt{-1}$, add the result to the first equation in (6.113e), and introduce the new complex variable

$$\xi(t) = x(t) + iy(t), \tag{6.113g}$$

to obtain the single complex equation

$$\ddot{\xi} + 2i\omega\dot{\xi} + p^2\xi = 0. \tag{6.113h}$$

The general solution of (6.113h) is

$$\xi(t) = A_1 e^{i\alpha_1 t} + A_2 e^{i\alpha_2 t}, \tag{6.113i}$$

in which A_1 and A_2 are integration constants, possibly complex, and the characteristic exponents are given by

$$\alpha_1 = -\omega - \omega^*, \qquad \alpha_2 = -\omega + \omega^*, \qquad \text{with} \quad \omega^* \equiv \sqrt{\omega^2 + p^2}. \tag{6.113j}$$

To determine the constants A_1 and A_2, let us suppose that the pendulum is released from rest at $x(0) = x_0$, $y(0) = 0$ at time $t = 0$. Then, by (6.113g), the initial values of the complex variable are $\xi(0) = x_0$, $\dot{\xi}(0) = 0$, and hence (6.113i) delivers

$$A_1 = \frac{x_0\alpha_2}{\alpha_2 - \alpha_1}, \qquad A_2 = -\frac{x_0\alpha_1}{\alpha_2 - \alpha_1}. \tag{6.113k}$$

Finally, use of (6.113j) in (6.113k) yields the real-valued constants

$$A_k = \frac{x_0}{2}\left(1 + (-1)^k \frac{\omega}{\omega^*}\right), \qquad k = 1, 2. \tag{6.113l}$$

We recall Euler's identity (6.49) to cast (6.113i) in the form

$$\xi(t) = (A_1 \cos\alpha_1 t + A_2 \cos\alpha_2 t) + i(A_1 \sin\alpha_1 t + A_2 \sin\alpha_2 t). \tag{6.113m}$$

It now follows with (6.113g) that the solution of the coupled equations in (6.113e) for the small amplitude motion of Foucault's pendulum is

$$\left.\begin{aligned} x(t) &= A_1 \cos\alpha_1 t + A_2 \cos\alpha_2 t, \\ y(t) &= A_1 \sin\alpha_1 t + A_2 \sin\alpha_2 t, \end{aligned}\right\} \tag{6.113n}$$

where the constants α_k and A_k are given by (6.113j) and (6.113l). Let the reader consider the following alternative procedure.

Exercise 6.8. Notice that (6.113h) is similar to the damped oscillator equation (6.83), and hence the solution method starting from (6.86a) is applicable. Begin with $\xi(t) = e^{\beta t}u(t)$, recall (6.113f) and (6.113j), and show that the general solution of (6.113h) for the assigned initial data for $\xi(t)$ yields the motion

$$\left.\begin{aligned} x(t) &= x_0\left(\cos\omega^* t \cos\omega t + \frac{\omega}{\omega^*}\sin\omega^* t \sin\omega t\right), \\ y(t) &= x_0\left(-\cos\omega^* t \sin\omega t + \frac{\omega}{\omega^*}\sin\omega^* t \cos\omega t\right). \end{aligned}\right\} \tag{6.113o}$$

Show that the same results follow from (6.113n). □

6.17.4. Physical Interpretation of the Solution

The motion (6.113n) is harmonic in time, but not simple, and it is not periodic unless α_1/α_2 is a rational number. Nevertheless, a period characteristic of the oscillation may be defined that will facilitate our physical understanding of the Foucault phenomenon.

The half-period $\tau/2$ is defined as the time required for the pendulum to complete its outward swing from its initial position. To find the period τ, we first determine all times $\hat{T} \neq 0$ for which $\dot{\mathbf{x}}(\hat{T}) = \mathbf{0}$. Differentiation of (6.113n) and use of (6.113k) shows that $\hat{T}$ must satisfy $\sin(\alpha_1\hat{T}) = \sin(\alpha_2\hat{T})$ and $\cos(\alpha_1\hat{T}) = \cos(\alpha_2\hat{T})$. Hence, $\hat{T}\alpha_2 = \hat{T}\alpha_1 \pm 2n\pi$ for all integers n. Use of (6.113j) in this expression yields the (positive) rest times

$$\hat{T}(n) = \frac{n\pi}{\omega^*} = \frac{n\pi}{\sqrt{\omega^2 + p^2}}, \qquad n = 1, 2, \ldots. \tag{6.113p}$$

At each time $\hat{T}(n)$, the bob attains a position of instantaneous rest. Thus, for the first outward swing, $\hat{T}(1) = \tau/2$, and hence the period of the oscillations is

$$\tau = \frac{2\pi}{\omega^*} = \frac{2\pi}{\sqrt{\omega^2 + p^2}}. \tag{6.113q}$$

Thus, ω^* defines the circular frequency of the oscillations, and the frequency is given by

$$f = \frac{1}{\tau} = \frac{\omega^*}{2\pi} = \frac{1}{2\pi}\sqrt{\omega^2 + p^2}. \tag{6.113r}$$

When the Earth's rotation is neglected so that $\omega = 0$, (6.113q) and (6.113r) reduce to the period and frequency for the simple pendulum. Otherwise, *the Earth's rotational effect on the oscillations of a pendulum is to increase its frequency* (*decrease its period*) *very slightly compared with that of the simple pendulum.* Moreover, in view of (6.113f), *the frequency is greatest* (*the period least*) *at the poles and least* (*greatest*) *at the equator where the effect vanishes to yield the simple pendulum, small amplitude value*. That is, the frequency varies from $p/2\pi$ at the equator to $(p^2 + \Omega^2)^{1/2}/2\pi$ at the poles.

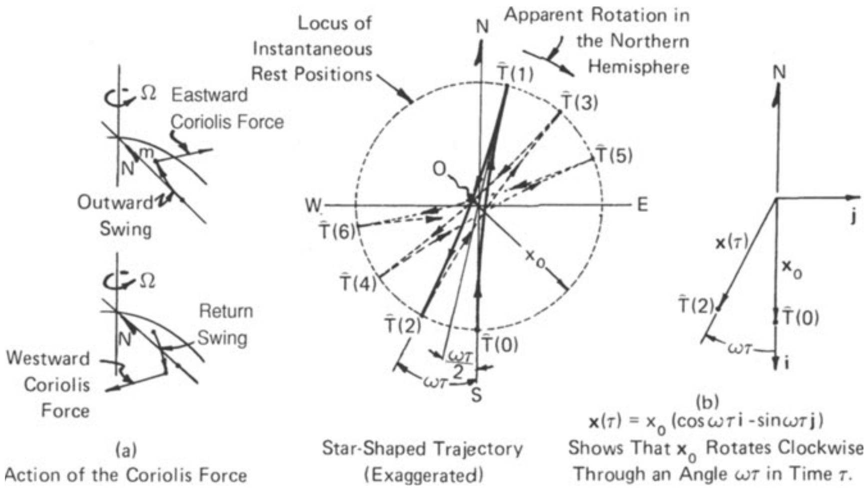


Figure 6.31. The Coriolis effect on the trajectory relative to the Earth of Foucault's pendulum viewed from its point of support at a place in the northern hemisphere where its apparent rotation is clockwise.

The rest positions of the bob at each half-period $\hat{T}(n) = n\tau/2 = n\pi/\omega^*$ may be obtained from (6.113o), which yields

$$x\left(\frac{n}{2}\tau\right) = (-1)^n x_0 \cos\left(\frac{n}{2}\omega\tau\right), \qquad y\left(\frac{n}{2}\tau\right) = (-1)^{n+1} x_0 \sin\left(\frac{n}{2}\omega\tau\right). \tag{6.113s}$$

In particular, the position of the bob after one full swing out and back is, for $n = 2$,

$$x(\tau) = x_0 \cos(\omega\tau), \qquad y(\tau) = -x_0 \sin(\omega\tau). \tag{6.113t}$$

Since $x(n\tau/2)^2 + y(n\tau/2)^2 = x_0^2$, it is seen that the locus of rest positions (6.113s) is a circle of radius x_0. Hence, (6.113t) shows that the initial position vector $\mathbf{x}_0 = x_0\mathbf{i}$, viewed from the point of support, has been rotated through an angle $\omega\tau$, which is clockwise when $\omega > 0$ and counterclockwise when $\omega < 0$. The second relation in (6.113f) shows that $\omega > 0$ in the northern hemisphere, $\omega < 0$ in the southern hemisphere, and $\omega = 0$ at the equator where the motion is always simple harmonic. Therefore, as first demonstrated by Foucault, *relative to the Earth, the plane of oscillation of a pendulum has an apparent clockwise rotation in the northern hemisphere, a counterclockwise rotation in the southern hemisphere, and no rotation at the equator*.

The motion is illustrated in Fig. 6.31 for the northern hemisphere. The pendulum starts from a southward displaced position of rest at a small distance x_0 from the plumb line. As the bob moves on its outward swing, it experiences a Coriolis force directed eastward; but on its return swing, the Coriolis force is directed westward. The deflection always is toward the right of the direction of the swing in the northern hemisphere. This is shown in Fig. 6.31a. Hence, the bob, after one period, has undergone a net displacement westward to the position $\mathbf{x}(\tau) = x_0(\cos \omega\tau \mathbf{i} - \sin \omega\tau \mathbf{j})$, the same distance from the origin, but rotated clockwise through a small angle $\omega\tau$ from $\mathbf{x}_0$, as shown in Fig. 6.31b. At each time $\hat{T}(n) = n\tau/2$, the same thing is repeated over and over, so the bob traces the star shaped trajectory described by (6.113o) and illustrated in Fig. 6.31. The apparent motion in the southern hemisphere for which $\lambda < 0$ is counterclockwise. The vertical plane of the pendulum's oscillations thus rotates relative to the Earth with Foucault's angular speed $\omega = \Omega \sin \lambda$, as indicated in (6.113t). The number of days $\tau_d(\lambda)$ required to complete one full revolution of the plane of oscillation of the pendulum is thus given by $\tau_d(\lambda) = 1/\sin \lambda$. *Consequently, Foucault's pendulum takes* 1 *day to complete its apparent turn at the poles where* $\lambda = \pm\pi/2$, *and this cyclic time increases as the latitude* λ *decreases toward the equator where the effect disappears.* Specifically, at $\lambda = \pi/6$, $\tau_d(\pi/6) = 2$ days/revolution, and at the equator $\tau_d(0) = \infty$ days/revolution, that is, the Foucault effect vanishes.

6.18. Relative Motion under a Constant Force

The scalar equations (6.109)–(6.111) for the motion of a particle relative to the Earth may be integrated exactly for any constant force components (Q, R, S). However, the general description of motion of a particle P relative to the Earth under a constant force $\mathbf{f} = \mathbf{F}/m$ per unit mass also may be derived as an easy approximate solution of the vector equation of motion (5.102), namely,

$$\frac{\delta \mathbf{v}}{\delta t} = \mathbf{f} - 2\boldsymbol{\Omega} \times \mathbf{v}. \tag{6.114a}$$

This is a first order, vector differential equation for the relative velocity $\mathbf{v} \equiv \mathbf{v}_\varphi(P, t) = \delta\mathbf{x}/\delta t$. Since $\boldsymbol{\Omega}$ is a constant vector, (6.114a) may be readily integrated to obtain

$$\mathbf{v}(P, t) = \frac{\delta \mathbf{x}}{\delta t} = \mathbf{f}t - 2\boldsymbol{\Omega} \times (\mathbf{x} - \mathbf{x}_0) + \mathbf{v}_0, \tag{6.114b}$$

in which $\mathbf{x}_0 \equiv \mathbf{x}(P, 0)$, $\mathbf{v}_0 \equiv \mathbf{v}(P, 0)$ are assigned initial values. For example, when gravity is the only force on a particle at rest initially at the origin, $\mathbf{f} = -g\mathbf{k}$, $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{v}_0 = \mathbf{0}$, and (6.114b) is then equivalent to the system of scalar equations (6.112a)–(6.112c) for the motion of a particle in free fall relative to the Earth.

The equation for the motion $\mathbf{x}(P, t)$ of P relative to the Earth under the general constant force $\mathbf{f}$ follows by use of (6.114b) in (6.114a); we find

$$\frac{\delta^2 \mathbf{x}}{\delta t^2} - 4\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) = \mathbf{f} - 2\boldsymbol{\Omega} \times (\mathbf{f}t + \mathbf{v}_0 + 2\boldsymbol{\Omega} \times \mathbf{x}_0). \tag{6.114c}$$

Upon discarding terms of order Ω^2, we obtain the easily integrable vector differential equation

$$\frac{\delta \mathbf{v}}{\delta t} = \frac{\delta^2 \mathbf{x}}{\delta t^2} = \mathbf{f} - 2\boldsymbol{\Omega} \times (\mathbf{f}t + \mathbf{v}_0). \tag{6.114d}$$

With the initial values $\mathbf{x}_0$ and $\mathbf{v}_0$ in mind, the first integral is

$$\mathbf{v} = \frac{\delta \mathbf{x}}{\delta t} = \mathbf{v}_0 + \mathbf{f}t - 2\boldsymbol{\Omega} \times \left(\frac{1}{2}\mathbf{f}t^2 + \mathbf{v}_0 t\right), \tag{6.114e}$$

and hence the approximate motion of P relative to the Earth is given by

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{f}t^2 - \boldsymbol{\Omega} \times \left(\frac{1}{3}\mathbf{f}t^3 + \mathbf{v}_0 t^2\right). \tag{6.114f}$$

To check the result, let us consider the motion of a particle in free fall from rest at the origin. Then $\mathbf{f} = \mathbf{g} = -g\mathbf{k}$ and (6.114f) simplifies to

$$\mathbf{x}(P, t) = \frac{1}{2}\mathbf{g}t^2 - \boldsymbol{\Omega} \times \frac{1}{3}\mathbf{g}t^3. \tag{6.114g}$$

Use of (6.107) yields (6.112k) derived earlier for the free fall case in which terms of order Ω^2 were neglected.

6.18.1. First Order Vector Solution for Projectile Motion

The approximate solution (6.114f) for the motion of a particle under a constant force is applied to investigate the Coriolis effect on the motion of a projectile P fired at $\mathbf{x}_0 = \mathbf{0}$ with a relative muzzle velocity

$$\mathbf{v}_0 = V(\cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}). \tag{6.115a}$$

Here α, β, γ are the direction angles of the gun in the frame $\varphi = \{O; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ defined in Fig. 6.28. The usual extraneous effects are neglected. Then only the body force $\mathbf{f} = \mathbf{g} = -g\mathbf{k}$ per unit mass acts on P. We thus recall (6.107) and (6.115a) to derive from (6.114f) the following estimate for the projectile's motion relative to the Earth:

$$\begin{aligned}\mathbf{x}(P, t) = {} & Vt(\cos\alpha + \Omega t \cos\beta \sin\lambda)\mathbf{i} \\ & + Vt\left[\cos\beta - \Omega t(\cos\gamma\cos\lambda + \cos\alpha\sin\lambda) + \frac{\Omega g t^2}{3V}\cos\lambda\right]\mathbf{j} \\ & + Vt\left(\cos\gamma + \Omega t\cos\beta\cos\lambda - \frac{gt}{2V}\right)\mathbf{k}.\end{aligned} \tag{6.115b}$$

Example 6.16. Determine the Coriolis deflection of a projectile fired eastward at latitude λ. Derive the classical relations for the motion and the range when the Earth's rotation is neglected.

Solution. Since the projectile is fired due east (the $\mathbf{j}$ direction in Fig. 6.28), the angle of elevation is β. Then $\alpha = \pi/2$, $\gamma = \frac{\pi}{2} - \beta$, and (6.115b) becomes

$$\begin{aligned}\mathbf{x}(P,t) &= \Omega V t^2 \cos\beta \sin\lambda \mathbf{i} \\ &\quad + Vt\left(\cos\beta - \Omega t \sin\beta\cos\lambda + \frac{\Omega g t^2}{3V}\cos\lambda\right)\mathbf{j} \\ &\quad + Vt\left(\sin\beta + \Omega t\cos\beta\cos\lambda - \frac{gt}{2V}\right)\mathbf{k}.\end{aligned} \tag{6.116a}$$

First consider the case when the Earth's rotation is neglected. With $\Omega = 0$, (6.116a) reduces to the classical elementary solution for projectile motion:

$$\mathbf{x}(P,t) = Vt\cos\beta\mathbf{j} + Vt\left(\sin\beta - \frac{gt}{2V}\right)\mathbf{k}. \tag{6.116b}$$

The time of flight $t^* = (2V\sin\beta)/g$ for which $z(t^*) = 0$ is then used to find the projectile's range $r \equiv y(t^*)$, namely,

$$r = \frac{V^2}{g}\sin 2\beta. \tag{6.116c}$$

Now consider the Earth's rotational effect. Equation (6.116a) indicates a lateral ($\mathbf{i}$-directed) Coriolis deflection of the projectile normal to its east directed range line, toward the south in the northern hemisphere and toward the north in the southern hemisphere. To find the deflection, we need the projectile's time of flight t^* given by $z(t^*) = 0$ in (6.116a). To the first order in Ω, we find for $V\Omega/g \ll 1$,

$$t^* = \frac{2V\sin\beta}{g}\left(1 + \frac{2V\Omega\cos\beta\cos\lambda}{g}\right). \tag{6.116d}$$

The lateral deflection $x^* \equiv x(t^*)$ and the range $r^* \equiv y(t^*)$ are now determined by the remaining components in (6.116a). The projectile's Coriolis deflection to first order in Ω, with $V\Omega/g \ll 1$, is thus given by

$$x^* = \frac{4\Omega V^3 \sin^2\beta\cos\beta\sin\lambda}{g^2}. \tag{6.116e}$$

The reader will explore the range effect in the exercise. □

Exercise 6.9. Show that to the first order in Ω, the variation $\delta r = r^* - r$ in the range due to the Earth's rotation when the gun is fired eastward with muzzle speed V at an elevation angle β and at latitude λ is

$$\begin{aligned}\delta r &= \frac{4\Omega V^3}{g^2}\cos\lambda\sin\beta\left(1-\frac{4}{3}\sin^2\beta\right)\\ &= \Omega\cos\lambda\sqrt{\frac{2r^3\cot\beta}{g}}\left(1-\frac{1}{3}\tan^2\beta\right).\end{aligned} \tag{6.116f}$$

□

Notice that $\delta r = 0$ when $\beta = 60°$; therefore, in the absence of air resistance, the Earth's rotation has no first order effect on the projectile's range when fired eastward at an elevation angle $\beta = 60°$. Otherwise, the Coriolis effect is to increase the range when $\beta < 60°$ and decrease it when $\beta > 60°$. The effect is the same in both hemispheres, and it may be considerable for high velocity projectiles or missiles. Large naval guns operate at fairly small angles of elevation, usually less than 15°; so the Earth's rotational effect is to increase their eastward directed range.

Finally, consider the Coriolis deflection (6.116e). In the northern hemisphere, $\sin\lambda > 0$ and $x^* > 0$; therefore, the projectile's lateral deflection from its eastward directed firing line is toward the right, southward. In the southern hemisphere, however, the deflection is toward the left, northward. *A correction for the effect in the northern hemisphere by directing the line of fire northward by an amount x^* without subsequent readjustment in the southern hemisphere at the opposite latitude would roughly double the northward deflection from the eastward directed line of fire.* While the projectile suffers no lateral deflection at the equator, $\lambda = 0$, the variation in the range with latitude given by (6.116f) is greatest there. Ballistic accuracy, therefore, requires that the Coriolis effect be accounted for in fire control and inertial guidance designs for long range, high velocity projectiles or missiles.

6.18.2. The Battle of the Falkland Islands

In late October 1914, Germany's (East Asiatic) China Squadron under the command of Vice Admiral von Spee patrolling in the Pacific Ocean was underway toward Cape Horn to harass British bases and shipping in the South Atlantic before attempting to return up the Atlantic to Germany.[††] Two heavy cruisers, von Spee's flagship, the *Scharnhorst*, and her sister ship, the *Gneisenau*, each mounting eight rapid-firing 8.2-in. guns, were accompanied by the three light cruisers *Nürnberg*, *Leipzig*, and *Dresden*, each with ten 4.1-in. batteries. The German gunners were well-trained, experienced, and most efficient.

[††] This account is adapted from the referenced reports by D. Howarth and Major R. N. Spafford, that by Howarth being more detailed. There are, however, a few minor discrepancies between them.

A British Squadron of older, slower ships, manned mostly by inexperienced reservists, based at the Falkland Islands under the command of Rear Admiral Sir Christopher Cradock, was at sea off the Pacific coast of Chile in search of von Spee. Cradock's flagship *Good Hope* mounted two 9.2-in. and sixteen 6-in. batteries; two light cruisers, the *Monmouth* and the *Glasgow*, each with several 6-in. guns; and an armored merchant ship, the *Otranto*, carried eight 4.7-inchers. A dilapidated battleship *Canopus* with four 12-inchers was too slow to keep pace with the others.

In the evening of November 1, Spee was found at Coronel off the coast of Chile. In heavy seas with winds near hurricane force, Cradock decided to run a parallel course and wait for an opportunity; but by 7 PM. Spee seized the initiative and engaged the British. The German light cruisers were outgunned and retreated from action, but Spee's two armored cruisers provided overwhelming rapid-fire power far superior to Cradock's. The *Scharnhorst* scored 35 hits on the *Good Hope*; the last struck the ship's magazine. An enormous explosion followed. Ablaze from stem to stern, almost instantly, the *Good Hope*, with Rear Admiral Sir Christopher Cradock and all 900 officers and crew, was gone. Later that night, following a relentless barrage by the *Gneisenau*, the *Monmouth* sank with all 754 hands. The *Glasgow* and *Otranto* fled southward to escape in the darkness and join the old battleship *Canopus*.

Not one man among the 1654 on board the two British cruisers survived the battle royal, while the Germans suffered only two wounded and six minor hits in the exchange. When word of this great tragedy and crushing defeat of the Royal Navy reached the British Admiralty, a superior British Squadron of eight warships was ordered to the Falklands to arrive on December 7, 1914. The dreadnoughts *Invincible* and *Inflexible*, two of the first heavily armored British battleships to have a large battery of eight 12-in. guns capable of being fired simultaneously in the same direction, five light cruisers, and an armed merchant vessel were directed to avenge the humiliating defeat at Coronel. The order: "Find Spee and destroy him!"

At dawn the next morning, December 8, the *Gneisenau* and the *Nürnberg* arrived at the Falklands to reconnoiter for a raid on the strategic coaling and wireless station at Port Stanley, expecting to find no ships of any importance stationed there. They were met instead with fire from the old *Canopus*, intentionally grounded in the harbor mud to serve as a Falkland fortress. One 12-in. shell hit the *Gneisenau*. Realizing the circumstances, von Spee's ships turned toward the open seas of the South Atlantic, unaware that any major British ships were in the area. Dreadnoughts with superior speed and fire power suddenly appeared on the horizon at the harbor entrance. At that moment, Spee realized his pending doom. At 12.45 PM that afternoon, in a calm sea with a clear sky, von Spee's *Scharnhorst, Gneisenau, Nürnberg*, and *Dresden* were overtaken and attacked by the superior British Squadron. Eight hours later, the fury ended. Vice Admiral Sir F. C. Doveton Sturdee's Royal Navy Squadron reported 6 killed and 19 wounded, while the Germans lost Vice Admiral Maximilian Graf von Spee, the Danish born

Pioneer of the German Navy, and 2260 other courageous officers and men.[††] The heavy cruiser *Scharnhorst*, the *Gneisenau*, and the light cruisers *Nürnberg* and *Leipzig* all sunk. Only the light cruiser *Dresden* escaped the British rage. Three months later, she was found at a small island off the Pacific coast of Chile. During negotiations for surrender and while flying the white flag from her foremast, the *Dresden* was scuttled by her crew on March 14, 1915.

Marion and Spafford report[§§] that at the start of this horrific battle, the British shells completely missed the German ships. Marion suggests that this was due to the double Coriolis effect, but precise details are not provided. It is a fact, however, that the British Isles are situated near 50° N latitude and the Falklands near 50° S latitude.

For south directed fire at an angle of elevation α, the transverse Coriolis deflection $y^* \equiv y(t^*)$ obtained from (6.115b) to first order in Ω is approximated by

$$y^* = -\frac{4\Omega V^3}{g^2}\sin^2\alpha\left(\frac{1}{3}\cos\lambda\sin\alpha + \sin\lambda\cos\alpha\right), \tag{6.117}$$

which varies with the latitude. On the other hand, we find no variation δr in the range at any latitude, when a projectile is fired either southward or northward, which may explain why the combatants ran a parallel course toward the east, firing toward the north and south. Notice that the Coriolis deflection (6.117) in a south directed shot is not symmetric in λ, so there is a slight difference in the magnitudes of the westward, northern hemisphere and eastward, southern hemisphere deflections. The maximum angle of elevation for large naval guns is about 15°. To estimate the Earth's rotational effect on a projectile's motion based only partially on circumstances reported for the Falklands engagement, let us suppose that at $\lambda = 50°$ N latitude a shell from a 12-in. gun is fired southward with a muzzle speed $V = 1800$ ft/sec (1227 mph) at an angle $\beta = 13°$. The reader will find that the range, which is given by the classical rule in (6.116c), is approximately 8.3 miles.

[††] The gallant Spee, his *Scharnhorst* seriously crippled and listing, rejected surrender to Sturdee. The *Scharnhorst* sank with Spee and all 765 hands. One hundred and ninety of the 850 man crew of the *Gneisenau* and only 23 sailors from both the *Nürnberg* and *Leipzig,* all sunk, were rescued from the frigid waters of the South Atlantic; but many of them subsequently died from their battle wounds or shock.

[§§] Marion (page 348) remarks on the Coriolis effect but provides no reference or calculation to support his claim that the British salvos fell 100 yards east of their southward targets. See the References and Spafford's report mentioned below.

The muzzle speeds used in the example presented below equation (6.117) and in Problem 6.76 are estimates obtained from general naval records: for a 5-in. gun, $V = 2650$ ft/sec, and for a 12-in. gun $V = 1800$ ft/sec and greater, depending on the model design. The range and latitude (actually closer to 51.5°S) are estimated from battle data described by Major R. N. Spafford, whose sketch of the battle plan of December 8, 1914 shows the British heading east, running a parallel course, 14,000 yards (8 miles) north of the Germans. By Spafford's account, *initial fire was exchanged but without effect,* except for a single German round that struck the *Invincible.* At the ideal range of 15,000 yards (8.5 miles) for the 12-in. guns of his battle cruisers, Sturdee found the target first and bombarded Spee's squadron.

Under these conditions, the deflection, according to (6.117), will be about 19 yards to the right, westward of the line of fire in the northern hemisphere and roughly 22 yards eastward in the southern hemisphere. A fire control system that corrects for the deflection only in the northern hemisphere (by pointing its sights eastward), when fired southward at 50° S latitude, will direct a shell about 41 yards to the left, east of its south directed target. These deviations increase substantially for larger muzzle speeds. (See Problem 6.76 and the remarks in the last footnote above.)

The weight of a projectile may vary considerably from roughly 70 lb for a 5-in. shell to about 1800 lb for a 16-in. shell. Since in this analysis gravitational force is the only force acting on the projectile, however, it is seen that the results are independent of the mass of the projectile or any of its design features. Introduction of drag force and aerodynamic body features would bring these additional characteristics into view. Of course, variations in the results arise from the lack of more precise data for the parameters, and the motion of the ships has been ignored.

Even though our model is not precise, it shows for a simple case that if initially the range of the British guns was erroneously set to correct for a westward Coriolis deflection (appropriate for battle in the northern hemisphere in the vicinity of the British Isles), when fired southward in similar circumstances at the opposite latitude in the South Atlantic Ocean, the barrage would fall to the left of its target, eastward, by a distance nearly double that deflection. If our simplified model is typical of the real circumstances, the actual gross effect must have appeared surprising to the British gunners when, in the situation described by Marion, their "accurately" aimed, southward directed salvos fell 100 yards to the east of the German ships.

6.18.3. Concluding Remarks

There are other kinds of subtle but measurable Coriolis effects. Instead of a single particle model, we may consider a stream of river particles flowing from the north toward the south, like the great Mississippi. The Coriolis force on a fluid particle in the Earth frame is directed westward. We thus see, if only heuristically, that the water will exert greater pressure on the west bank than the east. Geographers have established that this pressure causes greater erosion on the west bank and further that the water level also is slightly but measurably higher on the west bank. The same flow from north to south in the southern hemisphere would induce greater erosion and a higher water level on the river's east bank. The extent of the effect varies, of course, with the geographic latitude. A similar effect occurs for other directions of flow. The Coriolis effect on ocean and tidal currents is similar; the effect on atmospheric air flow and cyclonic motion is more pronounced. All of these measurable effects arise from the fact that the Earth is not an inertial reference frame, and all are predictable from Newton's basic principles of mechanics.

We have seen that the effects due to the Coriolis acceleration, though usually small, certainly are not always negligible. For the sake of simplicity and because the moving Earth frame closely approximates an inertial reference frame, henceforward, unless specified otherwise, the Earth's rotational motion is ignored in future applications. It is nonetheless important that the engineering analyst be aware of potential Coriolis effects and evaluate whether these should be safely excluded in problems of motion relative to the Earth.

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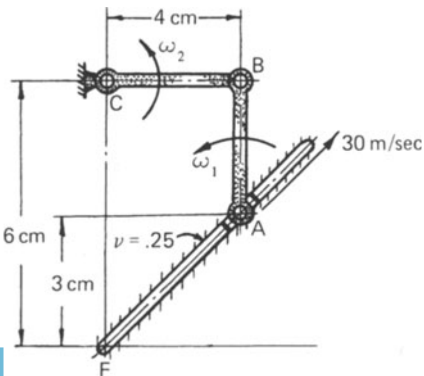
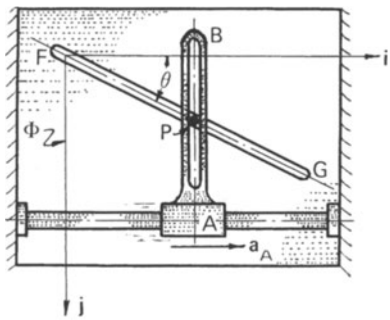
Problems

6.1. The slider block A of a mechanism has mass $m = 400$ gm and moves in the horizontal plane in a straight track with a dynamic coefficient of friction $\nu = 0.25$. At an instant of interest, the links AB and BC are in the positions shown in the figure, and A has a speed of 30 m/sec which is increasing at the rate of 20 m/sec^2. An instrument indicates that link AB is under tension. Find the forces that act on A in the plane of its motion at the moment of interest.



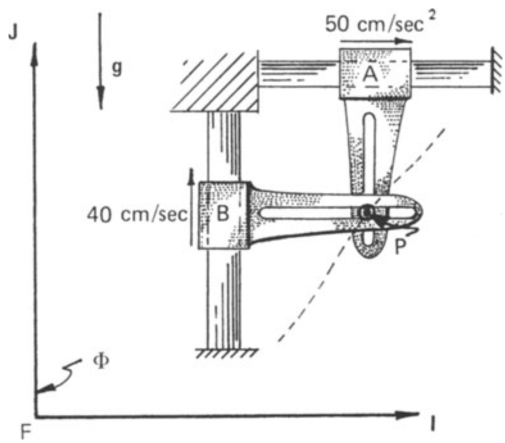
Problem 6.1.

6.2. A small pin P of mass m is constrained to move in a smooth, straight slot FG milled at an angle θ in a flat plate fixed in the horizontal plane. The motion of P is controlled by a smooth, slotted link AB that moves during an interval of interest with constant acceleration $\mathbf{a}_A$ in $\Phi = \{F; \mathbf{i}_k\}$, as shown in the figure. Find as functions of θ the force exerted on P by each slot, and show that the ratio of their magnitudes is a simple function of the angle θ alone. What is the acceleration of P relative to A?



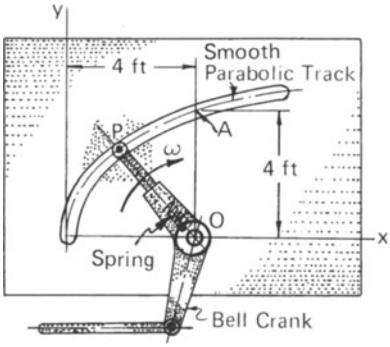
Problem 6.2.

6.3. Two slotted links shown in the figure move on smooth guide rails fixed at right angles to one another, their motion being controlled by a smooth pin of mass $m = 0.04$ kg. At a moment of interest in the machine frame $\Phi = \{F; \mathbf{I}_k\}$, the link A has an acceleration $\mathbf{a}_A = 50\mathbf{I}$ cm/sec^2, and the link B is moving upward with a speed of 40 cm/sec which is decreasing at the rate of 100 cm/sec^2. (a) What is the total force acting on P at the instant of interest? (b) Determine the force that each link exerts on P.



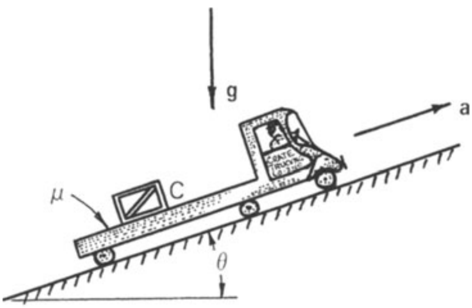
Problem 6.3.

6.4. A small guide pin P of mass 0.2 slug is attached to a spring loaded telescopic arm OP of a bell crank lever hinged at O. The guide pin moves in a smooth, horizontal parabolic track shown in the figure. At the track point A, the pin has a speed of 20 ft/sec, a rate of change of speed of 10 ft/sec^2, and the telescopic arm exerts a uniaxial compressive force on P. Determine for the instant of interest the magnitudes of all forces exerted on the pin P at A.



Problem 6.4.

6.5. The truck shown in the figure moves from rest with a constant acceleration **a** up an incline of angle θ. What is the greatest speed v that the truck can acquire in a distance d, if the crate C is not to slip on the truck bed? The coefficient of static friction is μ.

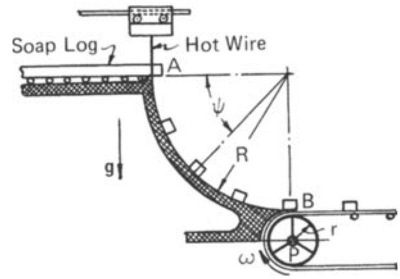


Problem 6.5.

6.6. A mass m is suspended from a point O by an inextensible cord of length ℓ, in a gravity field $\mathbf{g} = -g\mathbf{k}$ directed along the vertical axis through O. The mass rotates about the vertical axis with a constant angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$. (a) Apply cylindrical coordinates to determine the tension in the string and the vertical distance d from point O to the plane of motion of m. (b) Solve the problem by application of appropriate spherical coordinates.

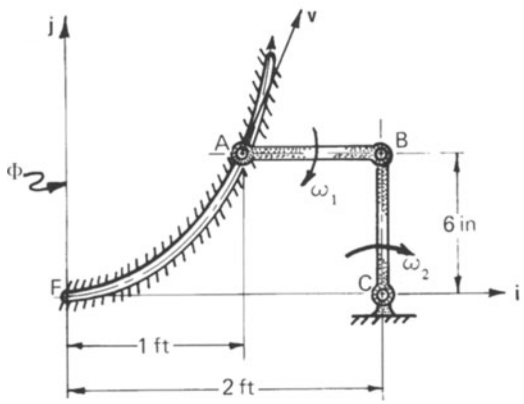
6.7. A small block of mass m rests on a rough, horizontal circular table that spins with a constant angular speed ω about its fixed central axis. What is the largest value that ω may have if the block is to remain at rest at the radial distance r from the center? Explain how this device might be used as an instrument to measure the coefficient of static friction.

6.8. Small bars of soap of equal weight W are cut from a continuous rectangular log by a moving hot wire at the point A in a packaging machine shown in the figure. Each bar is released from rest at A and slides down a smooth, circular chute of radius R to a conveyor belt at B. (a) Determine the constant angular speed of the belt pulley P so that continuous transfer of the bars to the conveyor will occur smoothly without sliding. (b) Find the contact force exerted on a bar of soap as a function of ψ and W. What force will a bar exert on the chute at the point B?



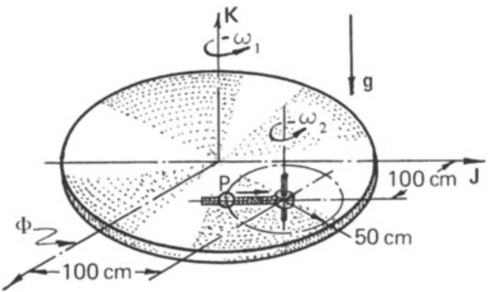
Problem 6.8.

6.9. A mechanism slider A of mass 0.2 slug moves in a horizontal plane in a smooth, parabolic slot defined by $2y = x^2$. At the instant shown in the diagram, A has a speed of 40 ft/sec, decreasing at the rate of $20\sqrt{2}$ ft/sec^2. All joints and surfaces are smooth. Find the plane forces that act on A.



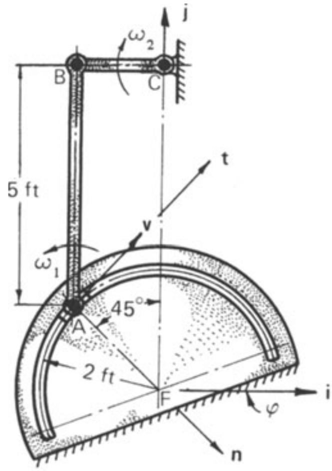
Problem 6.9.

6.10. A particle P weighing 1 N is free to slide on a smooth, rigid wire that rotates at a constant angular speed $\omega_2 = 30$ rad/sec relative to a platform. At the instant shown, the platform has an angular speed $\omega_1 = 15$ rad/sec that is decreasing at the rate of 5 rad/sec^2 relative to the ground frame Φ. The particle is initially at rest on the wire. What central directed force **F** is needed to impart to P an instantaneous initial acceleration of 1 m/sec^2 relative to the wire?



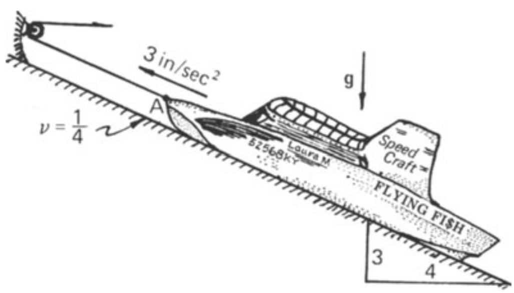
Problem 6.10.

6.11. The slider block A shown in the figure moves in a smooth, circular slot of radius 2 ft milled in a horizontal plate. The slider has a speed of 10 ft/sec, increasing at the rate of 20 ft/sec^2 at the instant when the links AB and BC are perpendicular. The link AB exerts a uniaxial tensile force on A, whose mass $m = 0.10$ slug. (a) Find the forces in the horizontal plane that act on A at this instant. (b) Suppose that the circular slot is rough with coefficient of dynamic friction $\nu = 0.30$, all other conditions being the same as before. Find the forces that act on A at the moment of interest.



Problem 6.11.

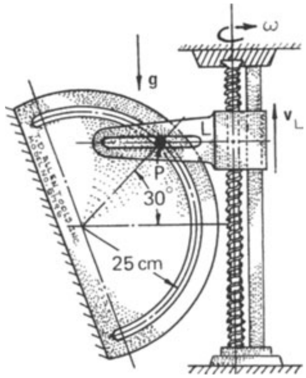
6.12. A 2560 lb boat is being dragged from its place of rest with a constant acceleration of 3 in./sec^2 up a steep inclined boat ramp shown in the figure. The dynamic coefficient of friction is $\nu = 1/4$ and at this place $g = 32$ ft/sec^2. (a) Find the tension in the cable at the connector A. (b) After 8 sec, the connector breaks. How much farther will the boat move up the plane?



Problem 6.12.

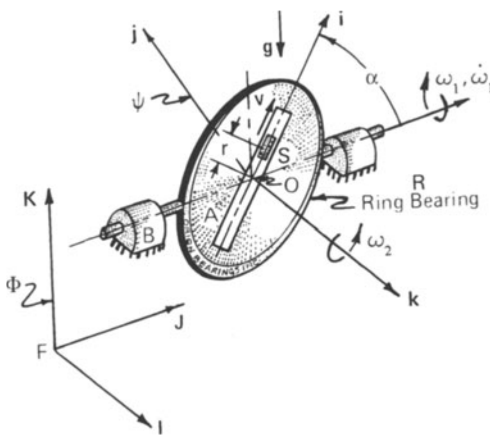
6.13. A guide link L is controlled by a drive screw to move a pin P of mass 50 gm in a circular slot in the vertical plane. The screw has a right-handed pitch $p = 5$ mm and is turning at a constant rate $\omega = 120$ rpm, as described in the figure. Ignore friction. What is the magnitude of the force exerted by the circular slot on the pin at the position shown?

6.14. Suppose that the drive screw described in the previous problem is turning at the rate $\omega = 150$ rpm, as shown, but is slowing down at the rate of 30 rpm each second. Calculate the magnitude of the force exerted by the circular slot on the pin at the position shown. What is the intensity of the force exerted on the pin by the guide link?



Problem 6.13.

6.15. A slider block S of mass $m = 0.5$ slug is constrained to move within a straight cylindrical tube attached to a large disk A supported in a ring bearing R. The machine is situated on the planet Vulcan where $g = 20$ ft/sec^2. The slider maintains a constant speed $v = 2$ ft/sec relative to A, which has a constant angular speed $\omega_2 = \dot{\alpha} = 2$ rad/sec relative to R. At an instant of interest shown in the figure, $\alpha = \tan^{-1}(3/4)$, $r = 2$ ft, and the ring bearing is turning about its horizontal shaft B with angular speed $\omega_1 = 10$ rad/sec and angular acceleration $\dot{\omega}_1 = 5$ rad/sec^2 in the inertial frame $\Phi = \{F; \mathbf{I}_k\}$. Determine the instantaneous value of the total contact force $\mathbf{F}_c$ exerted on S, referred to the frame $\psi = \{O; \mathbf{i}_k\}$ fixed in A.

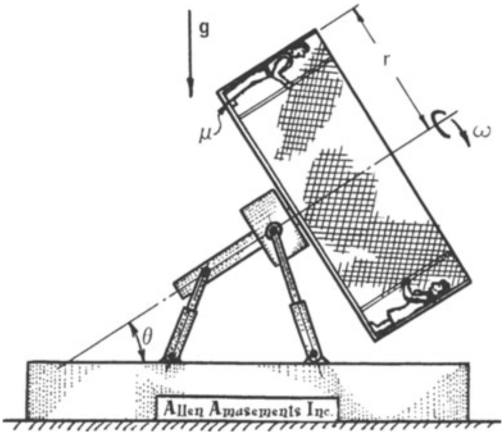


Problem 6.15.

6.16. A small object of mass m rests at the top of a smooth cylinder of radius r. Under the influence of gravity, a negligible disturbance causes the object to slide down the side of cylinder. Determine the angle ϕ_0 and the speed at which the object leaves the cylinder.

6.17. A constant total force $\mathbf{F} = 35\mathbf{i}$ lb acts for 3 sec at the center of mass particle C of a body $\mathscr{B}$ that weighs 161 lb. The initial velocity of C is $\mathbf{v}_0 = 9\mathbf{i} + 40\mathbf{j}$ ft/sec at $\mathbf{x}_0 = 16\mathbf{j}$ ft in the inertial frame $\Phi = \{O; \mathbf{i}_k\}$. (a) Find the velocity and the motion of C as functions of time in Φ. (b) Determine the speed of C after 3 sec. (c) What is its location 2 sec later, and how far did C move during that time? (d) Solve the problem for the same details by application of singularity functions. See Volume 1, Chapter 1, page 47.

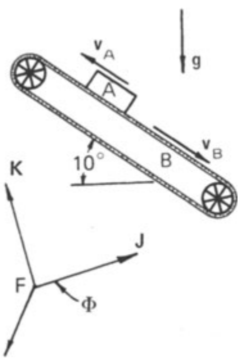
6.18. An amusement park centrifuge shown in the figure consists of a large circular cylindrical cage of radius r that rotates about its axis. People stand against the cylindrical wall, and after the cage has reached a certain constant angular speed ω, to further excite the riders the cage is rotated from its initial horizontal position to an inclined position at an angle θ. Determine the minimum angular speed in order that a passenger will not fall when reaching the highest point in the motion. The coefficient of static friction at the floor is μ.



Problem 6.18.

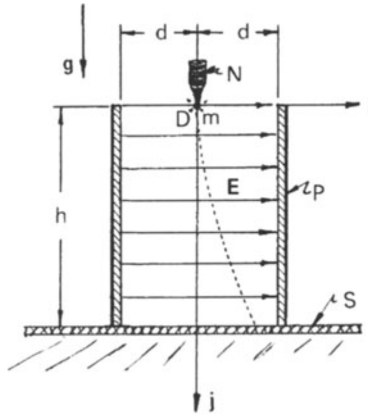
6.19. A small object of mass m rests on a smooth conical surface having an apex angle 2β. The cone turns about its vertical axis with a constant angular velocity $\boldsymbol{\omega} = \omega \mathbf{k}$ in a gravity field $\mathbf{g} = -g\mathbf{k}$. The object is restrained from sliding by an inextensible string of length ℓ attached at the apex on the axis of rotation. Identify appropriate spherical coordinates and apply (6.5) to determine the critical angular speed at which the object will leave the surface. What is the tension in the string at the critical speed?

6.20. The figure shows a box A moving upward on a loading belt B inclined at 10° and moving downward with a constant speed of 150 cm/sec relative to the ground frame $\Phi = \{F; \mathbf{I}_k\}$. At the initial instant, the speed of A is 50 cm/sec relative to Φ; and the coefficient of friction between the sliding bodies is $\nu = 0.3$. How long does it take to reduce the relative speed between the bodies to 25 cm/sec? Frame Φ should be suitably oriented for convenience.



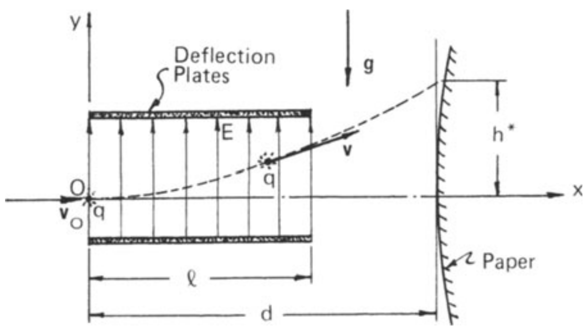
Problem 6.20.

6.21. An electrically conducting droplet of paint D of mass m and charge q, initially at rest at the tip of a nozzle N, falls through a uniform electric field of strength $\mathbf{E}$, directed as shown in the figure, and ultimately impacts a flat sheet S beneath it. The field deflection plates P have width $2d$ and height h. (a) Derive the equation of the path traveled by D. What is the maximum intensity of $\mathbf{E}$ that will still allow a droplet to impact S? (b) If the droplet D has an initial speed of 40 cm/sec, what electric field strength, directed as before, must be applied to produce a motion $\mathbf{x}(D, t) = 6t^2\mathbf{i} + (\beta t + \gamma t^2)\mathbf{j}$ cm, in which β and γ are constants? Determine β and γ. (c) Find the free fall droplet trajectory when the apparatus is tilted counterclockwise to an angle θ from the vertical axis.



Problem 6.21.

6.22. The deflection plates of an ink jet printer are arranged as shown in the figure. A charged ink droplet q enters the constant electric field $\mathbf{E}$ with the initial horizontal velocity $\mathbf{v}_0$ at point O. Find the trajectory $y = y(x)$ of q for $0 \leq x \leq d$ and determine the droplet deflection h^* at the paper surface, approximated as a plane. Show that, independent of g, the deflection h derived in (6.28d) for the case when $\ell = d$ is larger than the deflection h^* by an amount $h - h^* = (cE/2v_0^2)(d - \ell)^2$, where $c = q/m$.



Problem 6.22.

6.23. A bullet of mass m is fired directly into a fluid that exerts on the bullet a drag force that is proportional to its linear momentum. The gun has a muzzle velocity $\mathbf{v}_0$. Neglect gravity and other fluid forces. Determine the total distance traveled by the bullet.

6.24. Water exerts a drag force on a boat which is proportional to the cube of its speed. When the power is cut off, the boat's speed decreases from v_0 to $v(t)$ in time t. Find the distance traveled by the boat and determine t.

6.25. Consider a particle Q initially at rest at O in frame $\Phi = \{O; \mathbf{i}_k\}$ and acted upon by a constant force $\mathbf{f} = 4\mathbf{i} - \mathbf{j} + 32\mathbf{k}$ lb/unit mass and by a drag force $\mathbf{f}_D = -\dot{x}\mathbf{i} - 2\dot{y}\mathbf{j} - 0.4\dot{z}\mathbf{k}$ lb/unit mass. Find the velocity and the place in Φ occupied by Q after 2 sec.

6.26. A shell fired vertically upward from the ground with a muzzle velocity $\mathbf{v}_0$ experiences air resistance proportional to the square of its speed. (a) Determine the shells speed and altitude as functions of time. (b) What is the maximum altitude attained by the shell? (c) Find the time t^* required to reach the maximum height and show that no matter how large $\mathbf{v}_0$ may be, t^* cannot exceed $\pi\tau/2$. Identify the time constant τ.

6.27. A ball dropped from rest at the origin experiences air resistance proportional to the square of its speed. (a) Find its speed after the ball has fallen a distance h. What is its terminal speed? (b) Determine as functions of time the speed and the distance through which the ball has fallen. Sketch and label a nondimensionalized graph of the speed versus time and describe the results in a manner similar to Example 6.11, page 120.

6.28. Consider the following integral

$$u(t) = \int_{g(t)}^{h(t)} F(\tau; t)d\tau, \tag{P6.28a}$$

wherein $F(\tau; t)$ is an integrable function of τ and also depends continuously on a parameter t. Notice that the limits of integration are continuous functions $g(t)$ and $h(t)$ of t. (a) Use the definition of the derivative of a function $u(t)$, namely,

$$\frac{du(t)}{dt} = \underset{\Delta t \to 0}{\text{limit}} \frac{u(t + \Delta t) - u(t)}{\Delta t}, \tag{P6.28b}$$

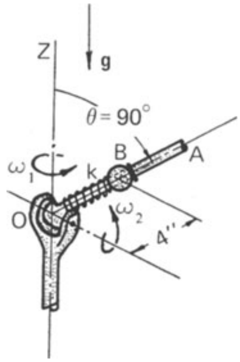
apply the mean value theorem of integral calculus, and derive Leibniz's formula for the derivative of the integral (P6.28a):

$$\frac{du(t)}{dt} = F(h(t); t)\frac{dh(t)}{dt} - F(g(t); t)\frac{dg(t)}{dt} + \int_{g(t)}^{h(t)} \frac{dF(\tau; t)}{dt} d\tau. \tag{P6.28c}$$

(b) Apply this rule to show that (6.47) is a particular solution of the differential equation (6.39).

6.29. Derive from (6.47) the particular solution (6.45b) of the differential equation (6.39) when $h(t)$ is given by (6.45a).

6.30. A ball governor of a speed control device consists of an arm OA hinged at O to a vertical shaft OZ that rotates relative to the machine with a constant angular speed $\omega_1 = 9$ rad/sec, as shown. At the same time, OA is elevated at a constant angular rate $\omega_2 = 3$ rad/sec relative to the shaft and a ball B of mass 0.02 slug slides on the smooth arm. The ball is attached to a spring, the other end of which is fastened to the arm. Design criteria specify that the shut-off position at which the ball comes to rest on OA must be 4 in. from O; and for the position shown at $\theta = 90°$, the spring must elongate 2 in. to achieve shut-off. Find the spring constant in units of lb/in. that will satisfy the design shut-off criteria. What force is exerted by the spring in this position?



Problem 6.30.

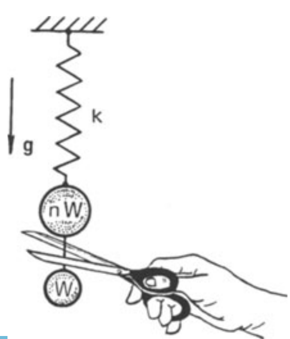
6.31. Consider a rigid body rotating through an angle $\theta(t)$ about a fixed axis with unit direction $\boldsymbol{\alpha}$. Notice that the velocity vector $\dot{\mathbf{x}}(P,t) = \dot{\theta}\boldsymbol{\alpha} \times \mathbf{x}$ of a body point P at $\mathbf{x}(P,t)$ from a point O on $\boldsymbol{\alpha}$ yields the equation $d\mathbf{x}/d\theta = \boldsymbol{\alpha} \times \mathbf{x}$ relating $\mathbf{x}$ and θ. (a) If initially $\mathbf{x}(P,0) = \mathbf{x}_0$ and $\theta(0) = 0$, prove that $\mathbf{x} \cdot \boldsymbol{\alpha}$ is a constant, and derive the relation $d^2\mathbf{x}/d\theta^2 + \mathbf{x} = (\boldsymbol{\alpha} \cdot \mathbf{x}_0)\boldsymbol{\alpha}$. Hint: Notice that $d\mathbf{x}/d\theta$ is perpendicular to $\boldsymbol{\alpha}$. (b) Determine the general solution of this vector differential equation. This involves two constant vectors of integration, say $\mathbf{A}$ and $\mathbf{B}$. (c) Find $\mathbf{A}$ and $\mathbf{B}$ and thus show that the solution yields (2.7), Volume 1, for the displacement of a particle P of a rigid body in its finite rotation about the fixed line.

6.32. The motion of a particle P initially at rest at the origin is governed by the equation $\ddot{x} - q^2 x = e^{qt}$. Find the motion of P.

6.33. The motions of two particles P and Q are governed by the following scalar equations of motion: $\ddot{x}(P,t) + p^2 x(P,t) = g$ and $\ddot{x}(Q,t) - p^2 x(Q,t) = g$, in which p and g are constants. Initially, each particle is started separately at the place $x(0) = x_0$ with a speed v_0. Find the motions of P and Q and discuss their physical nature. Determine their common motion when $p = 0$.

6.34. A linear spring-mass system shown in its natural state in Fig. 6.13, page 134, is given an instantaneous initial speed $v_0 = 3$ ft/sec on a smooth horizontal surface. The mass $m = 8\ \text{lb}_m$ and the spring stiffness $k = 3$ lb/in. Suppose that $g = 32\ \text{ft/sec}^2$. What is the maximum displacement of m? Caution: See Chapter 5 remarks on measure units, page 86.

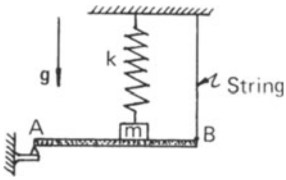
6.35. A linear spring of stiffness k supports weights W and nW connected by a cord, as shown. Initially, the system is at rest. (a) Determine the acceleration of the load nW immediately



Problem 6.35.

after the cord supporting the load W is cut. (b) Find the motion $z(t)$ of the load nW from the undeformed natural state of the spring. (c) Determine the motion $x(t)$ from the initial stretched state of the spring. (d) What is the motion $\xi(t)$ from the static equilibrium state of the load nW? (e) Which of the three motions is the simpler? Are they equivalent? How are they related?

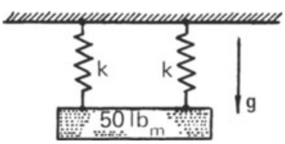
6.36. The figure shows an unstretched linear spring of stiffness k attached to a small block of mass m at rest on a horizontal board simply supported at A and suspended by a string at B. The string is cut and the board falls clear of m. Derive the equation of motion for the mass and determine its subsequent motion. How long does it take for m to return to its initial position?



Problem 6.36.

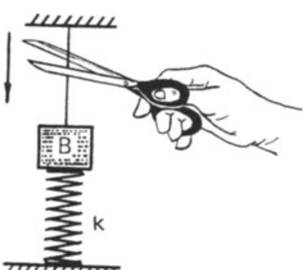
6.37. A certain simple harmonic oscillator has mass $m = 2$ slug and an equivalent spring constant $k_e = 600$ lb/in. The load is released at $u_0 = 2$ in. with a speed $\dot{u}_0 = -20$ in./sec directed toward its equilibrium position. Determine the frequency, amplitude, and initial phase of its motion $u(t)$.

6.38. A load $m = 50$ lb$_m$ is supported as shown by two linear springs having the same elasticity $k = 25$ lb/in. (a) Find the static stretch of each spring from its natural state and determine the stiffness of a single equivalent spring that may replace the parallel pair. (b) The mass is given an additional 2 in. displacement and released. Find its maximum speed and determine its greatest height from the equilibrium position. How long does it take to first attain these states? Compare these times with the period of the vibration.



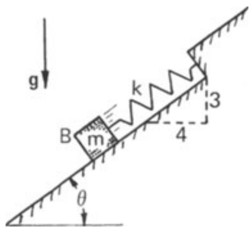
Problem 6.38.

6.39. The figure shows a block B weighing 25 N suspended by a string and attached to a linear spring of stiffness $k = 20$ N/cm in its natural state. Determine the amplitude, the frequency, and the position about which the vibration will occur when the string is suddenly cut.



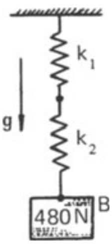
Problem 6.39.

6.40. A small block B of mass $m = 0.25$ slug is attached to a linear spring of stiffness $k = 16$ lb/ft in a gravity field of strength $g = 32$ ft/sec^2. The spring is compressed 6 in. from its natural state and the mass is released to execute oscillations on a smooth plane inclined as shown in the figure. Find the motion as a function of time and determine its frequency and amplitude.



Problem 6.40.

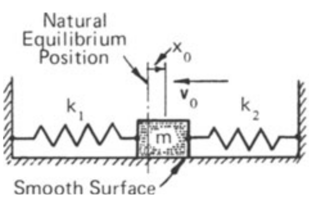
6.41. The figure shows a box B of weight 480 N supported by uniaxial linear springs having constant elasticities $k_1 = 40$ N/cm and $k_2 = 60$ N/cm. (a) Find the static displacement δ of B and determine the stiffness of a single equivalent spring that may replace the series pair. (b) The box is displaced an additional 5 cm from δ and released. What is the period of its vibration? (c) What is the location of the box from its static state 2 sec after its release?



Problem 6.41.

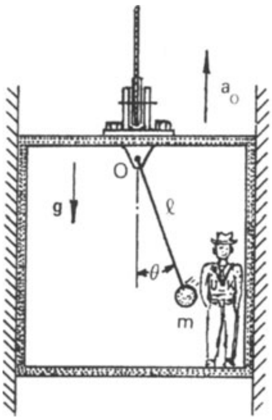
6.42. A particle P of mass m and charge q moves in an electromagnetic field of constant field strengths $\mathbf{E} = E\mathbf{i}$ and $\mathbf{B} = B\mathbf{k}$ in an inertial frame $\Phi = \{O; \mathbf{i}_k\}$. Initially, P is at rest at O. Find the motion $\mathbf{x}(P, t)$ of P in Φ and characterize its path. Neglect gravity.

6.43. Two unstretched, linear springs having moduli k_1 and k_2 are fastened, as shown, to a slider mass m that rests on a smooth horizontal surface. The slider is displaced a distance x_0 and released with speed v_0 directed toward the natural state. (a) What are the circular frequency, the period, and the amplitude of the vibration? (b) Derive the subsequent motion of m. Sketch the motion as a function of $\theta = pt$ and label its major features.



Problem 6.43.

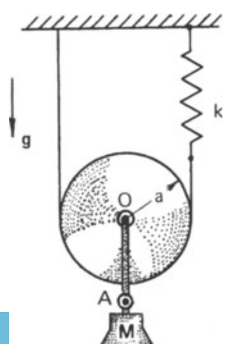
6.44. A simple pendulum shown in the figure is supported by a light, hinged rod hung from the ceiling in an elevator which moves upward with a constant acceleration $\mathbf{a}_O$. A curious person displaces the bob a finite angular amount θ_0 and releases it. (a) Find as a function of θ the ratio of the tension in the rod to the weight of the bob. (b) For small placements θ_0, what will be the circular frequency and the period of the pendulum motion witnessed by the person? (c) How will these results be changed if the elevator accelerates downward at the same rate? Describe any potentially unusual effects.



Problem 6.44.

6.45. According to elasticity theory, the infinitesimal circumferential engineering strain ϵ of a homogeneous, thin circular ring undergoing pure radial oscillations in its horizontal plane is given by $\epsilon = u/r$, where r denotes the undeformed radius and u is the infinitesimal radial displacement of the ring. The ring has uniform cross sectional area A and mass density μ per unit length. (a) Consider a circumferential ring element of mass $dm(P)$ at a material point P. Apply Hooke's law for the uniform circumferential engineering stress $\sigma = E\epsilon$, where E is Young's modulus, and derive the equation for the radial motion. Recall that the circumferential force F is defined by $F = \sigma A$. (b) Determine the circular frequency and period. (c) What is the stiffness of an equivalent linear spring-mass system that will produce the same vibrational frequency of a load equal to total mass of the ring?

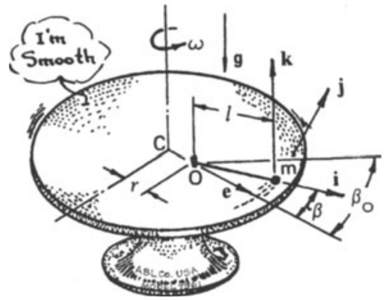
6.46. The spring and pulley system shown in the figure supports a load of mass M. The spring has stiffness k and the masses of the cable, pulley, and load support bar are negligible.



Problem 6.46.

Neglect friction and determine the circular frequency and period of the free vibration of the load in its vertical displacement $x(t)$ from the static equilibrium state of the system.

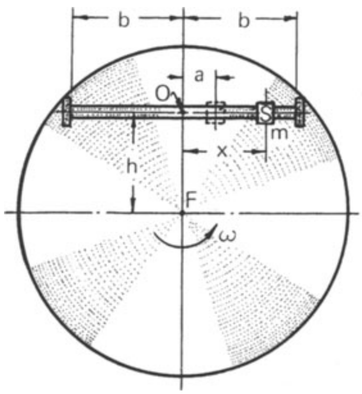
6.47. A pendulum bob of mass $m = 0.01$ kg is fastened by a string of length $l = 16$ cm to a hinge pin at $r = 4$ cm from the center of a smooth horizontal table on which the bob rests. The table turns with a constant angular speed ω, as shown in the figure. Relative to an observer in the table reference frame, the pendulum executes oscillations of small amplitude β_O and period $\tau = 0.5$ sec. Find the angular speed of the table and compute the string tension T when $\beta = \beta_O$.



Problem 6.47.

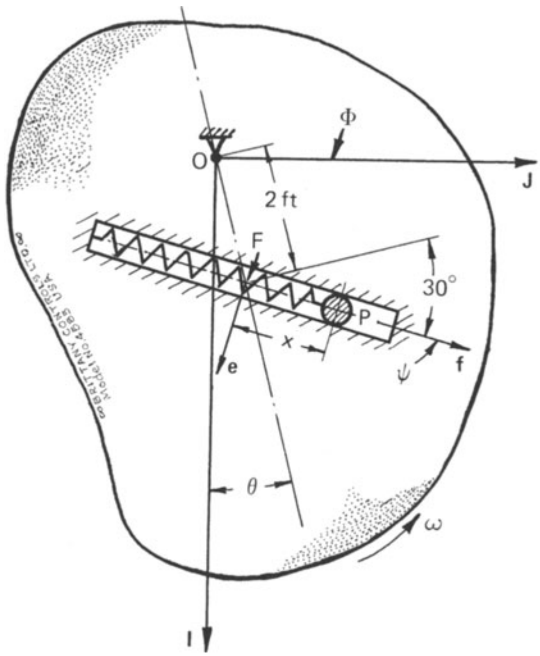
6.48. Gravitational attraction by a fixed, homogeneous, thin ring of radius R and mass M induces a particle P of mass m to move along its normal central axis, as shown in Fig. 5.13. (See Example 5.6, page 38.) (a) Derive the differential equation of motion for P. (b) Show that for sufficiently small displacements $\mathbf{X}(P, t)$ from the center O, the motion of P is simple harmonic. What is the frequency of its small oscillations?

6.49. A smooth, rigid rod of length $2b$ is attached to a table that turns in the horizontal plane with a constant angular velocity ω, as shown. A slider block S of mass m is released from rest relative to the rod at a distance a from its midpoint O. (a) Determine the horizontal force $\mathbf{R}$ exerted by the rod on the slider as a function of its distance x from O. (b) Find of the motion of S relative to the table.



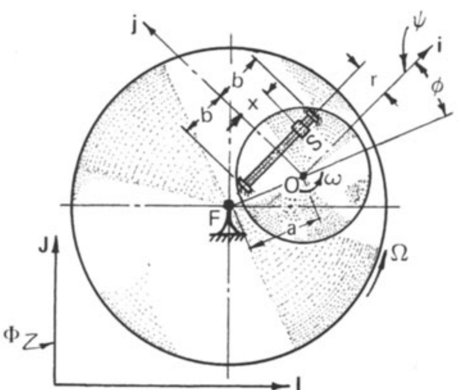
Problem 6.49.

6.50. A small ball P of mass m slides in a smooth slot cut in a flat plate, as described in the diagram. The plate rotates in the horizontal plane with a constant angular speed $\dot{\theta} = \omega$ about an axle at O in frame $\Phi = \{O; \mathbf{I}, \mathbf{J}\}$ fixed in the plane space. The ball is attached to a linear spring of modulus k, which initially is unstretched when P is released from rest relative to the plate at F. (a) Find the motion $\mathbf{x}(P, t)$ of P relative to the plate for all constant values of the angular speed ω. (b) Determine the force exerted on P by the slot as a function of x and as a function of t. (c) Characterize all physical aspects of the motion of P for all values of ω. Refer all quantities to the plate frame $\psi = \{F; \mathbf{e}, \mathbf{f}\}$.



Problem 6.50.

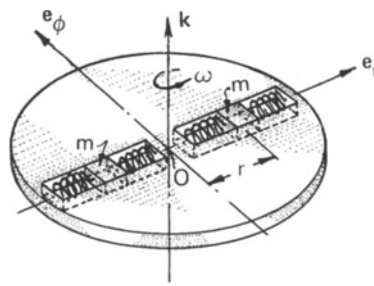
6.51. A block S of mass m is free to slide on a smooth rod of length $2b$ shown in the figure. The rod is fastened to a circular disk that rotates about an axle at O with a constant angular velocity



Problem 6.51.

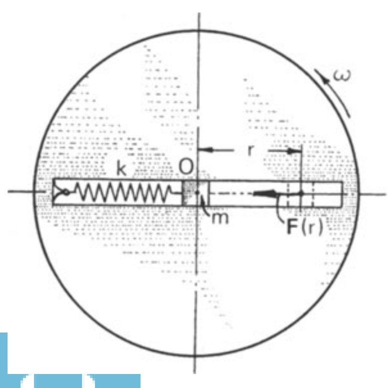
$\boldsymbol{\omega} = \omega \mathbf{k}$ relative to a turntable. The turntable spins in the horizontal plane with a constant angular velocity $\boldsymbol{\Omega} = \Omega \mathbf{k}$ about an axle at F in the ground frame. (a) Account for all forces that act on S and derive its scalar equations of motion referred to the disk frame $\psi = \{O; \mathbf{i}_k\}$. What unknown quantities do these equations determine? (b) Suppose that S is initially at ease at $x = 0$. Determine the unknown quantities as functions of time.

6.52. The diagram shows two slider blocks of equal mass m attached to precompressed springs of equal stiffness $k/2$. The blocks are confined to slide horizontally in smooth radial slots in a table that spins counterclockwise with a constant angular speed ω. Each block is positioned at a distance ℓ from the center O when $\omega = 0$. If each spring is always under compression, determine the equilibrium position $r = r_S$ of each block relative to the table. Examine the stability of this relative equilibrium state.



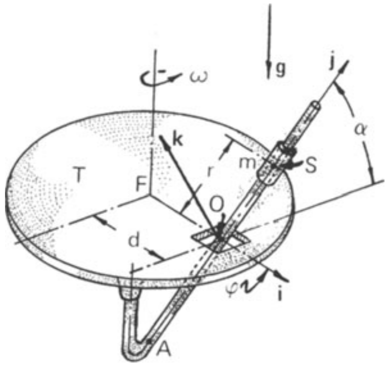
Problem 6.52.

6.53. The figure shows a slider block of mass m attached to a spring of stiffness k in its natural state at the center of a smooth rotating table upon which it rests in the horizontal plane. The table turns with a constant anticlockwise angular speed ω. (a) Determine the equation for the motion $r(t)$ of the slider and examine the stability of the relative equilibrium states. (b) Note that Hooke's spring law (6.64) is the same in every reference frame and for every observer, that is, the same extension of the spring in a fixed reference frame and in any other reference frame having an arbitrary motion gives rise to the same force. The spring force is an internal action. The inertial forces induced by the motion of the frame are external actions of the environment on the system. The rotating observer, however, may perceive a pseudo-spring force $\mathbf{F}(r)$ with stiffness k^* that includes these inertial effects of the environment. What pseudo-spring force and apparent stiffness are perceived by an observer in the table frame? (c) Discuss the character of the motion as ω is gradually varied.



Problem 6.53.

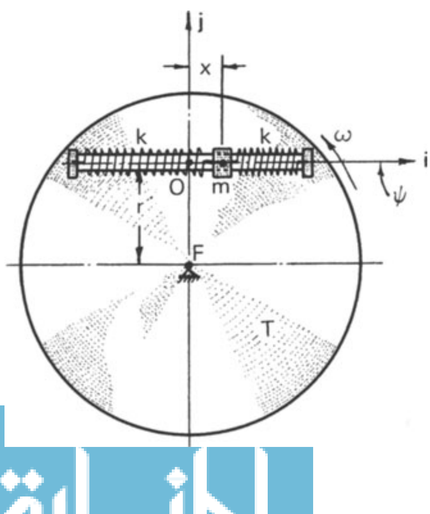
6.54. A block S of mass m slides freely on a smooth rigid rod inclined at an angle α with the horizontal plane of a rotating table T to which the rod is fastened, as shown in the figure. The table turns with a constant angular velocity $\boldsymbol{\omega}$ about a fixed vertical axis. If S is projected upward from point O in the plane of T with an initial speed v_0 relative to T, determine its subsequent position as a function $r(t)$. Find the initial force exerted on S by the rod. Refer all quantities to the frame $\varphi = \{O; \mathbf{i}_k\}$ fixed in the rod.



Problem 6.54.

6.55. Suppose in the previous problem that a coaxial spring of elasticity k is attached to the smooth rod at point A and to the block S. The spring is unstretched when S is at O where its initial speed is v_0, as before. (a) Determine the relative equilibrium positions r_S of S. (b) Find the motion $r(t)$ of S relative to the table for all values of the angular speed ω. (c) Discuss the stability of the relative equilibrium states of S. Refer all quantities to the rod frame $\varphi = \{O; \mathbf{i}_k\}$.

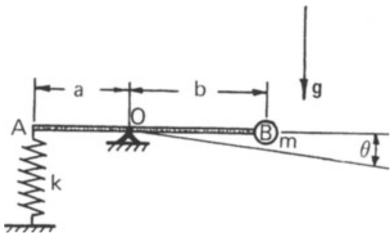
6.56. A smooth rigid rod, whose geometry is described in the figure, is attached to a table T that rotates in the horizontal plane with a constant angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$. A slider block of mass m, supported symmetrically by identical springs of elasticity k, is released from rest relative to the rod at a distance a from the natural state at point O. (a) Determine the rod reaction force on m as a function of its distance x from O. (b) Determine the critical angular speed ω^*



Problem 6.56.

of the table for which a simple harmonic motion is not possible. (c) Find the motion $x(m, t)$ for the three cases for which $\omega < \omega^*$, $\omega = \omega^*$, and $\omega > \omega^*$. What are the period and the amplitude of the motion of m in the oscillatory case? Use the table frame $\psi = \{O; \mathbf{i}_k\}$ as reference.

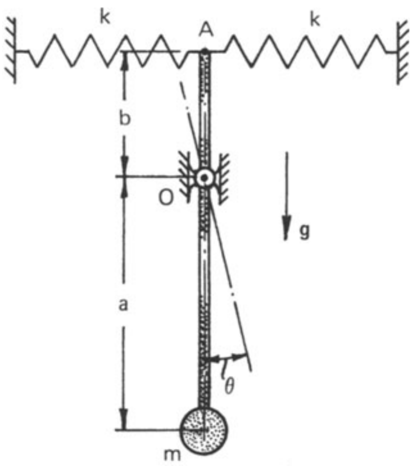
6.57. A mass m is attached to one end of a rigid rod supported by a smooth hinge at O and by a spring of stiffness k at A. The rod has negligible mass and the system is in equilibrium in the horizontal position shown in the figure. The mass is given a small angular placement and released. Apply the moment of momentum principle to derive the equation for the angular motion $\theta(t)$ of m and find the frequency of its small oscillations.



Problem 6.57.

6.58. Apply the moment of momentum relation (6.80) for a moving point O to derive the equation of motion of the pendulum bob in Problem 6.47.

6.59. A pendulum bob of mass m is attached to one end of a thin, rigid rod suspended vertically by a smooth hinge at an intermediate point O. The rod is fastened at its other end to identical springs of stiffness k, shown in the figure in their undeformed configuration. The pendulum is given a small angular placement θ_0 and released with a small angular speed ω_0 toward the vertical equilibrium state. Ignore the mass of the rod. Find the motion $\theta(t)$ of the bob and describe its physical characteristics.

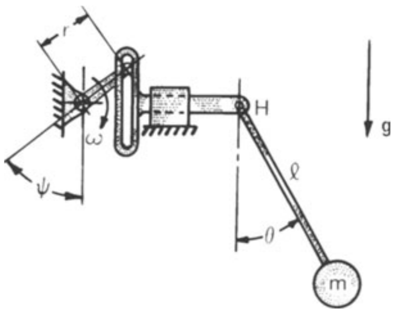


Problem 6.59.

6.60. Problem 4.48 in Volume 1 illustrates a simple pendulum of mass m and length ℓ hung from a sliding support that oscillates vertically with a motion $x(S, t) = a + b \sin pt$, where a, b, p are constants. Derive the scalar equations of motion for the bob. What quantities do these equations determine? This is a difficult nonlinear problem whose exact solution is unknown. For small amplitude pendulum oscillations, however, the motion $\theta(t)$ is described by Mathieu's linear differential equation, whose analysis, though well-studied, is not elementary. Let $2z = pt + \pi/2$ and thus show that the Mathieu form of the equation of motion for small angular placements is

$$\frac{d^2\theta}{dz^2} + \left(\frac{4g}{p^2\ell} - \frac{4b}{\ell}\cos 2z\right)\theta = 0. \tag{P6.60}$$

6.61. The hinge support H for a simple pendulum of mass m and length ℓ is attached to a Scotch mechanism. The crank has radius r and turns with a constant angular speed ω, as illustrated. (a) Derive the differential equation of motion for the bob m. (b) This equation has no known exact solution. Show, however, that for a small angular motion $\theta(t)$ the differential equation reduces to the equation of motion for the forced vibration of an undamped, harmonic oscillator. Find its solution when the pendulum is released at a small angle θ_0 with $\dot{\theta}(0) = 0$.

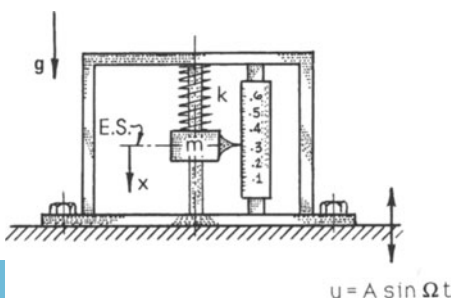


Problem 6.61.

6.62. Discuss the free vibrational motion (6.86l) of the heavily damped oscillator in relation to Fig. 6.22, page 156. Show that if the mass is released from rest, it can only creep back to its equilibrium position at $z = 0$ as $t \to \infty$ (similar to Curve 2). However, if released with initial velocity v_0, it is possible that the load may cross its equilibrium position at one and only one instant t_o, as suggested in Fig. 6.22. Find t_o.

6.63. Repeat the details of the last problem for the critically damped, free vibrational motion described in (6.86n).

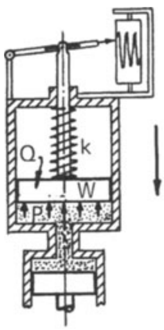
6.64. The pointer of a vibration instrument has mass m and is supported vertically by a spring of stiffness k. The base is subjected to a vertical motion $u = A\sin\Omega t$. (a) Derive the



Problem 6.64.

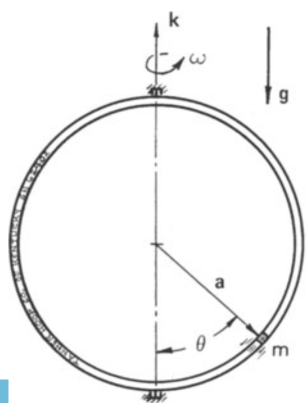
equation for the steady-state motion $x(t)$ of the pointer relative to the instrument and determine its amplitude. (b) Let $k = 5$ N/mm, $m = 2$ kg, and suppose that the pointer moves between the 0.35 and 0.45 scale marks when the base motion has frequency $\Omega = 100$ rad/sec. Determine the amplitude of the base motion. (c) Now suppose further that the base motion frequency is doubled while its amplitude is unchanged. What will be the response range of the pointer? Is the pointer amplitude increased or decreased? (d) Is the system operating above or below its resonant frequency?

6.65. The steam pressure indicator shown in the figure is an instrument that records the time varying cylinder pressure generated in an engine. The piston Q, with surface area A, is restrained by a spring of stiffness $k = 100$ lb/in. on one side and subjected to a periodically varying engine cylinder pressure $P = P_0 \cos \omega t$ on the other. The pressure produces forced vibrations of the piston which are recorded on a uniformly rotating drum. The design requires that the natural, free vibrational frequency p of the piston and recording pen assembly, which has a total effective weight W, shall be much greater than the cylinder pressure fluctuation frequency ω. Frictional effects may be considered negligible. Derive the equation of motion for the piston assembly relative to its static equilibrium position and estimate the weight limit of the assembly if the pressure fluctuation frequency is not to exceed 10 Hz.



Problem 6.65.

6.66. A heavy bead of mass m slides freely in a smooth circular tube of radius a in the vertical plane. The tube spins with constant angular speed about the vertical axis, as shown.

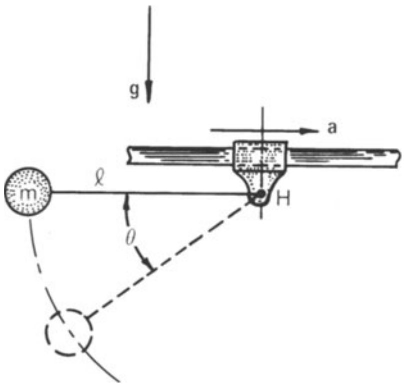


Problem 6.66.

(a) Derive the equation of motion two ways: (i) by use of the moment of momentum principle and (ii) by application of the Newton–Euler law. (b) Examine the infinitesimal stability of all relative equilibrium positions of the bead.

6.67. Experiment shows that the undamped, forced horizontal motion of the system shown in Fig. 6.20, page 152, has a steady-state amplitude H_1 when the driving frequency is Ω_1. When the machine is speeded up to double the driving frequency, the amplitude is reduced to 20% of its previous value. What is the resonant frequency of the system? Was the test data obtained above or below the resonant frequency?

6.68. The supporting hinge H of a simple pendulum of mass m and length ℓ is attached to a horizontal slider that has a constant acceleration **a**. The pendulum is released from rest in a horizontal position relative to the slider, as shown in the figure. (a) Find the pendulum string tension $T(\theta)$ as a function of its angular displacement θ. (b) Show that the other extreme position of the pendulum is given by $\theta_e = 2\tan^{-1}(g/a)$ and determine the string tension in terms of $a = |\mathbf{a}|$ at both extremes. (c) Derive an equation for the time t_e required to attain the position θ_e. (d) Determine all positions of relative equilibrium and examine their infinitesimal stability in terms of the assigned parameters only. Refer all quantities to the natural intrinsic frame for m.

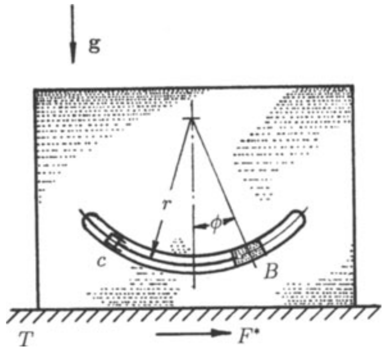


Problem 6.68.

6.69. A slider block B of mass m oscillates in a smooth circular groove of radius r milled in a plate in the vertical plane. The slider is attached to a linear viscous damper of circular design and damping coefficient c. The assembly is mounted on a shaker table T that exerts a horizontal driving force $F^* = F_0 \sin \Omega t$, as shown. (a) Derive the differential equation for the finite amplitude motion of B about its vertical equilibrium position and find an equation for the force exerted by the groove on the slider. (b) Now suppose that the shaker table is arrested and the damper is removed so that $\Omega = 0$ and $c = 0$ in the equation of motion. The block is then released from rest at a finite angle $\phi(0) = \phi_0$. Derive an exact integral relation that determines the period of the finite motion as a function of ϕ_0. What is the period of the small amplitude motion?

6.70. Consider the shaker table (Problem 6.69) for the case when the angular placement $\phi(t)$ of the slider is small. (a) Find the steady-state and transient parts of the motion $\phi(t)$. (b) What is the resonant frequency of the system? (c) What is the amplitude at the resonant frequency? (d) Identify the amplitude factor for the system.

6.71. A small cylindrical block B of unit mass oscillates with a simple harmonic motion $y = a\cos pt$ in a smooth, straight cylindrical tube oriented in the east–west direction on the Earth's surface at north latitude λ. The parameters a and p are constants. Show that in addition

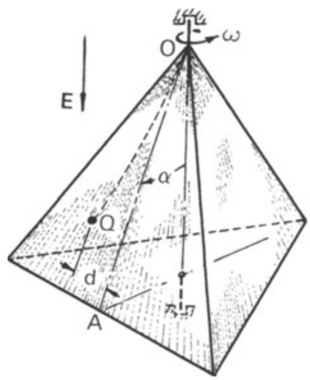


Problem 6.69.

to the weight of B, the Earth's rotation induces a tube reaction force on B which has both a north–south component and a vertical, radially directed component. Although these additional force components are very small compared with the weight of B, over a period of time they eventually induce wear of the tube surface, for example.

6.72. The motion of a particle on a smooth plane inclined at an angle γ is determined by the coupled equations $\ddot{x} - 2\dot{y}\omega\cos\gamma = g\sin\gamma$, $\ddot{y} + 2\dot{x}\omega\cos\gamma = 0$, in which ω is a small constant for which terms of $O(\omega^2)$ may be neglected and g is the acceleration of gravity. If the particle starts from rest at the origin of the inclined plane frame $\psi = \{O; \mathbf{i}, \mathbf{j}\}$, show that after a time t the particle has been deflected a distance $d(t) = \frac{1}{6}g\omega t^3 \sin 2\gamma$ from the $\mathbf{i}$-axis.

6.73. A particle Q of mass m and charge $q > 0$ moves in outer space down the side of a smooth, right pyramid that rotates with a small, constant angular speed ω about its fixed vertical axis in a constant electric field $\mathbf{E}$ directed as shown in the figure. Show that if the particle starts from rest at the apex O, its trajectory suffers a deflection $d(t)$ from the normal, altitude line OA which, to the first order in ω, is given by $d(t) = \frac{1}{6}\omega c E t^3 \sin 2\alpha$. Herein α denotes the surface inclination from the vertical axis and $c \equiv q/m$.



Problem 6.73.

6.74. A projectile is fired from the ground at north latitude λ with an initial velocity $\mathbf{v}_0$ directed skyward and it attains the ultimate altitude h. Neglect air resistance; assume that h is sufficiently small that effects due to altitude variations in $\mathbf{g}$ may be ignored; and include only first order effects of the Earth's rotation rate Ω. (a) Determine the Coriolis deflection $d^*(h)$ when the projectile reaches the height h. (b) Show that the projectile strikes the ground to the west of its launching site at a distance $d = \frac{8}{3}\Omega h \cos\lambda\sqrt{2h/g}$. (c) Find expressions for d^* and d in terms of the initial speed of the projectile.

6.75. A person seated at the wall in a cylindrical amusement park centrifuge of radius a tosses a ball B straight upward into the sky. The centrifuge has a constant angular velocity $\boldsymbol{\omega}$ relative to the Earth at north latitude λ. Derive the scalar equations of motion for B referred to the centrifuge frame $\psi = \{O; \mathbf{i}_k\}$. Include the effects of the Earth's rotation and identify the appropriate initial data. Check your result against the text solution in (6.109)–(6.111) for the free fall case when $\boldsymbol{\omega} = \mathbf{0}$. Show that when $\boldsymbol{\Omega} = \mathbf{0}$, $\dot{x}^2 + \dot{y}^2 = \omega^2(r^2 - a^2)$, where $r(t)$ is the radial distance of B from the centrifuge axis at time t.

6.76. British battle maps for the Falkland Islands conflict of 1914 show that the British directed their fire on the Germans from the north, almost directly southward, while heading east at a constant flank speed v^*. (a) Derive equation (6.117), for the Coriolis deflection relative to the ship, at 50 ° N and S latitudes; and find the projectile range and its Coriolis variation. (b) Determine the range and the Coriolis deflection for a shell fired with a muzzle velocity $V = 2650$ ft/sec at an angle of elevation $\alpha = 10°$. If the gun sight design corrected for the Coriolis effect only near 50 °N latitude, what is the total deflection by which the British shells would miss a German cruiser when fired at 50° S latitude? (c) Discuss any situations where the deflection may vanish when $\Omega \neq 0$.

7

Momentum, Work, and Energy

7.1. Introduction

Several methods of integration of the Newton–Euler vector equation of motion and its related scalar equations have been studied in a variety of applications in previous chapters. Although it is not possible to integrate these equations in general terms for all types of problems, certain kinds of problems do admit general first integrals that lead to several additional and useful basic principles of mechanics: the impulse–momentum principle, the torque–impulse principle, and the work–energy principle. Moreover, for certain kinds of forces, the work–energy principle may be reduced to a powerful fundamental law known as the principle of conservation of energy. The law of conservation of momentum and the law of conservation of moment of momentum are two more first integral principles that derive from the Newton–Euler law and the moment of momentum principle. This chapter concerns the development and application of these several additional principles.

7.2. The Impulse–Momentum Principle

The first integral of the equation of motion has been obtained in a variety of special problems where the force acting on a particle was a specified function of time. However, it is sometimes possible to obtain information about the motion even though a full specification of the force is not known. In particular, when a ball strikes a wall, the force exerted by the wall on the ball varies suddenly in time, and though we have no way of knowing the precise manner in which this impulsive force changes with time, we can still obtain useful information about the motion of the ball or the force exerted by the wall. To see how this may be done, we introduce the vector–valued integral function $\mathscr{I}(t; t_0)$, called the *impulse of the force* $\mathbf{F}(t)$,

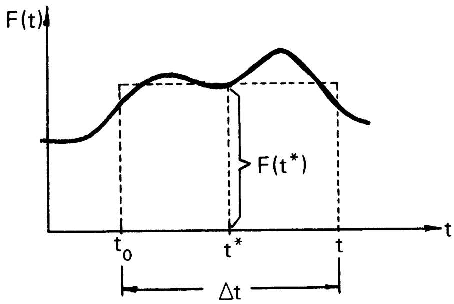


Figure 7.1. Graphical interpretation of the mean value theorem.

defined by

$$\mathscr{I}(t;t_0) \equiv \int_{t_0}^{t} \mathbf{F}(t)dt. \tag{7.1}$$

Then integrating the Newton–Euler equation of motion: $d\mathbf{p}/dt = \mathbf{F}(t)$ with respect to time on the interval $[t_0, t]$ in an inertial reference frame Φ and writing $\Delta\mathbf{p} \equiv \mathbf{p}(t) - \mathbf{p}(t_0)$ for the change in the momentum of the particle, we obtain the *impulse–momentum principle*:

$$\mathscr{I}(t;t_0) = \Delta\mathbf{p}. \tag{7.2}$$

In words, *the impulse of the force over the time interval* $[t_0, t]$ *is equal to the change in the linear momentum of the particle during that time.* We note that impulse has the physical dimensions $[\mathscr{I}] = [FT] = [MLT^{-1}]$.

The mean value of the force acting over the time interval $[t_0, t]$ is determined by the impulse. Consider first the one-dimensional graph shown in Fig. 7.1 for a force $F(t)$. According to the mean value theorem of integral calculus, there exists a value of t, say t^*, such that

$$\int_{t_0}^{t} F(t)dt = F(t^*)\Delta t, \tag{7.3}$$

wherein $t_0 \le t^* \le t$ and $\Delta t = t - t_0$. Geometrically, (7.3) shows that the area under the $F(t)$ curve on $[t_0, t]$ in Fig. 7.1 is equal to the area on $[t_0, t]$ of a rectangle of height $F(t^*)$. The value $F(t^*)$ is the *average value* of $F(t)$ on $[t_0, t]$. The same formula (7.3) may be applied to each force component. Therefore, more generally, the *average value* $\mathbf{F}^*$ of the force $\mathbf{F}(t)$ on the interval $[t_0, t]$ is defined by

$$\mathbf{F}^* \equiv \frac{1}{\Delta t}\int_{t_0}^{t} \mathbf{F}(t)dt = \frac{\mathscr{I}(t;t_0)}{\Delta t} = \frac{\Delta\mathbf{p}}{\Delta t}, \tag{7.4}$$

wherein (7.1) and (7.2) are introduced.

This result shows that although we may not know the actual impulsive force acting on the particle, its average value on the interval Δt is determined by the change in the linear momentum of the particle during that interval. Moreover, it is seen that in the limit as $t_0 \to t$, (7.4) returns the rule (5.34). The following example illustrates an average force calculation.

Example 7.1. A projectile S weighing 50 lb strikes a concrete bunker with a normal velocity of 1288 ft/sec (878 mph). The projectile imbeds itself in the wall and comes to rest in 10^{-2} sec. What is the average force exerted on the wall by the projectile during this time?

Solution. The change in the linear momentum of the projectile is

$$\Delta\mathbf{p} = -\frac{50}{32.2}(1288)\mathbf{n} = -2000\mathbf{n}\ \text{slug}\cdot\text{ft/sec}, \tag{7.5a}$$

in which $\mathbf{n}$ is the unit normal vector directed into the wall. The average force $\mathbf{F}_S^*$ acting on the projectile in the time $\Delta t = 10^{-2}$ sec is given by the last ratio in (7.4), and with (7.5a) we thus obtain

$$\mathbf{F}_S^* = -\frac{2000}{10^{-2}}\mathbf{n} = -2\times 10^5\mathbf{n}\ \text{lb} = -100\mathbf{n}\ \text{tons}. \tag{7.5b}$$

This estimates the total force exerted on the projectile by the concrete wall and gravity. Of course, the weight of the projectile compared with the total impulsive force (7.5b) is negligible, and hence the average force exerted on the wall by the projectile may be estimated by the equal and oppositely directed force $\mathbf{F}_W^* = 100\mathbf{n}$ tons. If the action time increment is smaller, the average force acting on the projectile or the penetration force acting on the wall grows larger. □

7.2.1. Instantaneous Impulse and Momentum

There are many physical situations in which a change in velocity induced by the exchange of deformation energy occurs so suddenly that it is very difficult to observe the transition from one state to another. When a cue strikes a billiard ball, for example, the ball experiences a finite change in velocity during an infinitesimally short interval of time. There is also no observable change in its position during the impact time. The same thing is true when a bullet strikes a block of wood and when an automobile impacts a pole. In these cases the impulse occurs virtually instantaneously. This physical idea of an instantaneous impulsive action is first characterized mathematically. Afterwards, the use of singularity functions to define the impulsive force and the instantaneous impulse are described.

In general, the average value $\mathbf{F}^*$ of the total force conveys no information about the nature or maximum intensity of the actual applied force $\mathbf{F}(t)$, rather, it provides only an estimate of $\mathbf{F}(t)$ that is independent of the duration of its application. A one-dimensional triangular loading, for example, has a mean value equal to one-half the height of the triangle regardless of the length of its time base. So, an average value estimate of $\mathbf{F}(t)$ might not be a very good one. On the other hand, when the impulse is almost instantaneous it is reasonable to imagine that the

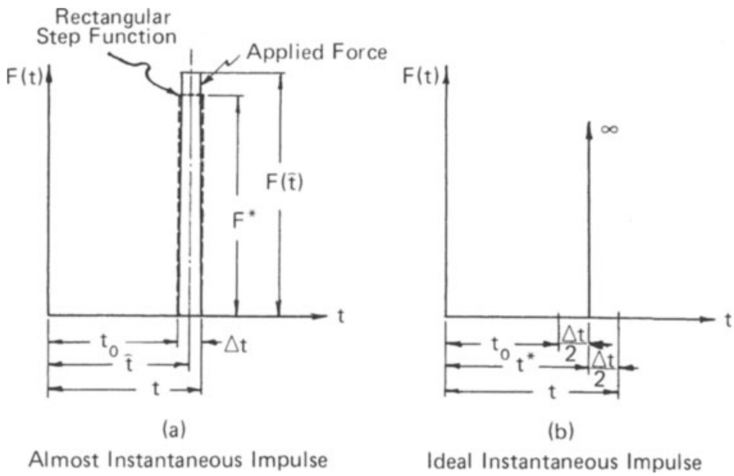


Figure 7.2. Graphical models of an almost instantaneous impulse and an ideal instantaneous impulse.

force–time graph, as shown in Fig. 7.2a for the one-dimensional case, is closely approximated by a rectangular step function. In this instance, the mean value $\mathbf{F}^*$ approximates very closely the extreme intensity $F(\hat{t})$ of the actual applied force. Of course, the impulse–momentum principle (7.2) holds for all time intervals Δt, large or small.

Furthermore, the particle's displacement $\Delta\mathbf{x} = \mathbf{x}(t) - \mathbf{x}(t_0)$ during any time interval $\Delta t = t - t_0$ is related to the *average value* $\mathbf{v}^*$ of its velocity $\mathbf{v}$ in accordance with the relation

$$\Delta\mathbf{x} = \int_{t_0}^{t} \mathbf{v}(t)dt = \mathbf{v}^*\Delta t. \tag{7.6}$$

Therefore, if a particle experiences a finite change in velocity in an infinitesimal time interval, it follows from (7.6) that the displacement during that interval must be infinitesimal, and as $\Delta t \to 0$, $\Delta\mathbf{x} \to \mathbf{0}$ also.

An instantaneous impulse, therefore, is characterized physically as a very large, suddenly applied force acting over a vanishing time interval, and resulting in an instantaneous but finite change in the particle's velocity with no change in its position. In accordance with (7.2), the *instantaneous impulse* $\mathscr{I}^*$ is defined by

$$\mathscr{I}^* \equiv \underset{t\to t_0}{\text{limit}} \int_{t_0}^{t} \mathbf{F}(t)dt = \Delta\mathbf{p}^*, \tag{7.7}$$

where $\Delta\mathbf{p}^*$ is the finite, but *instantaneous change in the linear momentum of the particle (or center of mass object)*. By (7.6), since the average velocity must be finite, there is no instantaneous change $\Delta\mathbf{x}^*$ in the particle's position at the impulsive instant:

$$\Delta \mathbf{x}^* \equiv \underset{t \to t_0}{\text{limit}}\ \Delta \mathbf{x} = \mathbf{0}. \tag{7.8}$$

Further, in the limit $\Delta t \to 0$, all *finite* forces that contribute to the total force that acts on a particle will vanish from (7.7). In order for this limit to be non-zero, the total force must become very large as Δt is indefinitely diminished. Therefore, when calculating the effect of an instantaneous impulsive force, we may neglect the effect of all other finite forces, such as gravity, spring or friction forces, that may act on the particle at the time of the instantaneous impulse. Of course, the instantaneous change in momentum occurs always in the direction of the instantaneous impulse.

Exercise 7.1. A particle P of unit mass is acted upon by a constant force $\mathbf{F}$. Prove that the average velocity $\mathbf{v}^*$ of P in a time interval Δt is equal to $\mathbf{v}(P, \Delta t/2)$, its velocity at the midpoint of the interval. □

The delta function (1.120) introduced in Volume 1 may be used to describe the instantaneous impulse. Let us write $\mathbf{F}(t) = \mathscr{I}^* \delta(t)$, where $\mathscr{I}^*$ is the instantaneous impulse at the instant $t^* \in [t_0, t]$ and $\delta(t) = \langle t - t^* \rangle_{-1}$ is the *Dirac delta function*. Then the *ideal instantaneous impulsive force* is described by

$$\mathbf{F}(t) = \mathscr{I}^* \langle t - t^* \rangle_{-1} = \begin{cases} \mathbf{0} \text{ if } t \neq t^*, \\ \infty \text{ if } t = t^*. \end{cases} \tag{7.9}$$

This ideal force is illustrated in Fig. 7.2b. In accordance with (1.133), we also obtain from (7.9)

$$\int_{-\infty}^{t} \mathbf{F}(t) dt = \mathscr{I}^* \int_{-\infty}^{t} \langle t - t^* \rangle_{-1} dt = \mathscr{I}^* \langle t - t^* \rangle^0, \tag{7.10}$$

wherein we recall the unit step function $u(t) = \langle t - t^* \rangle^0$ in (1.117). Hence,

$$\mathscr{I}^* \langle t - t^* \rangle^0 = \begin{cases} \mathbf{0} \text{ if } t < t^*, \\ \mathscr{I}^* \text{ if } t > t^*, \\ \text{undefined at } t = t^*. \end{cases} \tag{7.11}$$

To relate this to the instantaneous change of momentum in (7.7), we integrate (7.9) over an infinitesimal time interval about t^*, say from $t_0 = t^* - \frac{1}{2}\Delta t$ to $t = t^* + \frac{1}{2}\Delta t$, and note that the impulsive force $\mathbf{F}(t)$ vanishes outside this interval. Then letting $\Delta t = t - t_0 \to 0$, as $t \to t^*$ from above, we obtain with (7.10) the instantaneous impulse defined in (7.7).

7.2.2. Linear Momentum in an Instantaneous Impulse

Suppose that the impulse of the force exerted on a particle P_1 is due to its interaction with another particle P_2. It is not necessary that the particles come into contact, but they may. Let $\mathscr{I}_{12}$ denote the impulse acting on P_1 by P_2; then the third law requires that the impulse $\mathscr{I}_{21}$ acting on P_2 by P_1 be equal, but oppositely directed. Therefore, if $\Delta\mathbf{p}_1$ and $\Delta\mathbf{p}_2$ denote the changes in linear momenta of P_1 and P_2, respectively, the impulse–momentum principle (7.2) and the law of mutual action together imply that for the same time interval $\Delta\mathbf{p}_1 = \mathscr{I}_{12} = -\mathscr{I}_{21} = -\Delta\mathbf{p}_2$, that is, $\Delta(\mathbf{p}_1 + \mathbf{p}_2) = \mathbf{0}$. This yields *the law of conservation of instantaneous momentum for a system of two particles—the total instantaneous linear momentum for a system of two particles is constant during an impulsive interval*:

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{c}, \quad \text{a constant.} \tag{7.12}$$

If any other forces acting on either particle are finite and the impulse is *instantaneous*, these external forces contribute nothing to the impulse of the force on that particle and may be ignored. Then, (7.12) holds even though other external, but nonimpulsive forces may act on each particle. Clearly, forces whose resultant is zero may be ignored, too. If other impulsive forces act on either particle, however, (7.12) does not hold, rather, the impulse–momentum principle (7.7) must be applied separately to each particle, accounting for the total impulsive force that acts on each.

Example 7.2. A *ballistic pendulum* is a device used to determine the muzzle speed of a gun. A bag of wet sand of mass M is suspended by a rope, and a bullet of mass m is fired into the sand with unknown muzzle speed β. The pendulum then swings through a small angle θ_0 from its vertical position of rest as shown in Fig. 7.3. Replace the sack by its center of mass object, and find the muzzle speed of the gun.

Solution. During the infinitesimally small time interval of impact of the bullet with the sack of sand, the only external forces that act on the pair are their weight

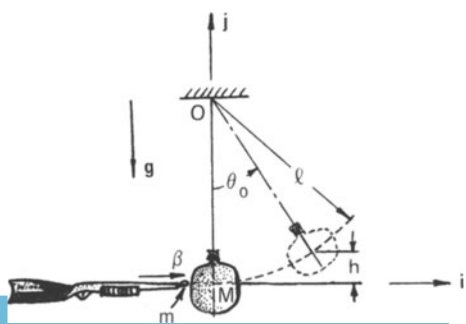


Figure 7.3. A ballistic pendulum model.

and the tension in the rope. These are finite external forces that contribute nothing to the instantaneous impulses and may be ignored. In fact, the resultant of the rope tension and the weight of the sand is zero during the impulsive instant. (See Exercise 7.3, page 229.) Hence, the total linear momentum at the instant of impact is constant. The bag being at rest, the linear momentum of the pair just prior to the instant t^* of impact is $m\beta\mathbf{i}$. Immediately afterward, when the bullet is lodged in the bag (captured by the center of mass object), which now has an instantaneous velocity $\mathbf{v}_0 = v_0\mathbf{i}$, the linear momentum is $(m + M)v_0\mathbf{i}$. Application of the momentum equation (7.12) yields $m\beta\mathbf{i} = (m + M)v_0\mathbf{i}$, and hence

$$\beta = \frac{m + M}{m} v_0. \tag{7.13a}$$

However, v_0 remains unknown. To find it, we consider a familiar problem.

After the impulse, the bag swings as a simple pendulum of length l and small amplitude θ_0 so that $h \ll \ell$ in Fig. 7.3. Therefore, the equation of motion for the bag carrying both the sand and the bullet is given by (6.67d), whose general solution for the initial conditions $\theta(0) = 0$ and $\ell\dot{\theta}(0) = v_0$ is provided by (6.67e). Hence, with $B = 0$ and $A = v_0/p\ell$, the solution is $\theta(t) = (v_0/p\ell)\sin pt$, from which the amplitude of the swing is $\theta_0 = v_0/p\ell$. With (6.67c), this yields $v_0 = \theta_0\sqrt{g\ell}$. Finally, use of this relation in (7.13a) delivers the muzzle speed of the gun:

$$\beta = \frac{m + M}{m} \theta_0\sqrt{g\ell}. \tag{7.13b}$$

Since $M \gg m$, the muzzle speed is closely estimated as $\beta = (M/m)\,\theta_0\sqrt{g\ell}$. □

7.3. The Torque–Impulse Principle

The moment of momentum principle (6.79) has the same analytical structure as the Newton–Euler law, so a parallel procedure is used to describe the impulse due to the moment of the force. The *impulse of the moment* $\mathbf{M}_O(t)$ *about the fixed point* O, called the *torque–impulse,* is the vector-valued integral function $\mathscr{M}_O(t; t_0)$ defined by

$$\mathscr{M}_O(t; t_0) \equiv \int_{t_0}^{t} \mathbf{M}_O(t)dt. \tag{7.14}$$

Let $\Delta\mathbf{h}_O \equiv \mathbf{h}_O(t) - \mathbf{h}_O(t_0)$ denote the change in the moment of momentum of the particle about O in the time interval $[t_0, t]$, and recall the moment of momentum principle: $d\mathbf{h}_O/dt = \mathbf{M}_O(t)$. Integrating this equation with respect to time, we obtain *the torque–impulse principle*:

$$\mathscr{M}_O(t; t_0) = \Delta\mathbf{h}_O. \tag{7.15}$$

Hence, the torque–impulse over the time interval $[t_0, t]$*, about a fixed point* O *in an inertial reference frame, is equal to the corresponding change in the particle's moment of momentum about* O. The torque–impulse and moment of momentum have the physical dimensions: $[\mathscr{M}_O] = [\mathbf{h}_O] = [FLT] = [ML^2T^{-1}]$.

The *average value* $\mathbf{M}_O^*$ of the torque $\mathbf{M}_O(t)$ on the interval $[t_0, t]$ is defined by

$$\mathbf{M}_O^* \equiv \frac{1}{\Delta t}\int_{t_0}^{t} \mathbf{M}_O(t)dt = \frac{\mathscr{M}_O(t; t_0)}{\Delta t} = \frac{\Delta \mathbf{h}_O}{\Delta t}, \tag{7.16}$$

wherein $\Delta t = t - t_0$ and we recall (7.14) and (7.15). In the limit as $\Delta t \to 0$, (7.16) returns the rule (6.79).

7.3.1. Instantaneous Torque–Impulse

When the torque is applied suddenly so that it results in a virtually instantaneous, finite change in the moment of momentum of the particle with no instantaneous change in its position, the torque–impulse is called an instantaneous torque–impulse. Symbolically, for a fixed point O in an inertial frame, the *instantaneous torque–impulse* $\mathscr{M}_O^*$ is defined by

$$\mathscr{M}_O^* \equiv \operatorname*{limit}_{t \to t_0} \int_{t_0}^{t} \mathbf{M}_O(t)dt = \Delta \mathbf{h}_O^*, \tag{7.17}$$

where $\Delta \mathbf{h}_O^*$ is the finite, but instantaneous change in the particle's moment of momentum about O.

The instantaneous torque–impulse about a fixed point O *is equal to the moment about* O *of the instantaneous impulse of the force acting on the particle, i.e.,*

$$\mathscr{M}_O^* = \mathbf{x}(t_0) \times \mathscr{I}^*. \tag{7.18}$$

To prove this, we recall that the particle's position vector from O does not change during an instantaneous impulse. Hence, with $\mathbf{x}_O = \mathbf{x}(t_0)$, (7.17) yields

$$\mathscr{M}_O^* = \operatorname*{limit}_{t \to t_0} \int_{t_0}^{t} \mathbf{x}_O(t) \times \mathbf{F}(t)dt = \mathbf{x}(t_0) \times \operatorname*{limit}_{t \to t_0} \int_{t_0}^{t} \mathbf{F}(t)dt,$$

from which (7.18) now follows by (7.7).

Exercise 7.2. Alternatively, form the difference $\Delta \mathbf{h}_O = \mathbf{h}_O(t) - \mathbf{h}_O(t_0)$, recall (5.31) and (7.2), and thus derive (7.18) differently. □

7.3.2. Moment of Momentum in an Instantaneous Torque–Impulse

Suppose that the impulse of the force exerted on a particle P_1 is due to its interaction with another particle P_2 so that $\mathscr{I}_{12}^* = -\mathscr{I}_{21}^*$. Let P_1 and P_2 have positions $\mathbf{x}_1$ and $\mathbf{x}_2$ from a fixed point O in an inertial frame Φ at the impulsive instant, and write $\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1$ for the vector of P_2 from P_1. Then (7.18) shows that for the system of two particles the total instantaneous torque–impulse about point O, defined by $\mathscr{M}_O^* = \mathscr{M}_{O1}^* + \mathscr{M}_{O2}^*$, is determined by

$$\mathscr{M}_O^* = \mathbf{x}_1 \times \mathscr{I}_{12}^* + \mathbf{x}_2 \times \mathscr{I}_{21}^* = \mathbf{r} \times \mathscr{I}_{21}^*. \tag{7.19}$$

We now assume that either (i) the particles collide, so that $\mathbf{r} = \mathbf{0}$, or (ii) they do not collide but their mutual interaction impulses are directed along the line joining the two particles, so that $\mathscr{I}_{21}^*$ is parallel to $\mathbf{r}$. In either case, the torque–impulse (7.19) vanishes; and, therefore, from (7.17), the change in the total moment of momentum of the system of two particles about the fixed point O, in accordance with (5.32), namely, $\Delta\mathbf{h}_O^* = \Delta(\mathbf{h}_{O1}^* + \mathbf{h}_{O2}^*)$, must vanish. This yields the *law of conservation of instantaneous moment of momentum for a system of two particles—at the instant of impulse, the total instantaneous moment of momentum about a fixed point O for a system of two particles is a constant vector*:

$$\mathbf{h}_{O1}^* + \mathbf{h}_{O2}^* = \mathbf{h}_O^*, \text{ a constant.} \tag{7.20}$$

Example 7.3. The rule (7.20) may be applied to find the muzzle speed of the bullet in the ballistic pendulum problem in Fig. 7.3. Just prior to impact, the moment of momentum of the bullet about point O is $\mathbf{h}_O^* = \ell m\beta\mathbf{k}$. Immediately thereafter, the moment of momentum of the system is $\mathbf{h}_O^* = \ell(m + M)v_0\mathbf{k}$. Hence, use of (7.20) yields (7.13a) giving the muzzle speed in terms of v_0. □

Exercise 7.3. Suppose the bullet enters the sack at a downward angle ψ from the horizontal axis in Fig. 7.3. The rope tension in this case will exert an additional instantaneous impulse on the sack. (a) Apply the impulse–momentum principle (7.7) to determine the muzzle speed β, and find the instantaneous impulse on the rope. (b) Apply the instantaneous torque–impulse principle (7.17) to find β. □

7.4. Work and Conservative Force

Consider a particle P in motion along a path $\mathscr{C}$ due to a force $\mathbf{F}(\mathbf{x})$ that varies only with the particle's position along $\mathscr{C}$, as shown in Fig. 7.4. The *work* $\mathscr{W}$ done by the force $\mathbf{F}(\mathbf{x})$ in moving P along $\mathscr{C}$ from the point $\mathbf{x}_1$ to the point $\mathbf{x}_2$ is defined by the *path,* or *line integral*

$$\mathscr{W} = \int_{\mathscr{C}} \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x}. \tag{7.21}$$

The physical dimensions of work are $[\mathscr{W}] = [FL]$. Thus, if force is measured in Newtons or pounds and length in meters or feet, the measure units of work are expressed as N · m or lb · ft (or also as ft · lb), respectively.

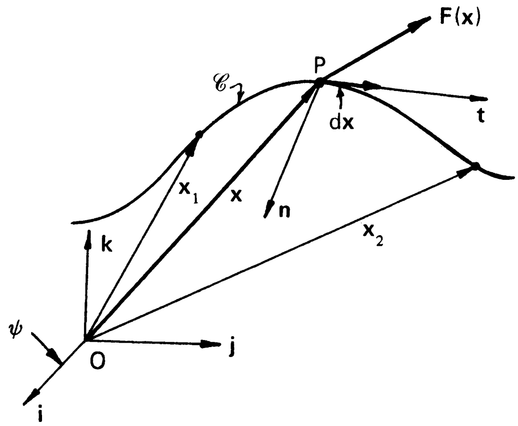


Figure 7.4. Schema for work done by a force $\mathbf{F}(\mathbf{x})$ acting on a particle over its path.

In accordance with (6.3), the total force referred to the intrinsic basis may be expressed as $\mathbf{F} = F_t\mathbf{t} + F_n\mathbf{n}$ and $d\mathbf{x} = ds\mathbf{t}$; therefore, in general, by (7.21), the work done by $\mathbf{F}$ is

$$\mathscr{W} = \int_{\mathscr{C}} F_t ds. \tag{7.22}$$

Consequently, only the component of force tangent to the path does work in moving P. Moreover, an increment of work $\Delta\mathscr{W} = F_t\Delta s$ is positive, negative, or zero according as the particle displacement Δs is in the same direction, the opposite direction, or perpendicular to the force acting on the particle, respectively. In particular, a propulsive force does positive work, whereas a drag force does negative work. The force (6.16) acting on a charged particle in a constant magnetic field does zero work; and the Coriolis force, $-2m\boldsymbol{\omega}\times\delta\mathbf{x}/\delta t$, being perpendicular to the relative velocity vector, does no work in the moving frame. Also, the property (7.8) of an instantaneous impulse shows that forces that act during an instantaneous impulse do no work.

The total work done by $\mathbf{F}$ depends not only on the end points, as emphasized by the second expression in (7.21), but generally also on the path traversed by P, as emphasized in the first expression. For certain forces, the work done is the same for every path joining the same end points, so the work done by these forces depends only on the values of the assigned end points. A force field $\mathbf{F}(\mathbf{x})$ that is independent of the path is called a *conservative force*. A force $\mathbf{F}(\mathbf{x})$ that is not conservative is called a *nonconservative force*; these are path dependent forces. Both propulsive and Coulomb friction forces, for example, always follow the motion along the specific path of the particle, so these are nonconservative forces that vary with the choice of path between fixed end states. Conservative forces are further characterized later on.

In rectangular Cartesian coordinates, we write $\mathbf{F}(\mathbf{x}) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$ and $d\mathbf{x} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$. Hence, for end states at $\mathbf{x}_j = (x_j, y_j, z_j)$, $j = 1, 2$, the second expression in (7.21) becomes

$$\mathscr{W} = \int_{(x_1,\, y_1,\, z_1)}^{(x_2,\, y_2,\, z_2)} (F_x dx + F_y dy + F_z dz). \tag{7.23}$$

This relation reveals the procedure for calculating the line integral. The force $\mathbf{F}(\mathbf{x})$ must be known as a function of position $\mathbf{x}$ along $\mathscr{C}$. In general, we must also know the equation of the path $\mathscr{C}$ so that the path variables x, y, z and their differentials can be related through this function; and, finally, the position coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) of the end points at $\mathbf{x}_1$ and $\mathbf{x}_2$ are required. This analysis is illustrated in the following problem.

Example 7.4. Find the work done by the force

$$\mathbf{F}(\mathbf{x}) = bxy\mathbf{i} + cy\mathbf{j}, \tag{7.24a}$$

in moving a particle from the origin (0, 0) to the point (1, a) along the paths defined by (i) a parabola $y = ax^2$ and (ii) the lines $x = 0$ and $y = a$. Here a, b, c are constants. (iii) What condition must be satisfied in order that the force (7.24a) may be conservative?

Solution of (i). The work done by the force (7.24a) in moving the particle from $(0, 0)$ to $(1, a)$ on any path is given by (7.23):

$$\mathscr{W} = \int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = \int_{(0,0)}^{(1,a)} bxydx + cydy. \tag{7.24b}$$

To integrate the first term in this equation, it is evident that we shall need to know how y is related to x. This means that the path must be specified, and hence the force (7.24a) is a nonconservative force.

For the parabolic path $y = ax^2$ shown in Fig. 7.5a, the path integral in (7.24b) becomes

$$\mathscr{W} = \int_{(0,0)}^{(1,a)} abx^3dx + cydy = \int_0^1 abx^3dx + \int_0^a cydy. \tag{7.24c}$$

Hence, the work done by the nonconservative force (7.24a) in moving the particle along the parabolic path $y = ax^2$ from $(0, 0)$ to $(1, a)$ is

$$\mathscr{W} = \frac{ab}{4} + \frac{ca^2}{2}. \tag{7.24d}$$

Solution of (ii). Consider the work done by the same force (7.24a) acting over the path defined by the lines $x = 0$ and $y = a$ joining the same end points,

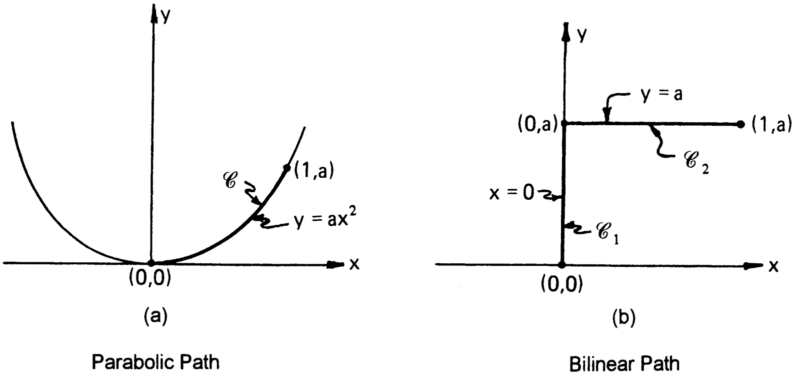


Figure 7.5. Distinct particle paths joining the same end points from (0, 0) to $(1, a)$.

as shown in Fig. 7.5b. The path $\mathscr{C}$ consists of two parts $\mathscr{C}_1$ and $\mathscr{C}_2$; hence, (7.21) is written as

$$\mathscr{W} = \int_{\mathscr{C}=\mathscr{C}_1\cup\mathscr{C}_2} \mathbf{F}\cdot d\mathbf{x} = \int_{\mathscr{C}_1} \mathbf{F}\cdot d\mathbf{x} + \int_{\mathscr{C}_2} \mathbf{F}\cdot d\mathbf{x}, \tag{7.25a}$$

wherein each integrand has the form of (7.24b). Now, $x = 0$ and y is variable on $\mathscr{C}_1$; hence, (7.24b) applied to $\mathscr{C}_1$ yields

$$\int_{\mathscr{C}_1} \mathbf{F}\cdot d\mathbf{x} = \int_{(0,0)}^{(0,a)} cy dy = \int_0^a cy dy = \frac{ca^2}{2}. \tag{7.25b}$$

Similarly, $y = a,\ dy = 0$ and x is variable on $\mathscr{C}_2$; hence, application of (7.24b) to $\mathscr{C}_2$ gives

$$\int_{\mathscr{C}_2} \mathbf{F}\cdot d\mathbf{x} = \int_{(0,0)}^{(1,a)} abx dx = \int_0^1 abx dx = \frac{ab}{2}. \tag{7.25c}$$

Therefore, the total work done by the force (7.24a) in moving the particle over the path $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$ between the same end states from (0, 0) to $(1, a)$, by (7.25a), is

$$\mathscr{W} = \frac{ab}{2} + \frac{ca^2}{2}. \tag{7.25d}$$

To conclude, let the reader consider the following additional example.

Exercise 7.4. Show that the work done by the force (7.24a) in moving the particle over the straight line path $y = ax$ joining (0, 0) to $(1, a)$ is given by

$$\mathscr{W} = \frac{ab}{3} + \frac{ca^2}{2}. \tag{7.25e}$$

Solution of (iii). Finally, we wish to determine the condition to be satisfied in order that (7.24a) may be conservative. The solutions (7.24d), (7.25d), and (7.25e) show that the work done by the same force will be the same for all paths considered above only if $b = 0$. This is the condition needed for the force (7.24a) to be conservative. Indeed, conversely, let us consider the force

$$\mathbf{F} = cy\mathbf{j}. \tag{7.26a}$$

Then the work done by $\mathbf{F}$ acting over a path $\mathscr{C}$ joining $(0, 0)$ to $(1, a)$ is given by

$$\mathscr{W} = \int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = \int_{(0,0)}^{(1,a)} cydy = \int_0^a cydy = \frac{ca^2}{2}. \tag{7.26b}$$

Note that in this integration there is no need to mention a specific path $\mathscr{C}$. This result is independent of the path; it depends only on values at the end points. The force (7.26a) is a conservative force. We thus find that the work done by the force (7.24a) is independent of the particle's path, when and only when $b = 0$. □

There are many kinds of conservative and nonconservative forces. Gravitational and linear spring forces are conservative; the Coulomb frictional force is not. The work done by each of these forces is discussed next.

7.4.1. Work Done by a Constant Force

It is easy to show by (7.21) that *a constant force* $\mathbf{F}_c$ *is conservative.* The work done by $\mathbf{F}_c$ acting between the point $\mathbf{x}_0$ and any other point $\mathbf{x}$ is given by

$$\mathscr{W} = \mathbf{F}_c \cdot \int_{\mathbf{x}_0}^{\mathbf{x}} d\mathbf{x} = \mathbf{F}_c \cdot \Delta\mathbf{x}, \tag{7.27}$$

wherein $\Delta\mathbf{x} \equiv \mathbf{x} - \mathbf{x}_0$. Clearly, the work done by the constant force is independent of the path joining the end points, so $\mathbf{F}_c$ is conservative.

The apparent gravitational force on a particle of mass m near the surface of the Earth is a constant force $\mathbf{F}_c = -mg\mathbf{k}$. Therefore, by (7.27), *the apparent gravitational force near the surface of the Earth is a conservative force that does work*

$$\mathscr{W}_g = -mg\Delta z = -mgh, \tag{7.28}$$

in which $h \equiv \Delta z = \mathbf{k} \cdot \Delta\mathbf{x}$ is the vertical change in elevation through which m moves along its path. If $h > 0$, then the particle increases its height from the Earth and $\mathscr{W} < 0$. This means that the gravitational force acts oppositely to the particle's vertical displacement, and hence work is done against the force of gravity to increase the particle's elevation. On the other hand, $h < 0$ means that the elevation has decreased, i.e. the particle has moved in the direction of the force of gravity which now does positive work. Because the gravitational force is perpendicular to the horizontal contribution of the total displacement $\Delta\mathbf{x} = \Delta x\mathbf{i} + \Delta y\mathbf{j} + \Delta z\mathbf{k}$, it does no work in any horizontal displacement whatsoever.

7.4.2. Work Done by the Coulomb Frictional Force

A Coulomb frictional force of constant magnitude must not be confused as a conservative force. A constant force must have both a constant magnitude and a constant direction. Although a Coulomb drag force may have a constant magnitude in some cases, its direction always varies with the path.

The work done by the Coulomb frictional force $\mathbf{f}_d = -\nu N(s)\mathbf{t}$ over any path $\mathscr{C}$ of length d is given by

$$\mathscr{W}_f = \int_0^d \mathbf{f}_d \cdot d\mathbf{x} = -\nu \int_0^d N(s)ds. \tag{7.29}$$

In general, the Coulomb force need not have a constant magnitude, $N(s)$ in (7.29) may vary along the path. This happens, for example, when the particle slides down a rough curved path in the vertical plane. On the other hand, *in any motion for which N is constant, the work done by the Coulomb frictional force is*

$$\mathscr{W}_f = -\nu N d = -f_d d. \tag{7.30}$$

This formula is valid for all paths, but for different paths joining the same end states, the distance d along the path connecting these states will be different—the Coulomb frictional force $\mathbf{f}_d$ is not conservative.

7.4.3. Work Done by a Linear Force

Finally, consider the linear force $\mathbf{F}_L = \alpha\mathbf{x}$, where α is a constant. Recall (7.21) and note that the integrand may be written as $\mathbf{F}_L \cdot d\mathbf{x} = \alpha\mathbf{x} \cdot d\mathbf{x} = \alpha d(\frac{1}{2}\mathbf{x} \cdot \mathbf{x})$. Then the work done by $\mathbf{F}_L$ in moving a particle over an arbitrary path from a point $\mathbf{x}_0$ to any other point $\mathbf{x}$ is given by

$$\mathscr{W} = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F}_L \cdot d\mathbf{x} = \frac{\alpha}{2}(\mathbf{x} \cdot \mathbf{x} - \mathbf{x}_0 \cdot \mathbf{x}_0), \tag{7.31}$$

which is independent of the path. *The linear force $\mathbf{F}_L = \alpha\mathbf{x} = \alpha(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ is a conservative force*. Note that the force in (7.26a) is a special linear force of this type. **Caution**: Not every force linear in the variables $x,\ y,\ z$ is conservative; the force $\mathbf{F} = \alpha y\mathbf{i}$, for example, is not conservative.

Let x denote the change of length of a linear spring from its undeformed state and recall (6.64). Then the uniaxial force required to stretch or compress the spring is given by $\mathbf{F}_H = kx\mathbf{i}$, which is a linear force of the type $\mathbf{F}_L$. Therefore, from (7.31), the work done in stretching a linear spring from any initial state of stretch $\mathbf{x}_0$ is

$$\mathscr{W} = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F}_H \cdot d\mathbf{x} = \frac{1}{2}k\left(x^2 - x_0^2\right). \tag{7.32}$$

If the initial state is the natural state, then $x_0 = 0$ and $\mathscr{W} = \frac{1}{2}kx^2$. *The force required to elongate or to compress a linear spring is a conservative force.*

Clearly, the equal but oppositely directed restoring force $\mathbf{F}_S = -kx\mathbf{i} = -\mathbf{F}_H$ exerted by the spring is a conservative force. By (7.32), *the work done by the elastic spring in its deformation from the natural state is given by*

$$\mathscr{W}_e = -\tfrac{1}{2}kx^2. \tag{7.33}$$

The work done by the spring force is negative, because the spring force opposes the displacement $d\mathbf{x}$.

7.5. The Work–Energy Principle

The concept of mechanical work is used to derive a general first integral of the equation of motion known as the work–energy principle. This principle is useful when the total force on a particle varies at most with its position $\mathbf{x}(P, t) = \mathbf{x}(s(t))$ along the path—the gravitational force of the Earth, the Coulomb frictional force, and the linear spring force being important examples. The notion of mechanical power is also introduced.

Let us consider the equation of motion when the total force is a function $\mathbf{F}(\mathbf{x})$ of the particle's position so that $m d\mathbf{v}/dt = \mathbf{F}(\mathbf{x})$. Now form its scalar product with $\mathbf{v}$, observe that $m\mathbf{v} \cdot d\mathbf{v}/dt = d(\frac{1}{2}m\mathbf{v} \cdot \mathbf{v})/dt$, and thereby obtain

$$\frac{dK(P,t)}{dt} = \mathbf{F}(\mathbf{x}) \cdot \mathbf{v}, \tag{7.34}$$

wherein, by definition, the new quantity

$$K(P,t) \equiv \tfrac{1}{2}m\mathbf{v} \cdot \mathbf{v} = \tfrac{1}{2}m\dot{s}^2, \tag{7.35}$$

is called the *kinetic energy* of the particle P. Then, recalling (7.21) and noting that $\mathbf{v}dt = d\mathbf{x}$, we integrate (7.34) over the path $\mathscr{C}$ traversed by the particle from $\mathbf{x}_0 = \mathbf{x}(t_0)$ to $\mathbf{x} = \mathbf{x}(t)$ to obtain the *work–energy principle*:

$$\mathscr{W} = \Delta K, \tag{7.36}$$

in which $\Delta K = K(P,t) - K(P,t_0)$ is the change in the kinetic energy of the particle that occurs in time $[t_0, t]$. *The work–energy equation states that the work done by the force* $\mathbf{F}(\mathbf{x})$ *acting on a particle over its path* $\mathscr{C}$ *from time* t_0 *to time* t *is equal to the change in the kinetic energy of the particle in that time.* It follows from (7.35) and (7.36) that $[K] = [MV^2] = [\mathscr{W}] = [FL]$.

The *mechanical power* $\mathscr{P}$ expended by the force is defined as the rate of working of the force:

$$\mathscr{P} = \frac{d\mathscr{W}}{dt} = \mathbf{F} \cdot \mathbf{v}. \tag{7.37}$$

With (7.36), this yields

$$\mathscr{P} = \frac{d\mathscr{W}}{dt} = \frac{dK}{dt}, \tag{7.38}$$

i.e. *the mechanical power expended by the force is equal to the time rate of change of the kinetic energy of the particle.* Power has the physical dimensions $[\mathscr{P}] = [FV] = [FLT^{-1}]$.

The work–energy equation is a single scalar equation, so it cannot replace equivalently the three scalar equations in the Newton–Euler vector equation of motion; rather, it often serves as a useful substitute for one of these equations integrated along the particle path. Since the work–energy equation was derived from the Newton–Euler law, we suspect that (7.36) may be applied conversely to derive the related single equivalent scalar equation of motion. In general, we see from (7.22) and (7.35) that in terms of intrinsic variables (7.36) may be written as

$$\frac{1}{2}m\dot{s}^2 - \frac{1}{2}mv_0^2 = \int_{s_0}^{s} F_t(s)ds, \tag{7.39}$$

where F_t is the tangential component of the total force $\mathbf{F}$ and $v_0 \equiv \dot{s}(t_0)$ is the particle's initial speed. Differentiation of this form of the work–energy equation with respect to the arc length parameter s and use of the relations $d(\frac{1}{2}m\dot{s}^2)/ds = m\ddot{s}$ and $d\mathscr{W}/ds = F_t$ show that

$$\Delta K = \mathscr{W} \iff m\ddot{s} = F_t. \tag{7.40}$$

(See (P6.28c) in Problem 6.28.) The reader may confirm that the same conclusion follows less directly by differentiation of (7.39) with respect to time.

The result (7.40) thus shows that the work–energy principle for a center of mass object is a convenient first integral of the tangential component of the Newton–Euler vector equation of motion, hence especially useful in single degree of freedom dynamical problems. On the other hand, if the value of $\mathscr{W}$ depends on the path, and we want to determine the particle's path, the work–energy rule might not be helpful. In other situations where the trajectory of the particle is known, or the force that acts on the particle does no work or is conservative so that its work is path independent, and especially when work is readily evaluated, the work–energy principle is most useful. The easy application of this rule is demonstrated in some examples that follow.

Example 7.5. Recall the ballistic pendulum problem in Fig. 7.3, page 226. Find the muzzle speed of the gun when the total angular placement may not be small enough to admit the approximate solution (7.13b) for which $h \ll \ell$.

Solution. The muzzle speed is still given by (7.13a), and v_0, the initial speed of the pendulum system, is the unknown of interest. The other end state condition and the path of the center of mass of this one-degree of freedom system are known. These facts strongly suggest that the work–energy principle will be helpful in this case. The total force that acts on this system is its total weight and the tension of the rope. The line tension is always normal to the circular path on which the center of mass moves, so it does no work as the system swings to its maximum placement θ_0. The work done by the constant force of gravity is determined by (7.28). Accordingly, in Fig. 7.3, the vertical height h through which the weight $(m + M)g$ is raised is $h = \ell(1 - \cos\theta_0)$, and hence $\mathscr{W} = -(m + M)g\ell(1 - \cos\theta_0)$. Since the system is at rest at θ_0 and has initial speed v_0, the change in the kinetic energy is $\Delta K = -\frac{1}{2}(m + M)v_0^2$. Thus, the work–energy principle (7.36) yields $-\frac{1}{2}(m + M)v_0^2 = -(m + M)g\ell(1 - \cos\theta_0)$. This gives the unknown v_0, and its use in (7.13a) provides the precise muzzle speed relation:

$$\beta = \frac{m + M}{m}\sqrt{2g\ell(1 - \cos\theta_0)}.$$

When $h/\ell = (1 - \cos\theta_0)$ is very small so that $1 - \cos\theta_0 = \frac{1}{2}\theta_0^2$, very nearly, the last equation reduces to our earlier approximate solution (7.13b). □

Example 7.6. A student is racing along in a sports car when suddenly, to avoid an impending collision, the driver slams on the brakes and skids along a straight line 200 ft to a stop in a 45 mph zone. Moments earlier, a policeman had checked the vehicle's speed on radar. Assume $\nu = 0.6$, ignore air resistance, and determine if the officer might give the student a ticket for exceeding the limit. Show for this example that the work–energy equation is the first integral of the equation of motion.

Solution. The assigned speed data in Fig. 7.6 show that the change in the kinetic energy of the car and its driver of total mass m is given by $\Delta K = -\frac{1}{2}mv^2$, and we want to find v. Because the path and the position varying nature of the forces acting on the system are known, we consider the work–energy method.

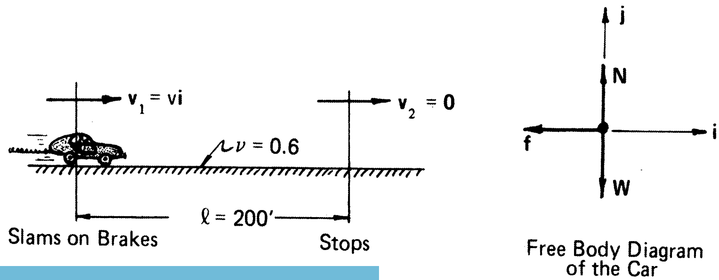


Figure 7.6. Motion of a braking vehicle over a rough road.

The forces that act on the car are shown in the free body diagram of Fig. 7.6. Both **N** and **W** do no work in the motion, and their magnitudes are equal: $N = W$. The work done by the nonconservative Coulomb frictional force $\mathbf{f}_d = -\nu N\mathbf{i}$ in the plane motion along a straight line is determined by (7.30) in which $d = \ell$, namely, $\mathscr{W}_f = -\nu W\ell$. The work–energy principle (7.36) thus yields the result $-\frac{1}{2}mv^2 = -\nu W\ell$, and with $W = mg$, the initial speed is determined by

$$v = \sqrt{2\nu g\ell}, \tag{7.41a}$$

which is *independent of the weight of the vehicle and its passenger.*

For the given data, $v = [2(.6)(32.2)(200)]^{1/2} = 87.91$ ft/sec, or very nearly 60 mph. In consequence, the student could receive a citation for speeding.

The first integral of the equation of motion $m\ddot{s} = d(\frac{1}{2}m\dot{s}^2)/ds = -\nu mg$ yields the general form of work–energy equation:

$$\tfrac{1}{2}m\dot{s}^2 - \tfrac{1}{2}mv^2 = -\nu mgs. \tag{7.41b}$$

When $s = \ell$, $\dot{s} = 0$, we obtain (7.41a). Conversely, differentiation of the work–energy equation (7.41b) with respect to either s or t yields the equivalent equation of motion. □

Example 7.7. A mass m is dropped from a height h onto a linear spring of constant stiffness k and negligible mass. Determine the maximum deflection δ of the spring, and compare this value with the static spring deflection δ_S produced by m. See Fig. 7.7. Assume that m maintains contact with the spring in its motion following the impact.

Solution. Since the velocity of m is zero at both its initial and terminal states at 1 and 3 in Fig. 7.7, the change in its kinetic energy on the path $\mathscr{C}$ is zero. Because the mass of the spring is negligible, its kinetic energy may be ignored. Moreover, all the forces that act on m are constant, workless, or vary only with the particle's position on $\mathscr{C}$. Therefore, the work–energy principle may be applied.

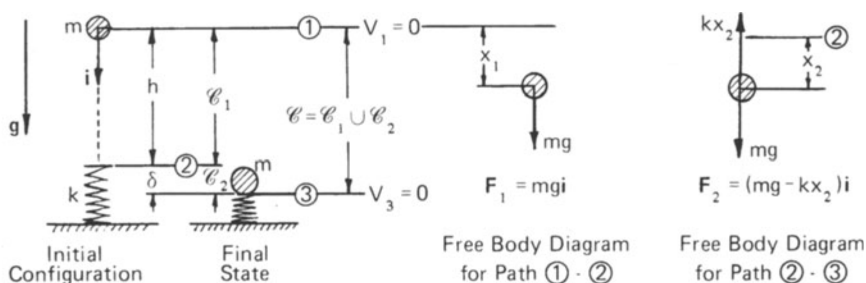


Figure 7.7. Spring deflection due to impact by a falling body.

The total work done by the forces acting on m from its initial position 1 to its final position 3 on $\mathscr{C}$ is determined by (7.21) in which $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$. We note that the instantaneous impulsive force of the spring does no work on m. The free body diagrams in Fig. 7.7 show that the force acting on m over $\mathscr{C}_1$ is $\mathbf{F}_1 = mg\mathbf{i}$ and over $\mathscr{C}_2$ is $\mathbf{F}_2 = (mg - kx_2)\mathbf{i}$. Hence,

$$\mathscr{W} = \int_{\mathscr{C}_1} \mathbf{F} \cdot d\mathbf{x} + \int_{\mathscr{C}_2} \mathbf{F} \cdot d\mathbf{x} = \int_0^h mg\,dx_1 + \int_0^\delta (mg - kx_2)\,dx_2. \tag{7.42a}$$

The work–energy equation (7.36) thus yields

$$\mathscr{W} = mgh + mg\delta - \tfrac{1}{2}k\delta^2 = \Delta K = 0, \tag{7.42b}$$

which determines the following deflection of the spring:

$$\delta = \frac{mg}{k} + \sqrt{\left(\frac{mg}{k}\right)^2 + \frac{2mgh}{k}}. \tag{7.42c}$$

The static deflection that would result from the weight alone is $\delta_S = mg/k$. Use of this relation in (7.42c) gives

$$\delta = \delta_S + \sqrt{\delta_S^2 + 2\delta_S h} \geq 2\delta_S. \tag{7.42d}$$

This formula shows that *the dynamic deflection δ is not less than twice the static deflection δ_S*. In particular, if m is released just at the top of the spring so that $h = 0$, then $\delta = 2\delta_S$.

The foregoing solution has illustrated the application of the work–energy equation when the work is calculated by use of the path integrals in (7.42a). Alternatively, we recognize that the work $\mathscr{W}_g$ done by the gravitational force acting over the entire path from 1 to 3 in the direction of the displacement is given by $\mathscr{W}_g = mg(h + \delta)$; the work $\mathscr{W}_e$ done by the elastic spring force acting on m over the path from 2 to 3 is $\mathscr{W}_e = -\frac{1}{2}k\delta^2$; and the impulsive force imposed on m at state 2 is workless. Hence, the total work done on m is $\mathscr{W} = \mathscr{W}_g + \mathscr{W}_e = mg(h + \delta) - \frac{1}{2}k\delta^2$, which is the same as (7.42b) derived above. □

Example 7.8. The work–energy equation for a relativistic particle may be derived from its intrinsic equation of motion (6.11). We form the scalar product of (6.11) with $\mathbf{v} = \dot{s}\mathbf{t}$ and recall equation (6.9) to obtain

$$\mathbf{F} \cdot \mathbf{v} = \frac{m}{1 - \beta^2}\ddot{s}\dot{s} = \frac{d}{dt}(mc^2). \tag{7.43a}$$

Integration with $d\mathbf{x} = \mathbf{v}dt$ leads to

$$\mathscr{W} = \int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = mc^2 - \alpha, \tag{7.43b}$$

where α is an arbitrary constant. The *relativistic kinetic energy* is defined by

$$E \equiv mc^2. \tag{7.43c}$$

Therefore, (7.43b) yields the *relativistic work–energy equation*:

$$\mathscr{W} = \Delta E. \tag{7.43d}$$

The work done in moving a particle from its rest state where its mass is m_o and $\alpha = m_o c^2$ is determined by the following change in the relativistic kinetic energy:

$$\Delta E = \Delta m c^2. \tag{7.43e}$$

Here $\Delta m = m - m_o$ is the increase in the relativistic mass of the particle over its rest mass. Equation (7.43e) is the famous Einstein relation connecting mass and energy in the special theory of relativity.

When the speed v of the particle is much smaller than the speed of light c, so that $\beta = c/v \ll 1$, (6.9) may be written as $m = m_o(1 + \frac{1}{2}\beta^2)$, approximately. In this case, $\Delta mc^2 = (m - m_o)c^2 = \frac{1}{2}m_o\beta^2 c^2$, and hence the change in the kinetic energy (7.43e) from the rest state of the particle coincides with the change in the nonrelativistic kinetic energy (7.35): $\Delta E = \frac{1}{2}m_o v^2 = \Delta K$. The work–energy equation (7.43d) then reduces to (7.36). □

This concludes our introduction to the work–energy equation and some applications. We shall return to this principle following discussion of some related topics.

7.6. Potential Energy

Suppose we are given a general, continuous force function $\mathbf{F}(\mathbf{x})$ defined over a space region $\mathscr{R}$. For certain force functions, the work done is independent of the path in $\mathscr{R}$, while for others it is not. To determine the special property that a force function must have in order that its work may be path independent, the concept of a potential energy function is introduced. As a consequence, a simple criterion necessary and sufficient for existence of a potential energy function emerges. If the given force function satisfies this criterion everywhere in a so-called simply connected region* $\mathscr{R}$, it passes the test and the force is conservative; otherwise it is not.

* The region $\mathscr{R}$ where $\mathbf{F}(\mathbf{x})$ is defined is called *connected* if any two given points in $\mathscr{R}$ can be joined by an arc all of whose points are in $\mathscr{R}$. A region $\mathscr{R} = \mathscr{R}_1 \cup \mathscr{R}_2 \cup \mathscr{R}_3$, where $\mathscr{R}_1$ and $\mathscr{R}_3$ are tracts of land separated by a river spanned by a bridge $\mathscr{R}_2$, is connected; but if the bridge is washed away by a flood, the new region $\mathscr{R}^* = \mathscr{R}_1 \cup \mathscr{R}_3$ is not connected. A curve $\mathbf{x}(t)$ that does not cross itself at any point $t \in (a, b)$ is called *simple*; and it is said to be *closed* when joined at its end points, i.e. when $\mathbf{x}(a) = \mathbf{x}(b)$. A connected region $\mathscr{R}$ is thus called *simply connected* if every simple closed curve in $\mathscr{R}$ can be continuously shrunk to a point of $\mathscr{R}$. A connected plane region $\mathscr{R}$ containing a hole of any kind, for example, is not simply connected, because any curve that encircles the hole cannot be shrunk to a point of $\mathscr{R}$. The hole in $\mathscr{R}$ may be a single point which has been excluded from $\mathscr{R}$.

First, we show that if the scalar-valued integrand in (7.21) is an exact differential of a single-valued function† $V(\mathbf{x})$, the work done is independent of the path. Indeed, suppose that the integrand in (7.21) is an exact differential so that $\mathbf{F} \cdot d\mathbf{x} = -dV(\mathbf{x})$. (The negative sign is introduced for future convenience in Section 7.8.4.) Then, if $\mathscr{C}$ is any smooth curve joining two points at $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathscr{R}$, we have

$$\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} -dV(\mathbf{x}) = -\Delta V, \tag{7.44}$$

where $\Delta V \equiv V(\mathbf{x}_2) - V(\mathbf{x}_1)$. The scalar-valued function $V(\mathbf{x})$ having this property is called the *potential energy*. Since $V(\mathbf{x})$ is single-valued, ΔV has a unique value determined only by the choice of end points. Then (7.21) shows that the work

$$\mathscr{W} = -\Delta V, \tag{7.45}$$

is independent of the path, and hence the force $\mathbf{F}(\mathbf{x})$ is conservative. *In consequence, the work done on a particle by a conservative force is equal to the decrease in the potential energy.*

7.6.1. Theorem on Conservative Force

But how are we to find this potential energy function? We show below that the components of a conservative force $\mathbf{F}$ are related to the partial derivatives of $V(\mathbf{x})$; and hence $V(\mathbf{x})$ may be found by integration of these equations. First, since $V(\mathbf{x}) = V(x, y, z)$, we have

$$dV(\mathbf{x}) = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = \nabla V(\mathbf{x}) \cdot d\mathbf{x}, \tag{7.46}$$

wherein, by definition, the vector

$$\nabla V(\mathbf{x}) \equiv \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}. \tag{7.47}$$

This vector is called the *gradient* of $V(\mathbf{x})$, and sometimes it is written as $\operatorname{grad} V(\mathbf{x})$. The ∇ symbol for the gradient operation is defined by

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \tag{7.48}$$

Consequently, if

$$\mathbf{F}(\mathbf{x}) = -\nabla V(\mathbf{x}), \tag{7.49}$$

† A function $y = f(x)$ is *single-valued* when $f(x)$ determines one and only one value y for each choice of x. The function $y = \sin x$, for example, is single-valued; but when its graph is turned through a right angle so that $y = \sin^{-1} x$, infinitely many values of y are determined for each choice of $x \in [-1, 1]$, and hence this function is *many-valued*. The parabola $y = x^2$ is a single-valued function for $x \in (-\infty, \infty)$, but the parabola $y = \pm x^{1/2}$ for $x > 0$, is not.

everywhere in a simply connected region $\mathscr{R}$, then $\mathbf{F}{\cdot}d\mathbf{x} = -dV$ follows from (7.46). In this case, the work done by $\mathbf{F}(\mathbf{x})$, as shown in (7.44), is independent of the path, and $\mathbf{F}(\mathbf{x})$ is conservative.

The relation (7.49) is a sufficient condition for $\mathbf{F}(\mathbf{x})$ to be conservative. Conversely, suppose that the work must be independent of the path in some region $\mathscr{R}$. We can then prove that there exists a scalar function $V(\mathbf{x})$, single-valued in $\mathscr{R}$, such that (7.49) holds everywhere in $\mathscr{R}$, except possibly at certain isolated points. Let $\mathbf{x}_1$ be an arbitrary fixed point and $\mathbf{x}$ a variable point in $\mathscr{R}$. Since $\mathscr{W}$ is independent of the path, the work integral over any curve $\mathscr{C}$ from $\mathbf{x}_1 = \mathbf{x}(s_1)$ to $\mathbf{x} = \mathbf{x}(s)$, where s denotes the distance along the path measured from any point on $\mathscr{C}$, say $\mathbf{x}_1$, is a single-valued function of the upper limit $\mathbf{x}$, and hence s alone. We thus write this work as

$$V(\mathbf{x}) = -\int_{\mathbf{x}_1}^{\mathbf{x}(s)} \mathbf{F}(\tilde{\mathbf{x}}) \cdot d\tilde{\mathbf{x}}, \tag{7.50}$$

where $\tilde{\mathbf{x}}$ is the dummy variable of integration introduced to avoid conflict with the variable limit. With the aid of Leibniz's rule (see Problem 6.28.), differentiation of the integral in (7.50) yields

$$\frac{dV(\mathbf{x})}{ds} = -\mathbf{F}(\mathbf{x}) \cdot \frac{d\mathbf{x}}{ds}. \tag{7.51}$$

An alternative derivation of (7.51) is left for the reader in the exercise below. It follows that $\mathbf{F}(\mathbf{x}) \cdot d\mathbf{x} = -dV(\mathbf{x}) = -\boldsymbol{\nabla} V(\mathbf{x}) \cdot d\mathbf{x}$ is an exact differential that must be independent of the path. Therefore, $(\mathbf{F} + \boldsymbol{\nabla} V) \cdot d\mathbf{x} = 0$ must hold for all $d\mathbf{x}(s)$; and hence (7.49) holds everywhere in $\mathscr{R}$. This completes the result summarized in the following theorem.

Theorem on conservative force: *A necessary and sufficient condition for* $\mathbf{F}(\mathbf{x})$ *to be conservative is that*

$$\mathbf{F}(\mathbf{x}) = -\boldsymbol{\nabla} V(\mathbf{x}). \tag{7.52}$$

With (7.47), it is seen that (7.52) is equivalent to the following three partial differential equations for $V(\mathbf{x})$:

$$\frac{\partial V}{\partial x} = -F_x, \qquad \frac{\partial V}{\partial y} = -F_y, \qquad \frac{\partial V}{\partial z} = -F_z. \tag{7.53}$$

Exercise 7.5. Consider a neighboring point at $\mathbf{x} + \Delta\mathbf{x}$ and apply (7.50) to obtain the unique increment $\Delta V \equiv V(\mathbf{x} + \Delta\mathbf{x}) - V(\mathbf{x})$ in V. Let $\mathbf{x}^* \in [\mathbf{x}, \mathbf{x} + \Delta\mathbf{x}]$ be a point on $\mathscr{C}$ within the arc length element $\Delta s = |\Delta\mathbf{x}|$. Apply the mean value theorem of integral calculus to derive $\Delta V/\Delta s = -\mathbf{F}(\mathbf{x}^*) \cdot d\mathbf{x}^*/ds$ at $\mathbf{x}^*$. Then in the limit as $\Delta s = |\Delta\mathbf{x}| \to 0$ and hence $\mathbf{x}^* \to \mathbf{x}$, derive (7.51). □

Example 7.9. The relationship (7.52) between the potential energy function and a conservative force is illustrated for two situations. (i) Given $V(\mathbf{x})$, find the conservative force $\mathbf{F}(\mathbf{x})$; and, conversely, (ii) given a conservative force $\mathbf{F}(\mathbf{x})$, find the potential energy function $V(\mathbf{x})$. When $V(\mathbf{x})$ is known, the work done is readily determined by (7.45).

(i) If $V(\mathbf{x})$ is given, then the force $\mathbf{F}(\mathbf{x})$ derived from (7.52) is a conservative force. For example, consider

$$V(\mathbf{x}) = -\frac{cy^2}{2} + d, \tag{7.54a}$$

where c and d are constants. Then (7.47) gives $\nabla V = -cy\mathbf{j}$, and (7.52) shows that the force $\mathbf{F} = -\nabla V = cy\mathbf{j}$ is conservative.

(ii) Conversely, suppose we know that a force $\mathbf{F}(\mathbf{x})$ is conservative. Then (7.52) holds and the potential energy function $V(\mathbf{x})$ may be found by integration of (7.53). For example, we know that $\mathbf{F}(\mathbf{x}) = cy\mathbf{j}$ is a conservative force, hence (7.53) become

$$\frac{\partial V}{\partial x} = 0, \qquad \frac{\partial V}{\partial y} = -cy, \qquad \frac{\partial V}{\partial z} = 0. \tag{7.54b}$$

The first and last of these equations show that V is independent of x and z. Hence, $V(\mathbf{x}) = V(y)$ is at most a function of y; and the second equation in (7.54b) becomes $dV/dy = -cy$. The solution of this equation is given by (7.54a) in which d is an arbitrary integration constant that may be chosen to meet any convenient purpose. For example, we may wish to define $V(0) = 0$; then $d = 0$ for this choice.

We recall that $\mathbf{F} = cy\mathbf{j}$ is the same force encountered earlier in (7.26a) and for which the work done in (7.26b) is independent of the path. An easier calculation for the work done by a conservative force is now provided by the rule (7.45). Let the potential energy be given by (7.54a), for example. Then, by (7.45), the work done on any path between the end points $\mathbf{x}_1 = (0, 0)$ and $\mathbf{x}_2 = (1, a)$ is

$$\mathscr{W} = -\Delta V = -[V(\mathbf{x}_2) - V(\mathbf{x}_1)] = -[-\frac{ca^2}{2} + d - d] = \frac{ca^2}{2}, \tag{7.54c}$$

precisely the result derived differently in (7.26b). Clearly, the physical dimensions of potential energy are those of work: $[V] = [\mathscr{W}] = [FL]$. □

Notice in this example that the constant potential energy $V_0 = d$ in (7.54a) is of no consequence whatsoever in evaluating either the force or the work done. This is typical—only differences in potential energy are relevant. A constant force $\mathbf{F}_c$, for example, is a conservative force that does work given by (7.27); so, by (7.45), the corresponding potential energy function may be written as $V(\mathbf{x}) = V_0 - \mathbf{F}_c \cdot \Delta\mathbf{x}$, where $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_0$ is the particle displacement vector. Similarly, by (7.45), the potential energy function for the conservative linear force $\mathbf{F}_L = \alpha\mathbf{x}$ may be read from (7.31): $V(\mathbf{x}) = V_0 - \frac{1}{2}\alpha(x^2 - x_0^2)$. In these and any other potential energy functions, the arbitrary constant potential energy V_0 is unimportant, it affects neither the force nor the work done. Therefore, sometimes the constant V_0 is suppressed in expressions for the potential energy, and sometimes its value is fixed for convenience. No generality is lost, if V_0 is discarded. For a constant force, this is equivalent to our assigning the value $V(\mathbf{x}_0) = V_0 = 0$ at $\mathbf{x} = \mathbf{x}_0$, say. In this case, $\mathbf{x} = \mathbf{x}_0$ becomes the datum point of zero potential energy with respect to which $V(\mathbf{x}) = -\mathbf{F}_c \cdot \Delta\mathbf{x}$. Another possibility is to absorb the constant part of the work terms in V_0 and write $V(\mathbf{x}) = \hat{V}_0 - \mathbf{F}_c \cdot \mathbf{x}$, or to choose $\hat{V}_0 \equiv V_0 + \mathbf{F}_c \cdot \mathbf{x}_0 = 0$ so that $V(\mathbf{x}) = -\mathbf{F}_c \cdot \mathbf{x}$ now vanishes at $\mathbf{x} = \mathbf{0}$. Then $\mathbf{x} = \mathbf{0}$ serves as the datum point of zero potential energy. A similar thing may be done for any other potential energy function.

7.6.2. The Needle in the Haystack

One very important question remains—How can we know if a given force $\mathbf{F}(\mathbf{x})$ admits a potential energy function or not, without our actually having to find it? Otherwise, the problem we face resembles the search for a needle in a haystack with no knowledge that there is a needle to be found. Therefore, before we begin looking for the needle, it would be a good idea to first locate some sort of detection device. Then, if we detect it, we might continue and try to find it; otherwise, we discontinue the search. In the same spirit, before we start searching for a potential energy function, it is best to find an easy test to which we may subject any suitably continuous force $\mathbf{F}(\mathbf{x})$ to determine first, if a corresponding potential energy function exists. We are going to show that the relation (7.52) can hold when and only when the force $\mathbf{F}(\mathbf{x})$ satisfies the condition

$$\nabla \times \mathbf{F} = \mathbf{0}, \tag{7.55}$$

everywhere in a simply connected region $\mathscr{R}$. (The importance of $\mathscr{R}$ being simply connected is shown in Problem 7.22.) The operation $\nabla \times \mathbf{F}$ is called the *curl of* $\mathbf{F}$, and sometimes (7.55) is written as $\operatorname{curl}\mathbf{F} = \mathbf{0}$. The curl operation is defined with the aid of (7.48): $\operatorname{curl}\mathbf{F} = (\mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y + \mathbf{k}\partial/\partial z) \times (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})$, which is more conveniently represented by the familiar determinant-like, vector product representation in its expansion across the top row:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}. \tag{7.56}$$

The rule (7.55) is a neat device for detecting the possible existence of our needle in the haystack—it provides the test for existence of a potential energy function based on the following useful theorem.

Criterion for existence of a potential energy function: *A force* $\mathbf{F}(\mathbf{x})$ *is conservative in a simply connected region* $\mathscr{R}$ *if and only if* $\operatorname{curl}\mathbf{F}(\mathbf{x}) = \mathbf{0}$ *everywhere in* $\mathscr{R}$. *In consequence, there exists a potential energy function such that* $\mathbf{F}(\mathbf{x}) = -\nabla V(\mathbf{x})$ *holds everywhere in* $\mathscr{R}$.

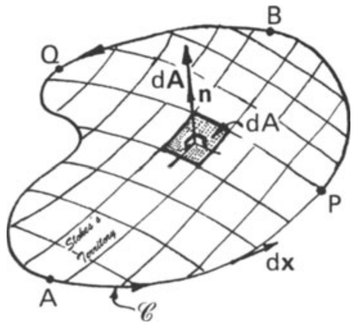


Figure 7.8. A simple closed curve bounding a region of area $\mathscr{A}$ in Stokes's theorem.

That (7.55) is necessary follows upon substitution of (7.53) into (7.56), which shows that (7.55) holds everywhere provided the continuous function $\mathbf{F}(\mathbf{x})$ is single-valued with continuous partial derivatives so that the order of the mixed partial derivatives of $V(\mathbf{x})$ may be reversed. Then, for example, $\partial F_y/\partial x - \partial F_x/\partial y = \partial^2 V/\partial x \partial y - \partial^2 V/\partial y \partial x = 0$.

Conversely, if (7.56) vanishes everywhere, the line integral of $\mathbf{F}(\mathbf{x})$ is independent of the path; hence, $\mathbf{F}$ has the form (7.52). The proof follows from Stokes's theorem of vector integral calculus, namely,

$$\mathscr{W} = \oint_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = \int_{\mathscr{A}} \operatorname{curl}\mathbf{F} \cdot d\mathbf{A}, \tag{7.57}$$

in which $d\mathbf{A}$ is the elemental vector area of the region $\mathscr{A}$ bounded by a simple closed curve $\mathscr{C}$, shown in Fig. 7.8. The symbol $\oint$ means that the integral is around the closed path $\mathscr{C}$, counterclockwise, with the region $\mathscr{A}$ on the left-hand side, as suggested in Fig. 7.8. The proof of (7.57) may be found in standard works on vector analysis. See, for example, the reference by Lass.

Stokes's theorem is applied to prove that $\operatorname{curl}\mathbf{F} = \mathbf{0} \implies \mathbf{F} = -\nabla V$. We thus require that (7.55) hold everywhere in a region containing $\mathscr{A}$, then (7.57) implies that

$$\oint_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = 0, \tag{7.58}$$

for every simple *closed* path in $\mathscr{R}$. Now, since only the unit tangent vector is reversed when a path is traversed in the opposite sense, we see in Fig. 7.8 that $\int_{BQA} \mathbf{F} \cdot d\mathbf{x} = -\int_{AQB} \mathbf{F} \cdot d\mathbf{x}$, where A and B are any two points on $\mathscr{C}$. Thus, by (7.58),

$$\int_{APB} \mathbf{F} \cdot d\mathbf{x} + \int_{BQA} \mathbf{F} \cdot d\mathbf{x} = \int_{APB} \mathbf{F} \cdot d\mathbf{x} - \int_{AQB} \mathbf{F} \cdot d\mathbf{x} = 0,$$

for any two points A and B on $\mathscr{C}$, that is,

$$\int_{APB} \mathbf{F} \cdot d\mathbf{x} = \int_{AQB} \mathbf{F} \cdot d\mathbf{x}.$$

Consequently, the condition (7.58) states, equivalently, that the work done by **F** is independent of the path joining A to B. However, this means that there exists a continuous single-valued, differentiable function $V(\mathbf{x})$ such that (7.52) holds everywhere in $\mathscr{R}$. This completes the proof. The thrust of the theorem is summarized as follows:

The work done by a force $\mathbf{F}(\mathbf{x})$ *is independent of the path, hence* $\mathbf{F}(\mathbf{x})$ *is conservative, if and only if the* curl$\mathbf{F}(\mathbf{x})$ *vanishes everywhere in the simply connected region* $\mathscr{R}$ *bounded by the closed path* $\mathscr{C}$*, and therefore if and only if* $\mathbf{F}(\mathbf{x})$ *is the gradient of a single-valued twice continuously differentiable potential energy function* $V(\mathbf{x})$. *That is, symbolically,*

$$\oint_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{x} = 0 \iff \nabla\times\mathbf{F} = \mathbf{0} \iff \mathbf{F} = -\boldsymbol{\nabla} V. \tag{7.59}$$

To illustrate our criterion for existence of a potential energy function, let us return to Example 7.4, page 231, and determine if $\mathbf{F}(\mathbf{x})$ in (7.24a) is conservative or not. We compute curl**F** using (7.56) and find

$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ bxy & cy & 0 \end{vmatrix} = \mathbf{k}(-bx) \neq \mathbf{0} \text{ everywhere.}$$

Therefore, no potential energy function exists; and hence **F** is not conservative. Indeed, **F** is conservative if and only if $b = 0$, as shown earlier. Only then does the force (7.24a) admit the potential energy function (7.54a).

7.7. Some Basic Potential Energy Functions

In anticipation of future applications, the potential energy function for the apparent gravitational force near the surface of the Earth, its variation with distance from the Earth, and the elastic potential energy for a linear spring are described next.

7.7.1. Gravitational Potential Energy

The work done by the constant gravitational force is given by (7.28); and by (7.45), we write $\Delta V_g = -\mathscr{W}_g$. *Therefore, the gravitational potential energy of a particle or center of mass object on or near the surface of the Earth is*

$$V_g = mgh, \tag{7.60}$$

in which $h = z - z_0$ *is measured vertically from* $z = z_0$*, the datum point of zero gravitational potential energy.* The gravitational potential energy increases with h above the datum level z_0 and decreases, becoming negative with h, below the z_0

reference level. The question of how close to the surface the particle must be for (7.60) to hold is discussed below.

The elementary rule (7.60) ignores the variation in the gravitational field strength with the distance from the Earth. To account for the elevation effect on the gravitational potential energy, let us consider a particle P of mass m that is free to move in the gravitational field of the Earth whose mass is M. In terms of its spherical coordinates in a Cartesian reference frame fixed at the center of the Earth at O, the gravitational force (5.58) exerted on P has the familiar form

$$\mathbf{F}(P;\mathbf{x}) = m\mathbf{g}(\mathbf{x}) = -\frac{GmM}{r^2}\mathbf{e}_r, \tag{7.61}$$

where $\mathbf{x} = r\mathbf{e}_r \neq \mathbf{0}$ is the radius vector of P from O. The reader may confirm through the following exercise that the gravitational force (7.61) is conservative.

Exercise 7.6. With $\mathbf{e}_r = \mathbf{x}/r$, express (7.61) in terms of rectangular Cartesian coordinates (x, y, z), and show that $\operatorname{curl}\mathbf{F} = \mathbf{0}$ everywhere. □

In accordance with (7.61), the conservative gravitational force at every point on the surface of a sphere of radius r is the same for every direction $\mathbf{e}_r = \mathbf{e}_r(\theta, \phi)$; so, by (7.61) and (7.52), the potential energy is at most a function of r. Therefore, with $\nabla V = \partial V(r)/\partial r\mathbf{e}_r$ and (7.61), equation (7.52) yields the relation $dV/dr = GmM/r^2$ whose integration delivers *the gravitational potential energy as a function of the distance r from the center of the Earth*:

$$V(r) = V_0 - \frac{GmM}{r}. \tag{7.62}$$

The reader may confirm the result (7.62) by an alternative derivation described in the following exercise based on an alternative proof that (7.61) is conservative.

Exercise 7.7. Show that the work done by the gravitational force (7.61) acting on a particle is independent of the path, and hence this force is conservative. Then deduce (7.62). □

The constant V_0 may be chosen so that $V(A) = 0$ at $r = A$, the surface of the Earth. Then $V_0 = GmM/A$ and (7.62) becomes

$$V(r) = GmM\left(\frac{1}{A} - \frac{1}{r}\right). \tag{7.63}$$

Recalling (5.61) for the acceleration of gravity, namely, $g = GM/A^2$, introducing $h(r) \equiv r - A$ into (7.63) and writing $V(r) = \hat{V}(h(r))$, we obtain the *gravitational potential energy as a function of the elevation h from the surface of the Earth*:

$$\hat{V}(h) = mgh\left(\frac{1}{1 + \dfrac{h}{A}}\right). \tag{7.64}$$

The elementary formula (7.60) thus follows from (7.64) when the particle is sufficiently close to the surface so that the term h/A may be considered negligible compared to unity. Otherwise, to account for the variation of gravity with elevation, the more precise relation (7.62) or (7.64) must be used. For example, at six miles (about 10 km) above the surface of the Earth, the approximate cruising altitude of a commercial jet airliner, the potential energy variance of (7.60) from (7.64) is only 0.16% (larger); and for a satellite at 250 miles (about 420 km) above the Earth, the variance is roughly 6.3% (larger). Hence, in most applications (7.60) is a very good approximation for motion on or near the surface of the Earth.

7.7.2. Elastic Potential Energy of a Spring

The elastic work done by a linear spring relative to its undeformed state is given by (7.33). Therefore, with (7.45), we write $\Delta V_e = -\mathscr{W}_e$ to obtain *the elastic potential energy stored by the spring*:

$$V_e = \tfrac{1}{2}kx^2, \tag{7.65}$$

wherein x is the change of length of the spring measured from its undeformed state, the reference state of vanishing potential energy. For a linear spring, the elastic potential energy function is the same in both compression $x < 0$ and tension $x > 0$.

7.8. General Conservation Principles

We now derive from the Newton–Euler law of motion and the work–energy principle for a center of mass particle the conservation principles of linear momentum, moment of momentum, and energy. We start with a general conservation theorem from which the two momentum conservation laws follow. The work–energy equation leads to the important principle of conservation of energy for a particle acted upon by conservative forces; and, finally, the general form of the work–energy principle in which the total work is split into its conservative and nonconservative parts is presented. Afterwards, these fundamental laws are illustrated in several further applications of physical interest.

7.8.1. A General Conservation Theorem

Let $\mathbf{A}(t)$ be a vector-valued function of time that is equal to the time derivative of another vector-valued function $\mathbf{u}(t)$:

$$\mathbf{A}(t) = \frac{d\mathbf{u}(t)}{dt}. \tag{7.66}$$

Let **e** be a constant unit vector and form the scalar product

$$\mathbf{A} \cdot \mathbf{e} = \frac{d}{dt}(\mathbf{u} \cdot \mathbf{e}). \tag{7.67}$$

We note that $\mathbf{A} \cdot \mathbf{e}$ and $\mathbf{u} \cdot \mathbf{e}$ are the respective components of **A** and **u** in the direction **e**. Thus, $\mathbf{A} \cdot \mathbf{e} = 0$, and hence **A** is perpendicular to **e**, if and only if $\mathbf{u} \cdot \mathbf{e} = constant$, in which case the quantity **u** is said to be *conserved in the direction* **e**. In summary, (7.67) reveals a useful theorem with application to the principles of mechanics.

General conservation theorem: *Consider the vector differential equation* $\mathbf{A(t)} = \dot{\mathbf{u}}(\mathbf{t})$ *and a fixed direction* **e**. *Then the component of* **A** *in the direction* **e** *is zero if and only if the component of* **u** *in the direction* **e** *is constant, that is, when and only when the quantity* **u** *is conserved in the direction* **e**.

7.8.2. The Principle of Conservation of Linear Momentum

Equation (5.34) has the form (7.66), so for any constant vector **e**,

$$\mathbf{F} \cdot \mathbf{e} = \frac{d}{dt}(\mathbf{p} \cdot \mathbf{e}), \tag{7.68}$$

which thus yields the following conservation law.

The principle of conservation of linear momentum: *The component of the force acting on a center of mass object in a fixed direction* **e** *vanishes for all time if and only if its momentum in the direction* **e** *is constant*:

$$\mathbf{F} \cdot \mathbf{e} = 0 \iff \mathbf{p} \cdot \mathbf{e} = \text{const.} \tag{7.69}$$

Further, the Newton–Euler law (5.34) shows that *the linear momentum is a constant vector when and only when the total force on the particle is zero*. The same result follows from (7.69) for all directions **e**. The principle (7.69), valid for *all time* in the motion, differs from our earlier conservation rule (7.12) for a system of *two* particles whose momentum is constant only during the impulsive instant. An easy application of the rule (7.69) follows.

Example 7.10. A particle of mass m is released from rest at A and slides down a smooth circular track of radius R shown in Fig. 7.9. At the lowest point B, the particle is projected horizontally and continues its motion until it strikes the ground at C. Determine the horizontal component of the particle's velocity at C.

Solution. The free body diagram in Fig. 7.9 on the path from B to C shows that no horizontal forces act on the particle. Therefore, the linear momentum in

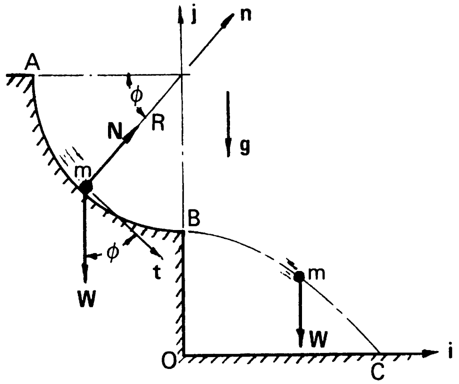


Figure 7.9. Application of the principles of conservation of momentum and work–energy.

the fixed horizontal direction **i** is conserved: $\mathbf{p} \cdot \mathbf{i} = m\dot{x} = \gamma$, a constant. Consequently, on the entire path BC, specifically at C, the horizontal component of the particle's velocity $\mathbf{v} \cdot \mathbf{i} = \dot{x}$ is a constant whose value is determined by its speed at point B. This value may be found by application of the work–energy principle.

The free body diagram of the particle on the circular path AB is shown in Fig. 7.9. The normal force **N** is workless on AB, while gravity does work $\mathscr{W}_g = mgR$ in reaching B. Hence, the total work done by the forces acting on m is $\mathscr{W} = mgR$. The increase in the kinetic energy as the particle slides from rest at A to the end state at B is $\Delta K = \frac{1}{2}mv_B^2$. The work-energy principle $\mathscr{W} = \Delta K$ determines the speed at B, and hence the horizontal component of the particle's velocity at C is given by

$$\dot{x} = v_B = \sqrt{2gR}. \qquad \square$$

Exercise 7.8. What is the normal force exerted on m by the surface at B? □

7.8.3. The Principle of Conservation of Moment of Momentum

The moment of momentum principle (6.79) also has the form (7.66). Therefore, for a fixed direction **e**,

$$\mathbf{M}_O \cdot \mathbf{e} = \frac{d}{dt}(\mathbf{h}_O \cdot \mathbf{e}), \tag{7.70}$$

where $\mathbf{M}_O \cdot \mathbf{e}$, the component of $\mathbf{M}_O$ in the direction **e**, characterizes the turning effect of the force about a line $\mathscr{L}$ through O having the direction **e**, as shown in Fig. 7.10. Thus, $\mathbf{M}_O \cdot \mathbf{e}$ is *the moment of the force about the axis* **e** *through* O. Similarly, $\mathbf{h}_O \cdot \mathbf{e}$ is *the moment of momentum about the axis* **e** *through* O. In these terms, the following conservation theorem is evident from (7.70).

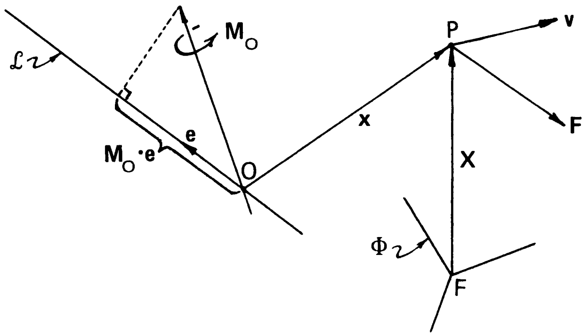


Figure 7.10. Schema for the torque about a line through the moment point O.

The principle of conservation of moment of momentum: *The moment of the force about an axis* **e** *through a fixed point O in an inertial frame* Φ *vanishes for all time when and only when the corresponding moment of momentum about that axis is constant*:

$$\mathbf{M}_O \cdot \mathbf{e} = 0 \Longleftrightarrow \mathbf{h}_O \cdot \mathbf{e} = \text{const.} \tag{7.71}$$

Moreover, the moment of momentum principle (6.79) shows that *the moment* $\mathbf{M}_O$ *about a fixed point vanishes for all time when and only when the moment of momentum is a constant vector.* The principle (7.71), valid for *all time* in the motion, differs from our earlier conservation rule (7.20) for a system of *two* particles whose moment of momentum is constant only during the impulsive instant. A classical application of our conservation law follows.

Example 7.11. *Central Force Motion and Kepler's Second Law.* A force directed invariably along a line through a fixed point is called a *central force.* A familiar example of a central force is the tension in a pendulum string; another is the gravitational force exerted by the Earth on a satellite shown in Fig. 7.11. Derive **Kepler's second law:** *A particle in motion under a central force alone must move in a plane; and if its path is not a straight line through the fixed central point* O*, its position vector from the fixed point sweeps out equal areas in equal intervals of time.*

Solution. Consider a central force **F** directed through the fixed origin O of an inertial frame $\varphi = \{O; \mathbf{e}_k\}$ in Fig. 7.11. Then the moment of **F** about O is zero: $\mathbf{M}_O = \mathbf{x} \times \mathbf{F} = 0$, wherein $\mathbf{x} = r\mathbf{e}_r$; and the moment of momentum principle (6.79) shows that the moment of momentum about O is a constant vector:

$$\mathbf{h}_O = \mathbf{x} \times m\mathbf{v}, \quad \text{a constant.} \tag{7.72a}$$

It follows that $\mathbf{h}_O = \mathbf{0}$ if and only if the position vector **x** is parallel to the velocity: $\mathbf{v} = k\mathbf{x}$, where k is a constant. In this instance the motion $\mathbf{x}(t) = \mathbf{x}_0 e^{kt}$, with $\mathbf{x}_0 = \mathbf{x}(0)$ initially, is along a straight line through O. Otherwise, when

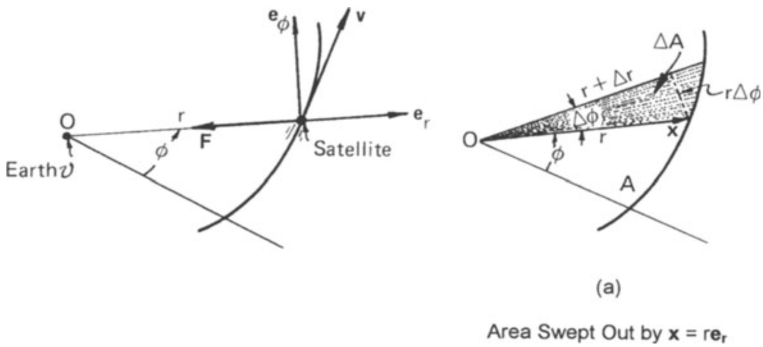


Figure 7.11. Satellite motion under a central force, and the orbital area swept out by the radius vector in an infinitesimal time.

$\mathbf{h}_O \neq \mathbf{0}$, both $\mathbf{x}$ and $\mathbf{v}$ are always perpendicular to the constant vector $\mathbf{h}_O$, and hence the particle must move in a plane whose vector equation is (7.72a). *Consequently, all central force motions are plane motions.*

To establish Kepler's equal area rule, we introduce polar coordinates and write $\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi$ and $\mathbf{x} = r\mathbf{e}_r$. Then $\mathbf{h}_O = h\mathbf{e}_z$ is normal to the plane of motion, and (7.72a) yields $mr^2\dot{\phi}\mathbf{e}_z = h\mathbf{e}_z$, that is,

$$r^2\dot{\phi} = \frac{h}{m} = \gamma, \quad \text{a constant.} \tag{7.72b}$$

Now, it may be seen in Fig. 7.11a that the element ΔA of the plane area swept out by the position vector is $\Delta A = \frac{1}{2}(r + \Delta r)r\Delta\phi$. Therefore, in the $\text{limit}_{\Delta t\to 0}\Delta A/\Delta t$, we have

$$\dot{A} = \tfrac{1}{2}r^2\dot{\phi}. \tag{7.72c}$$

This gives the rate of change of the area swept out by the position vector. Thus, with (7.72b), $\dot{A} = \gamma/2$; hence $A(t) = \frac{1}{2}\gamma t + A_0$, where $A_0 = A(0)$ is a constant, usually taken as zero. This is **Kepler's Second Law:** *The radius vector sweeps out equal areas in equal intervals of time.* □

Exercise 7.9. Apply the moment of momentum principle (6.79) to find $\mathbf{v}_B$ in Example 7.10, page 249. □

7.8.4. The Energy Principle

The important and useful principle of conservation of energy is derived next, and its equivalence with one of the Newton–Euler scalar equations of motion is demonstrated. Afterwards, the work–energy equation is recast in a form that separates the total work into its conservative and nonconservative parts. This procedure leads to the general energy principle, which includes the conservation law as a special case and shows clearly the roles of both nonconservative forces and workless normal forces in the work–energy equation.

7.8.4.1. The Principle of Conservation of Energy

The work–energy equation, $\mathscr{W} = \Delta K$, is valid for every conservative or nonconservative total force $\mathbf{F}(\mathbf{x})$. In addition, every conservative force is characterized by a scalar potential energy function $V(\mathbf{x})$ such that $\mathscr{W} = -\Delta V$. Therefore, when the total force acting on a particle (or center of mass object) is conservative, we have $\Delta K + \Delta V = \Delta(K + V) = 0$, from which the following conservation law is evident.

Principle of conservation of energy: *The sum of the kinetic and the potential energies of a particle acted upon by purely conservative forces is constant throughout the motion*:

$$K + V = E, \quad \text{a constant.} \tag{7.73}$$

Use of the negative sign for the potential energy in (7.44) was motivated by our desire at this point to assign a simple additive property in the conservation law (7.73). In general, the energy constant E is fixed by specified conditions at any instant in the motion. In accordance with (7.73), if the kinetic energy increases by some amount, the potential energy must decrease by the same amount, and vice versa. Indeed, when the kinetic energy in any motion of a conservative system attains its maximum value, the potential energy at that place must be least, and vice versa. Thus, for every conservative system

$$K_{\text{max}} + V_{\text{min}} = K_{\text{min}} + V_{\text{max}} = E, \quad \text{a constant.} \tag{7.74}$$

In a mechanical vibrations problem, for example, $K_{\text{min}} = 0$ at the extreme position of instantaneous rest in the oscillation of the load. The same thing holds in any problem where the motion of a particle begins from rest. In either case, at the corresponding instant in the motion, the energy constant in (7.74) is $E = V_{\text{max}}$. Specific examples are provided later on.

Since (7.73) was derived from the Newton–Euler law for a purely conservative total force $\mathbf{F}(\mathbf{x}) = -\boldsymbol{\nabla} V(\mathbf{x})$, this equation may be applied conversely to derive a single equivalent scalar equation of motion, as shown in (7.40) for the general work–energy equation. It is instructive to review this important result for a conservative system. In terms of intrinsic variables, we have $K = \frac{1}{2}m\dot{s}^2$ and $V = V(\mathbf{x}(s))$. Thus, by (7.73),

$$\tfrac{1}{2}m\dot{s}^2 + V(\mathbf{x}(s)) = E, \quad \text{a constant.} \tag{7.75}$$

This is to be compared with the general equation (7.39). Differentiation of (7.75) with respect to the arc length parameter s and use of the relation $dV/ds =$

$\partial V/\partial \mathbf{x} \cdot d\mathbf{x}/ds = -\mathbf{F} \cdot \mathbf{t} = -F_t$, where F_t is the tangential component of the total conservative force $\mathbf{F}$, yields $m\ddot{s} - F_t = 0$; hence,

$$K + V = E \iff m\ddot{s} = F_t. \tag{7.76}$$

The same result also may be obtained by differentiation of (7.75) with respect to time. Of course, this is the same as (7.40) applied to a conservative system. Thus, the principle of conservation of energy for a center of mass object is the first integral of the intrinsic, tangential component of the Newton–Euler law. In view of the equivalence relation (7.76), the principle of conservation of energy is especially useful in single degree of freedom dynamical problems. Notice that $V_{\min}$ occurs at places in the motion for which $dV/ds = 0$ holds, that is, at places for which $F_t = 0$. For conservative systems, these are static equilibrium positions of the particle.

Clearly, since forces of constraint perpendicular to the path do no work in the motion, these normal forces contribute nothing to the energy of an otherwise conservative system of forces. Accordingly, the energy principle can provide no information about such forces of constraint. The forgoing conclusions and remarks are illustrated in an example.

Example 7.12. (i) Apply the principle of conservation of energy to find the velocity $\mathbf{v}_B$ at which the particle in Example 7.10, page 249, is projected from point B shown in Fig. 7.9, and show that an arbitrary constant reference potential energy does not alter the conclusion. (ii) Derive from the energy equation the equivalent Newton–Euler scalar equation of motion for the mass on the circular path AB.

Solution of (i). First, we need to confirm that the energy principle (7.73) may be applied. The forces that act on the mass m on the circular path AB are shown in Fig. 7.9. Since the weight $\mathbf{W}$ is a conservative force and the normal surface reaction force does no work on AB, the total energy is conserved.

The point A is clearly a convenient datum for zero gravitational potential energy. However, we recall that only *differences* in the potential energy are relevant. Moreover, an arbitrary reference potential energy V_0 does not alter the energy balance in (7.73), for the same constant potential energy will appear in both sides of the equation. To demonstrate this, let us choose an arbitrary value V_0 for the reference potential energy at A. Since the particle is at rest at A, the kinetic energy at A is zero. Thus, initially the total energy is $E = K_A + V_A = V_0$. With A as the reference state, the potential energy of the mass m at point B is $V_B = V_0 - mgR$, and its kinetic energy is $K_B = \frac{1}{2}mv_B^2$, where $\mathbf{v}_B = v_B\mathbf{i}$ is the velocity of m when it projects from B. The energy principle (7.73) now yields

$$K_B + V_B = \frac{1}{2}mv_B^2 + V_0 - mgR = E = V_0, \tag{7.77a}$$

that is,

$$v_B = \sqrt{2gR}, \tag{7.77b}$$

so $\mathbf{v}_B = \sqrt{2gR}\mathbf{i}$, the same result found in Example 7.10 by application of the work–energy principle. Notice that the arbitrary reference potential energy V_0 cancels from (7.77a). Hence, the particular value assigned to the datum potential energy V_0 has no effect whatsoever on the solution.

Solution of (ii). The scalar equation of motion equivalent to the energy principle is readily derived from the energy equation. At an arbitrary point on the path AB, $K = \frac{1}{2}mR^2\dot{\phi}^2$ and $V = -mgR\sin\phi$, where we now fix $V_0 = 0$ at A. Since $E = 0$ at A, (7.73) yields the energy equation on the path AB,

$$\frac{1}{2}mR^2\dot{\phi}^2 - mgR\sin\phi = 0. \tag{7.77c}$$

Differentiation of (7.77c) with respect to the path variable ϕ (or with respect to t) yields the equivalent tangential component of the Newton–Euler equation of motion, namely,

$$mR\ddot{\phi} - W\cos\phi = 0. \tag{7.77d}$$

Notice, in agreement with (7.76), that $R\ddot{\phi} = \ddot{s}$ is the tangential component of the acceleration, and $W\cos\phi = F_t$ is the conservative tangential component of the total force $\mathbf{F} = \mathbf{W} + \mathbf{N}$ acting on m in Fig. 7.9, whose workless normal component is $F_n = N - W\sin\phi$. □

Let the reader consider the following example.

Exercise 7.10. Apply the principle of conservation of energy to solve Example 7.7, page 238. Derive the equation for the maximum spring deflection resulting from the impact by a mass m falling through a height h shown in Fig. 7.7. □

7.8.4.2. Remarks on Time Varying Potential Functions

The centripetal acceleration of a particle in a moving frame $\varphi = \{O; \mathbf{e}_k\}$ gives rise to a central directed, apparent centrifugal force $\mathbf{P}(\mathbf{x}, t) \equiv -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{x})$. So, consider a radial motion with $\mathbf{x} = r\mathbf{e}_r$ in φ and $\boldsymbol{\omega}(t) = \omega(t)\mathbf{e}_z = \alpha t\mathbf{e}_z$, say, then $\mathbf{P}(\mathbf{x}, t) = mr\omega^2\mathbf{e}_r$ in a cylindrical reference basis. Notice that this force has a potential function $\mathscr{V}(\mathbf{x}, t) = -\frac{1}{2}mr^2\omega^2$, such that $\mathbf{P} = -\nabla\mathscr{V}(\mathbf{x}, t) = -\partial\mathscr{V}/\partial r\mathbf{e}_r = mr\omega^2\mathbf{e}_r$. But this is not a conservative force, because the potential function $\mathscr{V}(\mathbf{x}, t)$ varies with both position and time, indeed, with $\omega = \alpha t$, $\partial\mathscr{V}/\partial t = -mr^2\alpha^2 t$. Moreover, with $d\mathbf{x} = dr\mathbf{e}_r$, the work done by this force, defined by $\mathscr{W}(\mathbf{x}, t) = \int \mathbf{P}(\mathbf{x}, t)\cdot d\mathbf{x} = m\omega^2\int r dr = -\mathscr{V}(\mathbf{x}, t)$, to within an arbitrary constant, also varies with time. In fact, it is possible to consider more general kinds of forces $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t)$ for which the work done is defined by $\mathscr{W}(\mathbf{x}, \dot{\mathbf{x}}, t) = \int \mathbf{P}(\mathbf{x}, \dot{\mathbf{x}}, t)\cdot d\mathbf{x}$. We shall not have an occasion to encounter these here.

Although it is possible to have a scalar-valued, time dependent function $V(\mathbf{x}, t)$ for which $\mathbf{F}(\mathbf{x}, t) = -\nabla V(\mathbf{x}, t)$, it is important to bear in mind that *the principle of conservation of energy holds only for conservative forces with potential functions that are independent of time*. To understand the reason for this, let us suppose that the potential energy function $V(\mathbf{x}, t)$ is time dependent. First, note that (7.34) holds for a force $\mathbf{F} = \mathbf{F}(\mathbf{x}, t)$. Consequently, in the present case,

$$\frac{dK(\mathbf{x}, t)}{dt} = \mathbf{F}(\mathbf{x}, t) \cdot \mathbf{v}(t) = -\nabla V(\mathbf{x}, t) \cdot \mathbf{v}(t). \tag{7.78}$$

However, since $V(\mathbf{x}, t)$ now depends explicitly on both the position vector $\mathbf{x}(t) = (x(t), y(t), z(t))$ and the time t, the total time rate of change of $V(\mathbf{x}, t)$ is

$$\frac{dV(\mathbf{x}, t)}{dt} = \frac{\partial V(\mathbf{x}, t)}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}(t)}{dt} + \frac{\partial V(\mathbf{x}, t)}{\partial t} = \nabla V(\mathbf{x}, t) \cdot \mathbf{v}(t) + \frac{\partial V(\mathbf{x}, t)}{\partial t}.$$

Adding this result to (7.78), we obtain

$$\frac{d}{dt}(K + V) = \frac{\partial V}{\partial t}. \tag{7.79}$$

Therefore, the time rate of change of the sum of the kinetic and potential energies is not zero, and hence the principle of conservation of energy does not hold when $V = V(\mathbf{x}, t)$. In fact, (7.79) shows that *the energy conservation law holds if and only if the potential energy is independent of time, i.e. when and only when* $V = V(\mathbf{x})$. Consequently, only those forces derivable from a potential energy $V = V(\mathbf{x})$, which is a function of position alone, are conservative forces; all other forces, even though they might have a potential function, are not conservative.

7.8.4.3. *The General Energy Principle*

It is useful to recast the work–energy equation in terms of the conservative and nonconservative parts of the total work done. We thus separate the total force $\mathbf{F}$ acting on a particle into its conservative part $\mathbf{F}_C = -\nabla V(\mathbf{x})$ and its nonconservative part $\mathbf{F}_N$. Clearly, any force perpendicular to the path, whatever its nature, necessarily is workless, and hence contributes nothing to the total work done. Therefore, with the aid of (7.21) and the work–energy equation (7.36), the total work done by $\mathbf{F} = \mathbf{F}_C + \mathbf{F}_N$ is related to the energy in accordance with

$$\Delta K = \int_{\mathscr{C}} \mathbf{F}_C \cdot d\mathbf{x} + \int_{\mathscr{C}} \mathbf{F}_N \cdot d\mathbf{x} = -\Delta V + \mathscr{W}_N,$$

in which $\mathscr{W}_N \equiv \int_{\mathscr{C}} \mathbf{F}_N \cdot d\mathbf{x}$ is the work done by the nonconservative force. Plainly, this is merely another form of (7.36). Let $\mathscr{E} \equiv K + V$ denote the *total energy*, and note that $\Delta K + \Delta V = \Delta(K + V) = \Delta\mathscr{E}$ is the change in the total energy. Consequently, the work–energy principle (7.36) yields the following equivalent law.

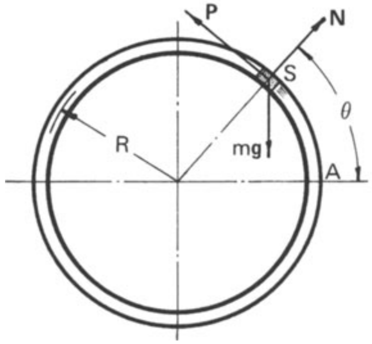


Figure 7.12. Propulsive motion of a slider block.

The general energy principle: *The change in the total energy is equal to the work done by the nonconservative part of the total force acting on the particle*:

$$\Delta\mathscr{E} = \mathscr{W}_N. \tag{7.80}$$

Hence, the total energy is constant if and only if the nonconservative part of the force does no work in the motion or, trivially, when nonconservative forces are absent.

It is useful to distinguish conservative and nonconservative forces, if possible; but if the nature of a force is uncertain, the ambiguous force is considered nonconservative until proven otherwise. The following example illustrates the straightforward application of the general energy principle (7.80).

Example 7.13. A propulsive force $\mathbf{P}$ of constant magnitude moves a slider S of mass m in a smooth circular track in the vertical plane, as shown in Fig. 7.12. The slider starts from rest at the horizontal position A. Determine the speed of S as a function of θ. What is its angular speed after n complete turns?

Solution. The total force that acts on S in the Fig. 7.12 consists of the workless normal reaction force $\mathbf{N}$ exerted by the smooth tube, the conservative gravitational force $\mathbf{F}_C = m\mathbf{g}$, and the nonconservative propulsive force $\mathbf{F}_N = \mathbf{P} = P\mathbf{t}$ which always is tangent to the path of S. The change in the potential energy of S is $\Delta V = mgR\sin\theta$, the datum being at A, and the change in the kinetic energy from the initial rest position at A is $\Delta K = \frac{1}{2}mv^2$. Therefore, with $\Delta\mathscr{E} = \Delta K + \Delta V$ and $\mathbf{F}_N \cdot d\mathbf{x} = P ds$, the general energy principle (7.80) yields

$$\frac{1}{2}mv^2 + mgR\sin\theta = P\int_0^{R\theta} ds = PR\theta, \tag{7.81a}$$

and hence the speed of S as a function of θ is given by

$$v(\theta) = \sqrt{\frac{2R}{m}(P\theta - mg\sin\theta)}. \tag{7.81b}$$

The angular speed of S is determined by $v = R\omega$. Thus, after n complete revolutions, $\theta = 2\pi n$ and (7.81b) provides the angular speed

$$\omega(n) = \sqrt{\frac{4n\pi P}{mR}}. \tag{7.81c}$$

□

7.9. Some Further Applications of the Fundamental Principles

Every problem of particle dynamics can be formulated entirely by use of the Newton–Euler law (5.34). Together with appropriate initial conditions, this law provides a complete system of three scalar equations for at most three unknown quantities. But the several auxiliary, first integral and moment of momentum principles derived from this law and discussed earlier in this chapter often deliver easily and more directly pieces of information that simplify the problem solution and often provide further physical insight as well. The remainder of this chapter is devoted to some further applications that demonstrate these attributes.

We begin with a familiar example that illustrates the joint application of the principles of conservation of momentum and energy in the elementary problem of projectile motion. The next example is an application of the principles of conservation of moment of momentum and energy in the formulation of the spherical pendulum problem. Finally, the phase plane curves for a simple harmonic oscillator and the motion of a spring-mass system are studied by the energy method. Some advanced topics are then presented in the sections that follow.

Example 7.14. *Application to projectile motion.* A projectile P is fired from a gun at O with muzzle speed v_0 at an elevation angle α from the horizontal ground plane in frame $\varphi = \{O; \mathbf{i}, \mathbf{j}\}$. Find the speed of the projectile as a function of its altitude; determine the maximum height h reached by P; and find its speed when it returns to the ground plane. Neglect air resistance.

Solution. A simple free body diagram of the projectile will show that the only force acting on P is the conservative gravitational force $m\mathbf{g} = -mg\mathbf{j}$ in frame φ. Consequently, the linear momentum in the horizontal direction $\mathbf{i}$ in φ, namely, $\mathbf{p} \cdot \mathbf{i} = m\dot{x}$, is constant. Initially, $\mathbf{p} \cdot \mathbf{i} = mv_0 \cos\alpha$; hence, for all time,

$$\dot{x} = v_0 \cos\alpha. \tag{7.82a}$$

This easy result provides auxiliary information for later use.

Clearly, the system is conservative, and with $y = 0$ as the zero reference for the potential energy, the total energy initially is $E = \frac{1}{2}mv_0^2$. At any subsequent position, the potential energy is $V(y) = mgy$ and the kinetic energy is $K(y) = \frac{1}{2}mv^2$. The energy principle (7.73) requires

$$\tfrac{1}{2}mv^2 + mgy = \tfrac{1}{2}mv_0^2, \tag{7.82b}$$

which determines the projectile's speed as a function of its altitude y:

$$v(y) = \sqrt{v_0^2 - 2gy}. \tag{7.82c}$$

The projectile's speed is independent of the gun's angle of elevation and the mass of the projectile.

To find the greatest height attained, we recall that $v^2 = \dot{x}^2 + \dot{y}^2$. Clearly, the projectile attains its maximum altitude h when $\dot{y} = 0$. With (7.82a), the speed $v(h) = \dot{x} = v_0 \cos\alpha$, and hence (7.82b) or (7.82c) yields the maximum altitude reached by P:

$$h = \frac{v_0^2}{2g}\sin^2\alpha. \tag{7.82d}$$

Consequently, the greatest height attained depends on the angle of elevation, but not the mass of the projectile.

Finally, when P returns to the ground at Q, $y = 0$ and (7.82c) shows that the shell lands with speed equal to its muzzle speed v_0. □

The following exercises are left for the reader.

Exercise 7.11. Show that the projectile's horizontal range is given by

$$r = \frac{v_0^2}{g}\sin 2\alpha. \tag{7.82e}$$

□

Exercise 7.12. Show that (7.82a) and (7.82b) are equivalent to two scalar equations of motion provided by the Newton–Euler law (5.34). Integrate these equations with respect to time to obtain $(x(t), y(t))$ in φ. (a) Introduce (7.82d) and (7.82e), and show that the projectile's trajectory, absent any environmental effects, is a parabola defined by

$$Y = -\frac{g}{2v_0^2\cos^2\alpha}X^2, \tag{7.82f}$$

where $Y \equiv y - h$ and $X \equiv x - r/2$ in the frame $\Phi = \{Q; \mathbf{I}_k\}$. Identify frame Φ. (b) Determine, as a function of 2α, the projectile's coordinates (x_m, y_m) at its greatest height in φ. Show that the loci of all points of maximum height is an ellipse

$$\frac{x_m^2}{a^2} + \frac{(y_m - b)^2}{b^2} = 1, \tag{7.82g}$$

where $a = v_0^2/2g$ and $b = a/2$. □

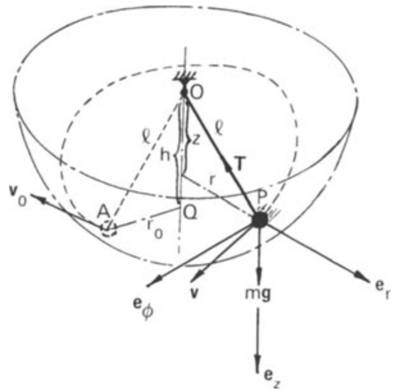


Figure 7.13. Spherical pendulum motion.

Example 7.15. *The spherical pendulum.* One end of a thin, rigid rod of length ℓ and negligible mass is fastened to a bob P of mass m, and its other end is attached to a smooth ball joint at O. In view of the constraint, the bob moves on a spherical surface of radius ℓ, so this device is called a *spherical pendulum*. The bob is given an arbitrary initial velocity $\mathbf{v}_0$ at a point A located in the horizontal plane at the distance h below O in Fig. 7.13. Find three equations that determine the velocity of P as a function of the vertical distance z below O, and describe how the motion $\mathbf{x}(P, t)$ may be found from the results.

Solution. To find $\mathbf{v}(z)$, it proves convenient to introduce cylindrical coordinates (r, ϕ, z) with origin at the ball joint O and basis directed as shown in Fig. 7.13, with $\mathbf{e}_z$ downward. The velocity of P is given by (see (4.59) in Volume 1)

$$\mathbf{v}(P, t) = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z. \tag{7.83a}$$

We wish to determine $\dot{r}$, $r\dot{\phi}$, and $\dot{z}$ as functions of z. Three equations are needed.

The first equation is obtained from the energy principle. The forces that act on P are its weight $m\mathbf{g}$ and the workless force $\mathbf{T}$ exerted by the rod. Therefore, the principle of conservation of energy (7.73), with $V_0 = 0$ at O, yields

$$\tfrac{1}{2}mv^2 - mgz = mE_0, \tag{7.83b}$$

where $E_0 \equiv E/m$ is the total energy per unit mass. The speed v of P is thus determined by (7.83a) and (7.83b):

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2 = 2(E_0 + gz). \tag{7.83c}$$

This provides one relation among the three unknown functions. The constant E_0 is fixed by the initial conditions at A, $E_0 = \frac{1}{2}v_0^2 - gh$ at $z = h$ in (7.83b).

Another equation may be obtained from the principle of conservation of moment of momentum. The rod tension $\mathbf{T}$ has no moment about O, and $m\mathbf{g}$ exerts no moment about the vertical OQ axis. Hence, by (7.71), the moment of momentum about this line is conserved: $\mathbf{h}_O \cdot \mathbf{e}_z = \eta$, a constant. Recalling (7.83a), we see that only the component $mr\dot{\phi}$ of the linear momentum $m\mathbf{v}$ has a moment about the line OQ, namely, $r(mr\dot{\phi})\mathbf{e}_z$. Therefore, $\mathbf{h}_O \cdot \mathbf{e}_z = mr^2\dot{\phi} = \eta$, and with $\gamma \equiv \eta/m$, we have

$$r^2\dot{\phi} = \gamma. \tag{7.83d}$$

This provides another equation relating the unknown functions. The constant γ is determined from the initial conditions. Let $\hat{\mathbf{e}}_\phi$ denote $\mathbf{e}_\phi$ at A. Then $m\mathbf{v}_0 \cdot \hat{\mathbf{e}}_\phi$ is the only component of the initial linear momentum having a moment about the line OQ, and hence $\gamma = r_0\mathbf{v}_0 \cdot \hat{\mathbf{e}}_\phi = r_0 v_0 \cos\langle \mathbf{v}_0, \hat{\mathbf{e}}_\phi\rangle$, wherein $r_0 = (\ell^2 - h^2)^{1/2}$ in Fig. 7.13.

The final equation is derived from the suspension constraint: $\ell^2 = r^2 + z^2$. This gives $r = (\ell^2 - z^2)^{1/2}$, and hence

$$\dot{r} = -\frac{z\dot{z}}{\sqrt{\ell^2 - z^2}}. \tag{7.83e}$$

A few moments reflection reveals that $\dot{r}$, $r\dot{\phi}$, and $\dot{z}$ are now known as functions of z. Indeed, upon substituting (7.83d) and (7.83e) into (7.83c), we reach

$$\dot{z}^2 = \frac{2g}{\ell^2}\left[(\ell^2 - z^2)(z + \frac{E_0}{g}) - \frac{\gamma^2}{2g}\right], \tag{7.83f}$$

which determines $\dot{z}(z)$. And with $r(z) = (\ell^2 - z^2)^{1/2}$, $\dot{r}(z)$ given by (7.83e), and $r\dot{\phi} = \gamma/r(z)$ from (7.83d), it is now a straightforward matter to write the velocity (7.83a) as a function of z alone. We omit these details.

Finally, we need to say how the motion $\mathbf{x}(P,t) = r\mathbf{e}_r + z\mathbf{e}_z$ may be read from the results. In principle, integration of (7.83f) determines $z(t)$, hence $r(t)$, and (7.83d) provides

$$\phi(t) = \int_0^t \frac{\gamma}{r^2}\,dt, \tag{7.83g}$$

to fix $\mathbf{e}_r(\phi)$, which thus determines the motion. The exact solution for $z(t)$ may be obtained from (7.83f) in terms of Jacobian elliptic functions introduced later; however, we shall not pursue this problem further. (See Synge and Griffith.) □

Exercise 7.13. Apply the Newton–Euler law to formulate the spherical pendulum problem. Hint: Show that $z\ddot{r} = -\frac{1}{2}r\,d\dot{r}^2/dz$. □

Example 7.16. *Constant energy curves in the phase plane.* Use the energy principle to derive the differential equation for the smooth, horizontal motion of the linear spring-mass system in Fig. 6.13, page 134. Show that the phase plane trajectories, the curves in the xv-plane, are curves of constant total energy.

Solution. The free body diagram is shown in Fig. 6.13a. The weight of the oscillator and the normal surface reaction do no work in any rectilinear motion along the smooth horizontal surface. The elastic potential energy of the linear spring force acting on m is given by (7.65): $V(x) = \frac{1}{2}kx^2$, wherein $V(0) = 0$ in the natural state $x = 0$. The system is conservative with kinetic energy $K = \frac{1}{2}m\dot{x}^2$, so the energy principle (7.73) yields

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = E. \tag{7.84a}$$

The equation of motion is obtained by differentiation of (7.84a) with respect to the path variable x or with respect to time; we find $m\ddot{x} + kx = 0$. This agrees with (6.65a) in which $p = \sqrt{k/m}$.

Now let us examine the curves in the xv-plane, called the *phase plane*. Because $k > 0$, the total energy E in (7.84a) is a positive constant determined from assigned initial data. Suppose that $x(0) = x_0$ and $\dot{x}(0) = v_0$ at $t = 0$, then $E = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2$. We introduce

$$\varepsilon^2 \equiv \frac{2E}{m}, \tag{7.84b}$$

and write $v = \dot{x}$ to cast (7.84a) in the form

$$\left(\frac{x}{\varepsilon/p}\right)^2 + \left(\frac{v}{\varepsilon}\right)^2 = 1. \tag{7.84c}$$

For any given spring-mass pair (m, k), the phase plane curve described by (7.84c) is an ellipse whose axes are determined by the constant ε. For each choice of initial data, ε has a different value; and hence (7.84c) describes a family of concentric ellipses each of which is traversed in the same time $\tau = 2\pi/p$, the period of the oscillation, and on each of which ε is a constant fixed by the total energy E. In consequence, the phase plane curves for a conservative dynamical system are called *energy curves*. In physical terms, (7.84c) shows that ε is equal to the maximum speed in the periodic motion, which occurs at the natural state $x = 0$, and $x_A \equiv \varepsilon/p$ is the symmetric amplitude of the oscillation, the maximum displacement from the natural state—it marks the extreme states in the motion at which $v = 0$. □

Example 7.17. *Motion and the energy of a spring-mass system.* An unstretched linear spring shown in Fig. 7.14 is attached to a mass m that rests on a hinged board supported by a string. Find the motion of the load when the string is cut and the board falls clear from under it. Describe the energy curve for the motion.

Solution. The free body diagram in Fig. 7.14a shows the gravitational and elastic forces that act on m when the string is cut. These are conservative forces with total potential energy $V(x) = -mgx + \frac{1}{2}kx^2$, wherein $V(0) = 0$. The kinetic

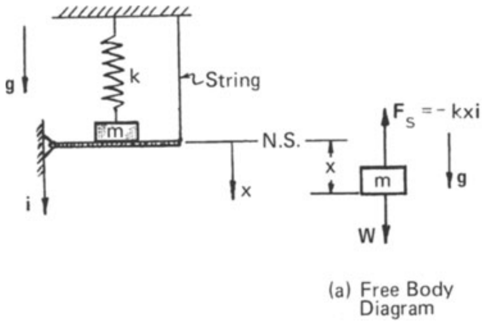


Figure 7.14. Gravity induced vibration of a simple harmonic oscillator.

energy is $K = \frac{1}{2}m\dot{x}^2$. Since the total energy initially is zero, the energy principle (7.73) gives

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 - mgx = 0. \tag{7.85a}$$

Because this is the first integral of the equation of motion for which the path variable is x, differentiation of (7.85a) with respect to x (or t) yields the familiar equation of motion:

$$m\ddot{x} + kx = mg. \tag{7.85b}$$

The motion is a gravity induced, free vibration of a simple harmonic oscillator. The general solution of this equation is

$$x = \frac{mg}{k} + A\sin pt + B\cos pt, \tag{7.85c}$$

in which $p = \sqrt{k/m}$. The initial circumstances $x(0) = 0$ and $\dot{x}(0) = 0$ require $A = 0$ and $B = -mg/k$ in (7.85c), so the motion of the mass is described by

$$x(t) = \frac{mg}{k}(1 - \cos pt). \tag{7.85d}$$

It is seen from (7.85b) that $x_S \equiv mg/k$ is the static equilibrium displacement. Hence, (7.85d) shows that the load oscillates about the equilibrium state with circular frequency $p = \sqrt{k/m}$ and amplitude equal to x_S. The reader may readily confirm that $dV/dx = 0$ at x_S, and hence $V_{\text{min}} = V(x_S) = -\frac{1}{2}kx_S^2$. As a consequence, $K_{\text{max}} + V_{\text{min}} = E = 0$ yields the maximum speed $v_{\text{max}} = px_S$. It is simpler, however, to note from (7.85d) that $\dot{x}(t) = px_S \sin pt$, hence $v_{\text{max}} = px_S$.

Now let us consider the energy curve. Due to the static displacement, the curve described by the energy equation (7.85a) in terms of x and $\dot{x}$ is an ellipse whose center is shifted a distance x_S along the x-axis. To see this, introduce the coordinate transformation $z = x - x_S$, which describes the motion of the load relative to its equilibrium position. Use of this relation in (7.85a) and (7.85b)

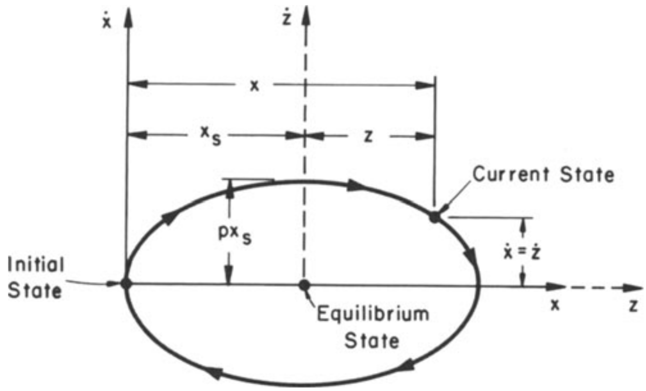


Figure 7.15. Phase plane diagram for the free vibration of a load on a linear spring.

yields the corresponding transformed equations:

$$\left(\frac{\dot{z}}{px_S}\right)^2 + \left(\frac{z}{x_S}\right)^2 = 1, \tag{7.85e}$$

$$\ddot{z} + p^2 z = 0. \tag{7.85f}$$

Clearly, the energy equation (7.85e) is an ellipse centered at the origin in the phase plane of z and $\dot{z}$; hence, the motion is periodic with circular frequency p and symmetric amplitude $z_{\max} = x_S$. The energy curve is the graph of (7.85e) shown in Fig. 7.15. Both the original and transformed variables are indicated. The geometry characterizes the motion of a simple harmonic oscillator described by (7.85f) whose solution is just the transformation of (7.85d) given by $z = -x_S \cos pt$. □

We now turn to some advanced applications. These include the finite amplitude oscillations of a simple pendulum, the plane motion of a particle on an arbitrary concave path, Huygens's isochronous pendulum, orbital motion, and Kepler's first and third laws. Elliptic functions and integrals are introduced along the way. In a first reading, however, these topics may be omitted without significant loss of continuity, if the reader may prefer to move forward to the next chapter.

7.10. The Simple Pendulum Revisited: The Exact Solution

Our earlier study of the simple pendulum focused on its small amplitude solution for which the motion is simple harmonic and hence isochronal, that is, the period is a constant, independent of the amplitude. Here we explore the exact solution for the large amplitude motion, which is neither simple harmonic nor isochronal. As a consequence, the accuracy of the approximation in the small amplitude solution is determined. The nonoscillatory, periodic motion of the revolving pendulum is also described. The importance of this classical problem lies in the parallel application of the method of analysis to a great variety of other physical systems described by a similar differential equation.

7.10.1. The Energy Equation and the Rod Tension

The energy equation will be applied to reformulate the finite motion problem for which the exact equation of motion is already given by the first equation in (6.67b). The energy equation, we recall, is the first integral of this equation of motion. An easy, exact result for the rod tension as a function of the angular motion $\theta(t)$ then follows immediately.

The problem geometry and the free body diagram of the bob are shown in Fig. 6.15, page 138. Recall that the supporting rod of length ℓ has negligible mass compared with the bob's mass m. Clearly, the variable rod tension $\mathbf{T}$ does no work in the motion, and the gravitational potential energy (7.60) is given by $V_g(\theta) = mg\ell(1 - \cos\theta)$. The system is conservative with kinetic energy $K = \frac{1}{2}m\ell^2\dot{\theta}^2$. Hence, the energy principle (7.73) yields

$$\tfrac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1 - \cos\theta) = E, \ \text{const.} \tag{7.86a}$$

The energy constant E may be evaluated from known conditions at any point in the motion, regardless of what the initial data might be. Here we consider an oscillatory motion with finite amplitude α so that $\dot{\theta} = 0$ when $\theta = \pm\alpha$. Then $E = mg\ell(1 - \cos\alpha)$, and (7.86a) becomes

$$\dot{\theta}^2 = 2p^2(\cos\theta - \cos\alpha), \tag{7.86b}$$

where $p = \sqrt{g/\ell}$. This is the exact integral of the first of the equations of motion in (6.67b). Therefore, substitution of (7.86b) into the second equation in (6.67b) yields the rod tension as a function of $\theta \in [-\alpha, \alpha]$, namely,

$$T(\theta) = W(3\cos\theta - 2\cos\alpha). \tag{7.86c}$$

7.10.2. The Finite Pendulum Motion and Its Period

We now turn to the exact analysis of the finite amplitude motion. The finite oscillatory motion of the pendulum and its period are essentially determined upon integration of (7.86b) to obtain

$$pt = \int_0^\theta \frac{d\theta}{\sqrt{2(\cos\theta - \cos\alpha)}}, \tag{7.87a}$$

wherein the value $\theta(0) = 0$ at $t = 0$ has been assigned for convenience, that is, time is measured from the instant when the bob is at its lowest vertical position.

To be consistent with the initial data, the positive root has been chosen in (7.86b). Equation (7.87a) thus provides the exact, but implicit solution for the periodic angular motion $\theta(t)$. The motion is periodic but not simple harmonic. The precise period of the finite oscillation, denoted by τ^*, follows from (7.87a). Let us write the integral in (7.87a) as $f(\theta)$ and note that $t = \tau^*/4$ is the time to reach the state $\theta = \alpha$ at the end of the pendulum's primary swing. Then the periodic time $\tau^* = 4f(\alpha)/p$. It is seen that the period varies, in fact increases, with the amplitude α, and hence the motion is not isochronous. This phenomenon is typical of nonlinear oscillation problems.

The integral in (7.87a) cannot be evaluated in terms of elementary functions, but it can be expressed in the form of an elliptic integral or a corresponding Jacobian elliptic function whose numerical value may be found from mathematical tables or computed directly. To cast the integral in its standard form, we introduce a new variable ϕ defined by the transformation

$$\sin\frac{\theta}{2} = k\sin\phi, \qquad k = \sin\frac{\alpha}{2}, \tag{7.87b}$$

so that $0 < k < 1$. Then $\cos\alpha = 1 - 2k^2$ and $\cos\theta = 1 - 2k^2\sin^2\phi$ follow from the familiar double angle trigonometric identities, and use of these relations in (7.87a) yields the standard formula

$$pt = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}. \tag{7.87c}$$

The integral in (7.87c) is called the *elliptic integral of the first kind*, usually denoted by

$$F(\phi; k) \equiv \int_0^\phi \frac{d\vartheta}{\sqrt{1 - k^2\sin^2\vartheta}}, \tag{7.87d}$$

for $0 < k < 1$. The variable limit ϕ is the *argument* of the integral whose dummy variable ϑ replaces ϕ in (7.87c). The constant k, defined by the second relation in (7.87b), is called the *modulus*. The two equations in (7.87b) determine the argument and the modulus in terms of the pendulum variable θ and its amplitude α. As the physical angular placement θ grows from 0 to α, the argument ϕ increases from 0 to $\pi/2$. The special integral obtained from (7.87d) at $\phi = \pi/2$, written as

$$K(k) \equiv F(\frac{\pi}{2}; k) = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - k^2\sin^2\vartheta}}, \tag{7.87e}$$

is called the *complete elliptic integral of the first kind*. The use of K for this integral is conventional and is not to be confused with the kinetic energy function, and, of course, F is not a force. The values of $F(\phi; k)$ and $K(k)$ are tabulated‡ in

‡ See, for example, P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, 2nd Edition, revised, Springer-Verlag, New York, 1971. This is an especially valuable

handbooks of mathematical tables, and nowadays may be routinely calculated by computer. The elliptic integral and the complete elliptic integral of the second kind are introduced later in Exercise 7.19, page 283. An elliptic integral and a complete elliptic integral of the third kind also arise in nonlinear dynamical problems, but we shall not encounter these here.

The period τ^* of the oscillation is now readily determined. With $t = \tau^*/4$ at $\theta = \alpha$, (7.87c) and (7.87e) yield the exact periodic time of the pendulum motion,

$$\tau^* = \frac{4}{p}K(k) = \frac{2\tau}{\pi}K(k). \tag{7.87f}$$

Here $\tau = 1/f = 2\pi/p = 2\pi\sqrt{\ell/g}$ denotes the constant small amplitude period of the pendulum defined in (6.67g). Recall that k in the second of (7.87b) depends on the amplitude α. The result (7.87f), therefore, describes the precise manner in which the period τ^*, and hence the frequency $f^* \equiv 1/\tau^*$, varies with the amplitude, and it renders explicit the relation of the exact period τ^* (frequency f^*) to the elementary period τ (frequency f). We can now determine which is greater.

Because $0 < k < 1$, (7.87e) shows that $K(k) > \int_0^{\pi/2} d\vartheta = \dfrac{\pi}{2}$, that is, $2K(k)/\pi > 1$. By (7.87f), therefore, $\tau^* > \tau$, so $f^* < f$: *The exact period (frequency) of the finite amplitude oscillations of a simple pendulum is always greater (smaller) than the period (frequency) of its small amplitude, simple harmonic motion.*

Further, notice in (7.87e) that $K(k)$ increases monotonically with $k \in (0, 1)$, while the second relation in (7.87b) shows that k increases with the amplitude angle α. Therefore, $K(k)$ increases with α. As perceived earlier, (7.87f) shows that *the period (frequency) of the finite pendulum motion increases (decreases) when the amplitude is increased.*

7.10.3. Introduction to Jacobian Elliptic Functions

The solution (7.87c) provides the travel time in terms of the argument $\phi(\theta)$ in accordance with $pt = F(\phi; k) = f(\theta)$. By the introduction of a Jacobian elliptic function, however, this integral relation may be inverted to obtain the solution in the closed form $\theta = \theta(t) \equiv f^{-1}(pt)$. To motivate the idea of the Jacobian elliptic functions, let us recall first the elementary integral

$$u \equiv \int_0^y \frac{d\vartheta}{\sqrt{1-\vartheta^2}} = \sin^{-1} y, \tag{7.88a}$$

where ϑ is a dummy variable in all of these standard integrals. Then the inverse of the integral u whose argument is y is the familiar circular function $y = \sin u$.

resource for formulas, identities, descriptions of properties and graphics for elliptic functions, and it provides transformations of general elliptic integrals of all sorts to their standard forms. It contains adequate explanatory material on elliptic functions and integrals for those unfamiliar with the subject.

Similarly, the inverse of the elementary integral

$$u \equiv \int_0^y \frac{d\vartheta}{1-\vartheta^2} = \tanh^{-1} y, \tag{7.88b}$$

is the hyperbolic function $y = \tanh u \equiv \sinh u / \cosh u$.

The same idea may be applied to invert the elliptic integral (7.87c). We first introduce a new argument

$$y = \sin\phi, \tag{7.88c}$$

into (7.87c) and replace the integrand variable y with the dummy variable ϑ to obtain the following alternate standard formula for the elliptic integral of the first kind:

$$F(\phi(y); k) \equiv u(y) = \int_0^y \frac{d\vartheta}{\sqrt{(1-\vartheta^2)(1-k^2\vartheta^2)}}, \tag{7.88d}$$

for $0 < k < 1$. Notice that this integral reduces to (7.88a) when $k = 0$ and to (7.88b) when $k = 1$. With $\phi = \pi/2$, we have $y = 1$; hence, in accordance with (7.87e), the alternate standard form of the complete elliptic integral of the first kind is

$$K(k) = u(1) = \int_0^1 \frac{d\vartheta}{\sqrt{(1-\vartheta^2)(1-k^2\vartheta^2)}}. \tag{7.88e}$$

Now, the *Jacobian elliptic sine function* $\mathrm{sn}u$, read as "$ess - en - u$", is similarly defined by

$$u(y) = \int_0^y \frac{d\vartheta}{\sqrt{(1-\vartheta^2)(1-k^2\vartheta^2)}} \equiv \mathrm{sn}^{-1} y. \tag{7.88f}$$

Therefore, the inverse of the elliptic integral u whose argument is y is the Jacobian elliptic sine function $y = \mathrm{sn}u$. With (7.88c),

$$y = \sin\phi = \mathrm{sn}u, \tag{7.88g}$$

yields the desired *inverse solution* $\phi = \sin^{-1}(\mathrm{sn}u)$.

We next establish the properties of the Jacobian elliptic sine function. In accordance with (7.88g), $\phi = 0$ implies $y = 0$, and (7.88f) gives $u = 0$; therefore, $\mathrm{sn}0 = 0$. Similarly, $\phi = \pi/2$ yields the corresponding amplitude $y = 1 = \mathrm{sn}u(1)$, and hence by (7.88e), $\mathrm{sn}K(k) = 1$ is the amplitude of the graph $\mathrm{sn}u$ at $u(1) = K(k)$. Because $\sin\phi$ is an odd periodic function with quarter period $\pi/2$, the elliptic sine function in (7.88g) also is an odd periodic function: $\mathrm{sn}(-u) = -\mathrm{sn}u$, but with quarter period $K(k)$ that varies with k. Hence, $\mathrm{sn}u$ has amplitude 1 and period $4K$. The graph of $\mathrm{sn}u$, therefore, is similar to the map of the familiar sine function to which it reduces when $k = 0$. The reader may find it helpful to sketch the graph of the periodic function $y = \text{sn}u$ to illustrate the properties of the Jacobian elliptic sine function, namely,

$$\text{sn}(qK) = \begin{cases} 0 \text{ if } q = 0, 2, 4, \\ 1 \text{ if } q = 1, \\ -1 \text{ if } q = 3. \end{cases} \tag{7.88h}$$

$$\text{sn}(u + 4K) = \text{sn}u, \qquad \text{sn}(-u) = -\text{sn}u. \tag{7.88i}$$

Two additional Jacobian elliptic functions are defined in terms of the elliptic sine function. Study of these functions and their properties is left for the student in Problems 7.54 and 7.55. We shall find no need for these additional functions as we continue discussion of the pendulum problem.

7.10.4. The Pendulum Motion in Terms of an Elliptic Function

With the aid of (7.87d) and (7.88d), the simple pendulum solution (7.87c) may be written as $pt = F(\phi; k) = u(y)$. Consequently, the inverse of the elliptic integral for the pendulum problem is provided by (7.88g):

$$y = \sin\phi = \text{sn}(pt). \tag{7.89a}$$

Finally, use of this result in the first equation in (7.87b) delivers, explicitly, *the oscillatory motion $\theta(t)$ of a simple pendulum with amplitude α*:

$$\theta(t) = 2\sin^{-1}[k\text{sn}(pt)], \tag{7.89b}$$

in which $p = \sqrt{g/\ell}$ and $k = \sin(\alpha/2)$.

The periodicity of sn$u$ in the first of (7.88i) shows that the physical period τ^* of the pendulum motion is given by $p\tau^* = 4K$, which agrees with (7.87f). And at the quarter period $t = \tau^*/4 = K/p$, we confirm that (7.89b) yields the amplitude $\theta(\tau^*/4) = \alpha$ in (7.87b). This concludes the introduction to elliptic functions and their application to the simple pendulum problem.

7.10.5. The Small Amplitude Motion

We now know precisely the manner in which the periodic time of a simple pendulum increases with the amplitude. The elementary small amplitude solution, on the other hand, predicts a smaller constant period $\tau = 2\pi\sqrt{\ell/g}$. With the exact solution in hand, we can now assess the accuracy of the simple harmonic approximation. For small amplitudes α and hence small modulus k, we may approximate the elliptic integral (7.87c) and its modulus in (7.87b) by the first few terms of their power series expansions. We begin with the elliptic integral.

Since $k^2 \sin^2 \phi < 1$, the binomial expansion of the elliptic integral (7.87c) yields

$$pt = F(\phi; k) = \int_0^{\phi} [1 + \frac{1}{2}k^2 \sin^2 \phi + \frac{1 \cdot 3}{2 \cdot 4}k^4 \sin^4 \phi + \cdots] d\phi,$$

and term by term integration provides the series solution

$$pt = \phi + \frac{k^2}{2}\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right) + O(k^4), \tag{7.90a}$$

accurate to terms of the order k^4. With $t = \tau^*/4$ at $\phi = \pi/2$, (7.90a) gives the period as an approximate function of the modulus k—*the period increases monotonically with* k^2:

$$\tau^* = \tau\left(1 + \frac{k^2}{4}\right) + O(k^4). \tag{7.90b}$$

The series expansion for $k(\alpha)$ in (7.87b) yields

$$k = \frac{\alpha}{2} - \frac{1}{6}\left(\frac{\alpha}{2}\right)^3 + O\left(\alpha^5\right), \tag{7.90c}$$

and hence (7.90b) delivers the estimate of the period as a function of the amplitude:

$$\tau^* = \tau\left(1 + \frac{\alpha^2}{16}\right) + O(\alpha^4), \tag{7.90d}$$

accurate to terms of the order α^4. *Consequently, for small amplitudes, the period increases with the square of the amplitude.*

When terms of order greater than α and hence terms of order larger than k may be considered negligible, (7.90a), (7.90d), and the first equation in (7.87b) yield the isochronal simple harmonic solution as the lowest order approximation:

$$\theta = \alpha \sin pt, \qquad \tau^* = \tau = \frac{2\pi}{p}. \tag{7.90e}$$

Finally, we now assess the error that results from use of the small amplitude period as compared with the second order approximation in (7.90d), rewritten as

$$\frac{\tau^* - \tau}{\tau} = \left(\frac{\alpha}{4}\right)^2. \tag{7.90f}$$

Suppose the estimated error is not to exceed 1%. Then with $\tau^*/\tau \leq 1.01$, by (7.90f), the amplitude ought not to exceed $\alpha = 0.4$ rad, or 23° very nearly. Therefore, the small amplitude approximation of the oscillatory pendulum motion as a simple harmonic motion with constant period $\tau = 2\pi/p$ is very good for amplitudes smaller than 23°. Indeed, the solution computed for the complete elliptic integral with $\sin^{-1} k = \alpha/2 = 11.5^\circ$ in the exact equation (7.87f) is $p\tau^*/4 = 1.5868$, that is, $\tau^*/\tau = 1.0101$. Therefore, the error involved in the second order approximation (7.90d) in comparison with the exact solution computed for an amplitude of 23° is insignificant. This analysis emphasizes that the "smallness role" of some quantities considered in approximations is not always so infinitesimally small as is sometimes imagined.

7.10.6. Nonoscillatory Motion of the Pendulum

If the initial angular speed $\dot{\theta}(0) \equiv \omega_0$ at $\theta(0) = 0$ was sufficiently great, the bob eventually could reach its highest point at $\alpha = \pi$ or swing past it. The energy equation (7.86a) shows that the angular speed $\omega_0 = \omega_z$ required for the bob to just reach its zenith is given by

$$\omega_z = 2p. \tag{7.91a}$$

Now consider starting the system at its lowest point $\theta(0) = 0$ with angular speed $\dot{\theta}(0) = \omega_0$. The corresponding amplitude function determined by (7.86b) and (7.91a) is $\cos\alpha = 1 - 2\omega_0^2/\omega_z^2$, and hence (7.86b) may be written as

$$\dot{\theta}^2 = \omega_0^2[1 - \kappa^2 \sin^2(\theta/2)], \qquad \kappa \equiv \frac{\omega_z}{\omega_0}. \tag{7.91b}$$

Integration of this equation provides the travel time as a function of $\theta \in [0, \pi]$:

$$t = \frac{1}{\omega_0}\int_0^\theta \frac{d\theta}{\sqrt{1 - \kappa^2 \sin^2(\theta/2)}}. \tag{7.91c}$$

The physical nature of the motion depends on whether $\kappa < 1, = 1$, or > 1.

First, consider the case $\kappa = 1$. Equation (7.91c) shows that $t \to \infty$ as $\theta \to \pi$. *Therefore, when $\omega_0 = \omega_z$, the pendulum approaches its zenith without reaching it in finite time, and hence the period is infinite.*

Now suppose that $\kappa > 1$ so that the initial angular speed $\omega_0 < \omega_z$. In this case, the bob cannot reach the zenith, so the motion is oscillatory with amplitude

$$\alpha = \cos^{-1}\left[1 - 2\left(\frac{\omega_0}{\omega_z}\right)^2\right]. \tag{7.91d}$$

With the aid of the second relation in (7.87b), we find $k = \kappa^{-1} = \omega_0/\omega_z < 1$ and $p = \omega_z/2$. The motion with amplitude (7.91d) and the corresponding period are respectively determined by (7.89b) and (7.87f).

Exercise 7.14. Show that for $\kappa > 1$ equation (7.91c) may be cast in the form (7.87c) for the oscillatory motion. □

Finally, suppose that $\kappa < 1$. Then $\omega_0 > \omega_z$ and now the bob turns past the zenith, because the initial kinetic energy $\frac{1}{2}ml^2\omega_0^2$ exceeds the gravitational potential energy $2mgl$ attained as the bob rises to its highest point. In this case, from (7.91b), $\dot{\theta}$ is positive for all θ, however large, and hence the angular speed $\omega \equiv |\dot{\theta}|$ varies from its greatest value ω_0 at $\theta = 0$ to its least value $\omega_0(1-\kappa^2)^{1/2}$ at $\theta = \pi$:

$$\omega_0\sqrt{1-\kappa^2} \le \omega \le \omega_0. \tag{7.91e}$$

We set $\phi = \theta/2$ in (7.91c) to obtain the travel time in the revolving pendulum motion:

$$t = \frac{2}{\omega_0}\int_0^{\frac{\theta}{2}} \frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}} = \frac{2}{\omega_0}F\left(\frac{\theta}{2};\kappa\right), \tag{7.91f}$$

in which $F(\theta/2;\kappa)$ is the elliptic integral of the first kind with modulus κ. Hence, the bob reaches its zenith at $\theta = \pi$ in the finite time t_z given by

$$t_z = \frac{2}{\omega_0}K(\kappa), \tag{7.91g}$$

in which $K(\kappa)$ is the complete elliptic integral of the first kind.

Thus, if $\omega_0 > \omega_z$, the pendulum, in the absence of any frictional and aerodynamic effects, spins forever in the same direction about its support. The angular speed varies periodically between the extremes (7.91e); the motion is periodic, but not oscillatory. Indeed, with the aid of (7.91g), *the periodic time* $\tau_o = 2t_z$, *the time for the whirling pendulum to complete one orbit about its support, is given by* $\tau_o = 4K(\kappa)/\omega_0$.

Exercise 7.15. Show that the orbital motion $\theta(t)$ of the revolving pendulum is described by $\theta(t) = 2\sin^{-1}(\mathrm{sn}\,\frac{1}{2}\omega_0 t)$. □

7.11. The Isochronous Pendulum

The finite amplitude motion of a simple pendulum is not isochronous; its period varies with the amplitude. Here we explore the existence of a pendulum whose finite amplitude oscillation is isochronal. The gravity induced, oscillatory motion of a particle on a smooth symmetric curve, concave upward in the vertical plane is studied. The equation for the finite amplitude motion is obtained from the energy equation, and the frequency for small amplitude, simple harmonic oscillations on an arbitrary concave curve is derived. To study the finite amplitude motion, however, the curve geometry must be specified. The finite amplitude motion on a cycloidal curve is investigated, and it is shown that the cycloidal oscillator is exactly simple harmonic, hence isochronous. Moreover, the cycloid is the only plane curve having this property.

7.11.1. Equation of Motion on an Arbitrary Concave Path

Consider a particle P of mass m free to slide on a smooth and concave upward, but otherwise arbitrary curve $\mathscr{C}$ in the vertical plane. The free body diagram of P is shown in Fig. 7.16a. The normal, surface reaction force $\mathbf{N}$ is workless, and the gravitational force has potential energy $V(y) = mgy$. The system is conservative with kinetic energy $K(P,t) = \frac{1}{2}m\dot{s}^2$, where $s(t)$ is the arc length along $\mathscr{C}$ measured from point O at $y = 0$, say. The energy principle (7.73) requires

$$\tfrac{1}{2}m\dot{s}^2 + mgy = E. \tag{7.92a}$$

Differentiation of (7.92a) with respect to s yields $\ddot{s} + g\,dy/ds = 0$. Noting in Fig. 7.16a that

$$\sin\gamma(s) = \frac{dy}{ds}, \qquad \text{where} \quad \gamma = \tan^{-1}\frac{dy}{dx}, \tag{7.92b}$$

we obtain the equation of motion of P on $\mathscr{C}$:

$$\ddot{s} + g\sin\gamma(s) = 0. \tag{7.92c}$$

This is the tangential component of the intrinsic equation of motion of P.

Let $v(s) \equiv \dot{s}(t)$ and $v_0 \equiv v(s_0)$, $s_0 \equiv s(0)$ at $y = y_0$ initially. Introducing these initial data in (7.92a), integrating the first relation in (7.92b), and noting that $ds/v(s) = dt$, we obtain the general solution in terms of $s(t)$:

$$v^2(s) = v_0^2 - 2g\int_{s_0}^{s}\sin\gamma(s)ds, \qquad t = \int_{s_0}^{s}\frac{ds}{v(s)}, \tag{7.92d}$$

To do more, we shall need to know the shape function $\gamma(s)$.

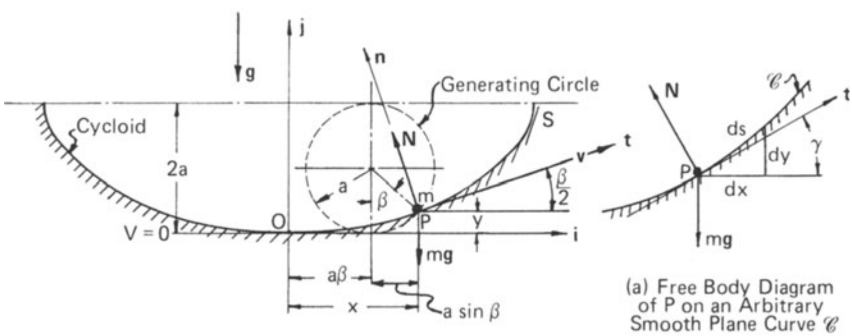


Figure 7.16. Motion on a smooth cycloid.

7.11.2. Small Oscillations on a Shallow, Concave Curve

Consider a shallow symmetric curve with a horizontal tangent at O where $s_0 = 0$, so that $\gamma(s)$ is a small inclination. Then $\sin\gamma = \gamma$, approximately. By (7.92c), the point O is the static equilibrium state at which $\gamma(s_0) = 0$. For a small amplitude oscillation of P about O, the power series expansion of the shape function $\gamma(s)$ about $s = s_0 = 0$ gives

$$\gamma(s) = \frac{d\gamma(0)}{ds}s + O(s^2). \tag{7.93a}$$

The path has the curvature $\kappa(s) = d\gamma(s)/ds$. Hence, to terms of the first order in s, (7.93a) yields a general, though approximate relation for $\gamma(s)$:

$$\gamma(s) = \kappa_0 s = \frac{1}{R_0}s, \tag{7.93b}$$

where $R_0 \equiv 1/\kappa_0 \equiv 1/\kappa(0)$ is the radius of curvature of $\mathscr{C}$ at the origin O.

Consequently, for small amplitude oscillations of a particle on a smooth, shallow and symmetric concave curve in the vertical plane, the equation of motion (7.92c) *reduces to the equation for the simple harmonic oscillator*:

$$\ddot{s} + p^2 s = 0, \qquad p = \sqrt{\frac{g}{R_0}} = \sqrt{g\kappa_0}. \tag{7.93c}$$

The small amplitude frequency $f = p/2\pi$ and period $\tau = 1/f$ are determined by the radius of curvature of the path at the equilibrium point O. For a circular arc of radius $R_0 = \ell$, (7.93c) describes the small amplitude oscillations of a simple pendulum of length ℓ, for example.

7.11.3. Finite Amplitude Oscillations on a Cycloid

Now consider the finite amplitude oscillations of a particle on a smooth cycloid generated by a point P, starting at O, on a circle of radius a, as shown in Fig. 7.16. As the circle rolls toward the right, without slipping on the horizontal line at $y = 2a$, the radial line turns counterclockwise through the angle $\beta \in [0, \pi]$ measured from its initial vertical direction at O. Hence, the parametric equations of the cycloid are described by the Cartesian coordinates of P, namely,

$$x = a(\beta + \sin\beta), \qquad y = a(1 - \cos\beta). \tag{7.94a}$$

Clearly, $y \in [0, 2a]$, and for symmetric oscillations, $\beta \in [-\pi, \pi]$ and $x \in [-\pi a, \pi a]$. (See Example 2.5, page 109, in Volume 1.)

The tangent angle $\gamma(s)$ in (7.92b) and the curvature $\kappa(s)$ of the cycloid are readily determined from (7.94a). With the aid of the double angle trigonometric identities, we first obtain

$$dx = 4a\cos^2\frac{\beta}{2}d\frac{\beta}{2}, \qquad dy = 4a\sin\frac{\beta}{2}\cos\frac{\beta}{2}d\frac{\beta}{2}.$$

These yield

$$\frac{dy}{dx} = \tan\frac{\beta}{2}, \qquad R = \frac{1}{\kappa} = \frac{ds}{d(\beta/2)} = 4a\cos\frac{\beta}{2}. \tag{7.94b}$$

Therefore, from the second relation in (7.92b), the tangent angle $\gamma(s)$ in Fig. 7.16 and the radius of curvature $R(\gamma)$ of the cycloid are given by

$$\gamma = \frac{\beta}{2}, \qquad R(\gamma) = \frac{ds}{d\gamma} = 4a\cos\gamma. \tag{7.94c}$$

Hence, $\gamma \in [-\pi/2, \pi/2]$ and $R(\gamma)$ decreases from $R(0) = 4a$ to $R(\pm\pi/2) = 0$. The greatest amplitude is restricted by the curve geometry shown in Fig. 7.16 for $\gamma \in [-\pi/2, \pi/2]$.

Integration of the last equation in (7.94c) determines the function $\gamma(s)$:

$$s = \int_0^{\gamma} 4a\cos\gamma\, d\gamma = 4a\sin\gamma(s). \tag{7.94d}$$

Use of this relation in (7.92c) yields *the exact equation of motion of a particle free to slide on a smooth cycloid in the vertical plane*:

$$\ddot{s} + p^2 s = 0, \qquad p = \sqrt{g/4a}. \tag{7.94e}$$

We thus find a most interesting result: *The finite amplitude, cycloidal motion is exactly simple harmonic and hence isochronous*. The period of the cycloidal pendulum for all amplitudes is a constant given by

$$\tau = 4\pi\sqrt{\frac{a}{g}}. \tag{7.94f}$$

The result (7.94f) is truly astonishing: *If a particle of arbitrary mass slides from a position of rest at any point whatsoever on a smooth cycloid, it reaches the bottom always in the same time $t^* = \tau/4 = \pi\sqrt{a/g}$.*

We notice from (7.94c) that $R_0 \equiv R(0) = 4a$ at the equilibrium position $\gamma = 0$. Hence, the small amplitude formulas (7.93c) are the same, of course, as the exact relations (7.94e) for arbitrary amplitudes.

Exercise 7.16. The analysis reveals some additional geometrical properties of the cycloid. Consider the cycloidal curve from O to its orthogonal intersection with the line $y = 2a$ at S in Fig. 7.16 and derive the following properties. (a) The length σ of the cycloid from O to S is equal to its radius of curvature at O: $\sigma = 4a = R_0$. (b) The slope of the cycloid at a point P situated at a distance s from O is equal to the product of the curvature $\kappa(s) = 1/R(s)$ and the arc length

s at P: $\tan\gamma = \kappa s = s/R$. (c) At a point P on a cycloid, the sum of squares of its radius of curvature and its arc length from O is a constant equal to the square of the radius of curvature $R_0 = 4a$ at its lowest point: $R^2 + s^2 = 16a^2$, and hence s and $R(s)$ at every point on a cycloid describe the same circle of radius R_0 in the Rs-plane. □

It is also known that the cycloid is the unique curve of quickest descent between two points in the vertical plane. This is the classical brachistochrone problem of the calculus of variations, a topic beyond the scope of our current studies. See Problem 7.68 for an example.

7.11.4. Uniqueness of the Isochronal, Cycloidal Pendulum

Glancing back to (7.92b) and noting the proportionality in (7.94d), we ask: Are there any curves besides the cycloid for which

$$\sin\gamma = \frac{dy}{ds} = cs, \qquad \cos\gamma = \frac{dx}{ds} = \sqrt{1 - c^2 s^2}, \qquad \gamma = \tan^{-1}\frac{dy}{dx}, \tag{7.95a}$$

where c is a constant? If so, (7.92c) becomes $\ddot{s} + cgs = 0$, and hence the motion on any such smooth curve is simple harmonic, hence isochronous. To address the question, we need to find the parametric equations of all plane curves characterized by (7.95a).

We fix the Cartesian origin at the equilibrium point defined by $\gamma = 0$, and integrate the first two equations in (7.95a) to obtain

$$y = \frac{cs^2}{2}, \qquad x = \frac{1}{2c}[cs\sqrt{1 - c^2 s^2} + \sin^{-1}(cs)].$$

Then we use (7.95a) to write these expressions in terms of the tangent angle γ, and afterwards introduce the double angle trigonometric identities to obtain

$$x = \frac{1}{4c}[2\gamma + \sin 2\gamma], \qquad y = \frac{1}{4c}(1 - \cos 2\gamma). \tag{7.95b}$$

The parametric equations (7.95b) describe a cycloid whose generating circle is described by

$$\left(x - \frac{2\gamma}{4c}\right)^2 + \left(y - \frac{1}{4c}\right)^2 = \left(\frac{1}{4c}\right)^2. \tag{7.95c}$$

The radius is $a \equiv 1/4c$. As the circle turns counterclockwise, rolling on the line $y = 2a$, its center thus moves horizontally toward the right a distance $2\gamma/4c = 2\gamma a$, and hence the circle turns through an angle $\beta \equiv 2\gamma$. Therefore, we find exactly our original parametric equations (7.94a). The relation (7.94d) between the arc length s and the tangent angle $\gamma(s)$ is uniquely characteristic of the cycloid.

In summary, *the unique plane curve on which the motion of a particle is simple harmonic for all amplitudes is the cycloid. Hence, the cycloidal pendulum is the only exact isochronal pendulum.*

The reader may consider the following similar problem.

Exercise 7.17. Apply (7.92b) to prove that the unique plane curve whose tangent angle is proportional to its arc length is a circle. Show in this case that (7.92c), valid for the finite amplitude oscillations of a particle on any smooth, concave curve in the vertical plane, yields the equation for the finite motion of a simple pendulum. □

7.11.5. Huygens's Isochronous Clock

The isochronous, cycloidal pendulum was invented in 1673 by the Dutch scientist and ingenious clockmaker, Christian Huygens (1629–1695). The idea was used in construction of a pendulum clock to assure that its period would not change with variations in the amplitude of its swing. Huygens was able to produce a cycloidal motion of the bob by applying the property that the evolute of a cycloid is another cycloid of the same kind as the generating curve. The evolute of the cycloid is the path traced by the center of curvature of the generating cycloid. In Fig. 7.17, the evolute of the cycloid arc OS is the similar cycloid arc QS, both are generated by a circle of radius a. As P moves from S toward O, the center of curvature T of the arc OS traces the arc from S to Q. In other words, if a string of length $4a$ is tied to a fixed point Q that forms the cusp of an inverted cycloidal curve in Fig. 7.17, and the string is pulled over the contour arc QS to point S where the bob is released from rest, the bob will describe the same cycloidal path OS as our sliding particle in Fig. 7.16. On the basis of these unique

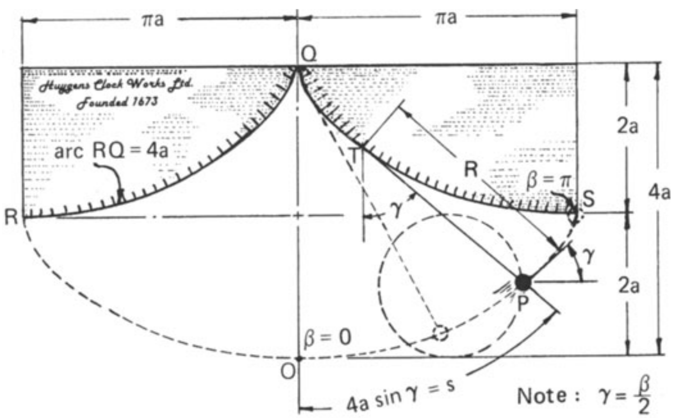


Figure 7.17. Huygens's isochronal pendulum.

properties of the cycloid, Huygens's isochronal pendulum may be constructed with shortened cycloidal surfaces at the cusp support Q so that the bob P moves on a shorter cycloidal path of some practical design dimensions. Subsequent inventors introduced certain drive control devices to adjust for energy losses due to frictional effects that would otherwise lead to variations in the amplitude.

7.12. Orbital Motion and Kepler's Laws

Consider a body P of mass m moving relative to an inertial frame $\varphi = \{O; \mathbf{e}_k\}$ under a central directed gravitational force (7.61) due to a body S of mass M with its center of mass fixed at O. For example, S might be the Sun and P a planet, or S the Earth and P the Moon or a satellite. It is natural to model the two bodies as a system of two center of mass particles, interactions with all other bodies being ignored. Then P moves in a plane such that, by (7.72b), $r^2\dot{\phi} = \gamma$, a constant. This has the geometrical interpretation that the radius vector of P sweeps out the same area in equal time intervals, the second of three laws deduced empirically by the German astronomer and mathematician Johannes Kepler (1571–1630) based on precise astronomical observations of the positions of stars and planets by the Danish astronomer Tycho Brahe (1546–1601), Kepler's mentor. More than half a century later, Kepler's laws were deduced by Newton (1642–1727) from his mathematical theory of planetary motion. Here we determine the motion of an orbital body, characterize its path, and derive Kepler's first and third laws of planetary motion.

7.12.1. Equation of the Path

We introduce cylindrical coordinates identified in Fig. 7.11, page 252, with origin at O in the inertial frame $\varphi = \{O; \mathbf{e}_r, \mathbf{e}_\phi\}$ and with respect to which the polar coordinate equation of the path of a particle P is described by $r = r(\phi)$. The only force acting on P is the conservative gravitational force with potential energy given in (7.62). The constant $V_0 = 0$ may be chosen so that $V \to 0$ when $r \to \infty$, and hence $V = -\mu m/r$, where the constant $\mu \equiv GM$. The kinetic energy of P is given by $K = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)$. The energy principle together with (7.72b) yields

$$\dot{r}^2 + \frac{\gamma^2}{r^2} - \frac{2\mu}{r} = \frac{2E}{m}, \tag{7.96a}$$

in which γ is the constant moment of momentum per unit mass of P and E is the constant total energy.

To find $r(\phi)$, it is convenient to introduce a change of variable

$$u(\phi) = \frac{1}{r(\phi)}. \tag{7.96b}$$

Then, by (7.72b), $\dot{\phi} = \gamma u^2(\phi)$, and from (7.96b), $\dot{r} = -\gamma du/d\phi$. Now (7.96a) may be recast as

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2\mu}{\gamma^2}u = \frac{2E}{m\gamma^2}. \tag{7.96c}$$

Although this form of the energy equation may be readily integrated for $u(\phi)$, it is easier to first differentiate (7.96c) with respect to ϕ to obtain the equation of motion:

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{\gamma^2}, \tag{7.96d}$$

whose easy general solution is

$$u(\phi) = \frac{1}{r(\phi)} = \frac{\mu}{\gamma^2} + C\cos(\phi - \phi_0), \tag{7.96e}$$

in which C and ϕ_0 are integration constants. The base line for ϕ may be chosen so that $\phi_0 = 0$, and C may be expressed in terms of γ and E by substitution of (7.96e) into (7.96c). We find

$$C^2 = \frac{\mu^2}{\gamma^4}\left(1 + \frac{2E\gamma^2}{m\mu^2}\right). \tag{7.96f}$$

Hence, (7.96e) yields the path equation

$$r(\phi) = \frac{d}{1 + e\cos\phi}, \tag{7.96g}$$

in which, by definition,

$$d \equiv \frac{\gamma^2}{\mu}, \qquad e \equiv \sqrt{1 + \frac{2E\gamma^2}{m\mu^2}}. \tag{7.96h}$$

Thus, the plane motion of P for all time is given by

$$\mathbf{x}(P, t) = r(\phi)\mathbf{e}_r(\phi) = \frac{d}{1 + e\cos\phi}\mathbf{e}_r(\phi). \tag{7.96i}$$

7.12.2. Geometry of the Orbit and Kepler's First Law

The total energy E in the second relation of (7.96h) may be positive, negative, or zero. Thus, in particular, when $E = -m\mu/2d < 0$, $e = 0$ and, by (7.96g), the orbit is a circle of radius $r = d$. Otherwise, (7.96g) describes the polar equation of a conic section in Fig. 7.18—defined as the locus of a point P that moves in a plane in such a way that the ratio of its distance $|\overline{OP}|$ from a fixed point O in the plane to its distance $|\overline{DP}|$ from a fixed line is constant. The fixed point is called the *focus*. The fixed line is known as the *directrix*, and the constant ratio

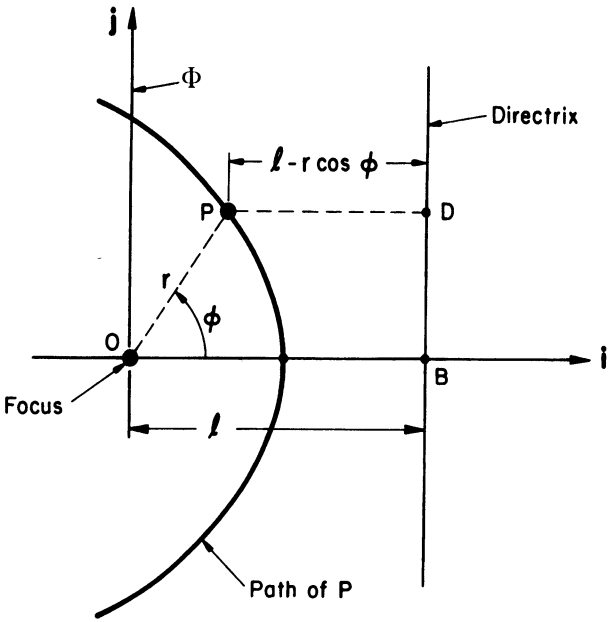


Figure 7.18. Geometry of a conic section.

of the two distances is called the *eccentricity*. With the focus at the origin O of frame $\Phi = \{O; \mathbf{i}, \mathbf{j}\}$ in Fig. 7.18, the directrix is a straight line BD parallel to $\mathbf{j}$ at a distance ℓ from O along $\mathbf{i}$. The eccentricity is defined by

$$e = \frac{|\overline{OP}|}{|\overline{DP}|} = \frac{r}{\ell - r\cos\phi} > 0. \tag{7.97a}$$

Solving this relation for $r(\phi)$, we obtain the general equation (7.96g) in which

$$d \equiv \ell e. \tag{7.97b}$$

Equation (7.96g) shows that $r(-\phi) = r(\phi)$, so the path is symmetric about the line $\phi = 0$, the $\mathbf{i}$-axis. The chord along the $\mathbf{j}$-axis (parallel to the directrix) through the focus O is called the *latus rectum*. When $\phi = \pi/2$, (7.96g) shows that $r(\pi/2) = d$, and hence $2d$ is the length of the latus rectum.

If $e > 1$, the conic described by (7.96g) is a hyperbola; if $e = 1$, the conic is a parabola; and if $e < 1$, the conic is an ellipse. The circle is a degenerate ellipse for which $e = 0$. It is amazing that the type of conic trajectory is uniquely characterized in terms of the total energy E in accordance with (7.96h), namely,

ellipse	$e < 1$ if $E < 0$
circle	$e = 0$ if $E = -m\mu/(2d) < 0$
parabola	$e = 1$ if $E = 0$
hyperbola	$e > 1$ if $E > 0$.

(7.97c)

The circle is a degenerate ellipse for which $E < 0$, and whose eccentricity vanishes when the total energy, $E = -m\mu/(2d) = \frac{1}{2}V(\pi/2)$, is one-half the potential energy at the semi-latus rectum.

A planet or satellite having a parabolic or hyperbolic path ultimately would leave the solar system forever. Astronomical observations, however, dictate that the planets have closed orbits around the Sun, and Newton's theory proves that these orbits are elliptical with the Sun situated at one focus O. This is **Kepler's first law**: *The planets travel on elliptical paths with the sun at one focus.*

7.12.3. Kepler's Third Law

We now turn to Kepler's third law on the orbital period. The orbital geometry is described in Fig. 7.19. To determine the *periodic time* τ in which P describes its elliptical orbit, we recall Kepler's second law relating the area swept out to the time, namely,

$$A = \tfrac{1}{2}\gamma t. \tag{7.98a}$$

The area $A = \pi ab$ enclosed by the elliptical path is thus covered in the time

$$\tau = \frac{2\pi ab}{\gamma}, \tag{7.98b}$$

where a and b are the respective semi-major and semi-minor axes. We prefer, however, to express this result in terms of the geometrical and gravitational constants.

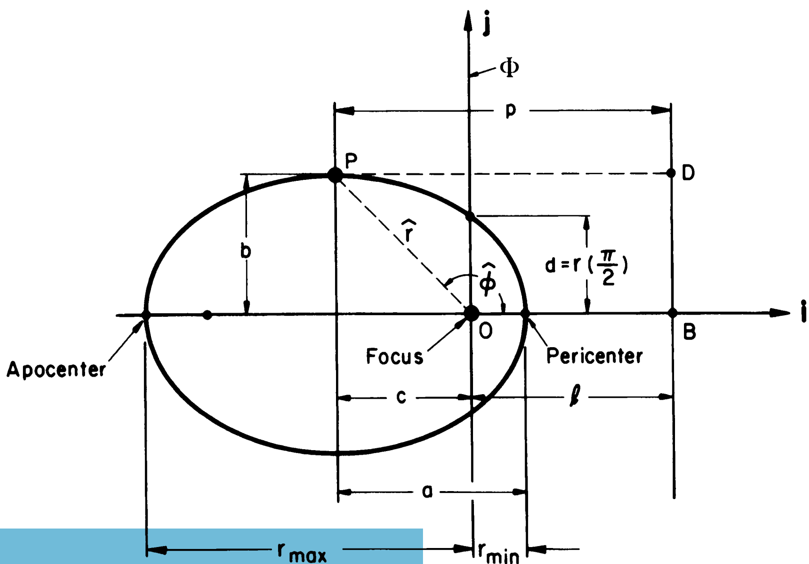


Figure 7.19. Geometry of an elliptical orbit.

The constant γ is related to the gravitational constant through (7.96h): $\gamma = \sqrt{\mu d}$, and hence the next step is to relate d, a, and b.

The ratio d/a may be found by use of (7.96g). The focus O is on the major axis, and the nearest location of P to O, called the *pericenter*, is at $\phi = 0$ in Fig. 7.19. Thus, by (7.96g), $r(0) = r_{\text{min}} = d/(1+e)$. Similarly, at $\phi = \pi$, the greatest distance of P from O, named[§] the *apocenter*, is $r(\pi) = r_{\text{max}} = d/(1-e)$. The length of the major axis, therefore, is $2a = r_{\text{min}} + r_{\text{max}}$, and hence

$$\frac{d}{a} = 1 - e^2. \tag{7.98c}$$

We next seek a relation for b/a. Equation (7.97a) applied to the point on the minor axis in Fig. 7.19 gives $e = \hat{r}/(c + \ell)$, wherein $c \equiv a - r_{\text{min}} = a - d/(1+e)$. Hence, by (7.98c), $c = ae$; and, with the aid of (7.97b), we have $\hat{r} = e(c + \ell) = a$. Now observe from Fig. 7.19 that $\hat{r}^2 = b^2 + c^2$, introduce $\hat{r}$ and c, and thus derive the ratio

$$\frac{b}{a} = \sqrt{1 - e^2}. \tag{7.98d}$$

Finally, use of (7.98c) in (7.98d) gives $b = \sqrt{ad}$. We now return to (7.98b), recall the first equation in (7.96h) to obtain $b/\gamma = \sqrt{a/\mu}$, and thus derive the periodic time for the elliptical orbit:

$$\tau = 2\pi\sqrt{\frac{a^3}{\mu}}. \tag{7.98e}$$

It is remarkable that the orbital period involves only one geometrical constant a and the physical constant $\mu = GM$. Therefore, the ratio τ^2/a^3 is the same for all planets in motion about the Sun. This is **Kepler's third law:** *The square of the periodic time of a planet is proportional to the cube of the semi-major axis of its orbit.*

Exercise 7.18. (a) It is remarkable also that the third law may be cast in terms of only one dynamical constant, the total energy E. Show that

$$\tau = \frac{2\pi\mu}{\sqrt{(-2E/m)^3}}, \tag{7.98f}$$

in which $E < 0$ for an elliptical orbit. (b) Show that the squared speed of P is given by

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right), \tag{7.98g}$$

which is independent of its mass. Hint: Use (7.96h) and (7.96a). □

[§] These general terms are used when no specific focal body is identified. However, when the body at O is the Earth and the orbital body P is the Moon or a satellite, the point nearest the Earth in the orbit of P is called the *perigee*, and the point farthest from the Earth is termed the *apogee*. The point nearest the Sun in the orbit of a planet or another body is known as the *perihelion*, and the most remote point in its orbit is called the *aphelion*.

Exercise 7.19. *Distance on an elliptical orbit.* Let $\psi = \{C; \mathbf{i}_k\}$ be a Cartesian reference frame at the center of an ellipse whose parametric equations are given by $x = a \sin\phi,\ \ y = b\cos\phi$, where ϕ denotes the central, clockwise angle from the y-axis and a and $b < a$ are the corresponding semi-axes of the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1. \tag{7.99a}$$

(a) Show that the area A enclosed by this ellipse is $A = \pi ab$.

(b) The name elliptic integral derives from the following problem of determining the length of an elliptic arc. Starting at $(x, y) = (0, b)$, show that the distance s traveled on an elliptical orbit is given by

$$s(\phi) = aE(k; \phi), \tag{7.99b}$$

in which $E(k; \phi)$, not to be confused with the constant total energy, is standard notation for the *elliptic integral of the second kind,* defined by

$$E(k; \phi) \equiv \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \vartheta}\, d\vartheta, \tag{7.99c}$$

in which $k^2 \equiv (a^2 - b^2)/a^2 = e^2$, so that $0 < k < 1$. By (7.99b), the circumference Γ of the elliptical orbit is thus determined by

$$\Gamma \equiv 4s(\pi/2) = 4aE(k; \pi/2), \tag{7.99d}$$

wherein $E(k; \pi/2)$ is the *complete elliptic integral of the second kind.* In particular, for a circle of radius a, $k = e = 0$; hence, (7.99b) yields the circular arc length $s(\phi) = aE(0; \phi) = a\phi$ and (7.99d) gives the circumference $\Gamma = 4as(\pi/2) = 2\pi a$. In general, the value of E for a given modulus $k \in (0, 1)$ and a specified angle $\phi \in [0, \pi/2]$ may be found from tables of elliptic integrals or by computation.

(c) Consider an elliptical orbit for which $a = 2b$. Find in terms of a the distance traveled when $\phi = \pi/4, \pi/2$, and 2π. □

The foregoing theory requires that the focal body S be fixed while the moving body P is attracted only by S. Of course, Newton's law of gravitation holds for every pair of bodies in the world, and disturbances induced by the mutual attractions with other bodies have been ignored. An accurate dynamical treatment of the solar system, the major problem of celestial mechanics, entails far greater complexities than those embodied in the simple model studied here. If the ratio of m/M of the masses of P and S is small, and their mutual distance and their separation from all other bodies is great, the elementary model gives a very close estimate of the facts. On the other hand, it is natural to question what may be said about the motion of a system of two or more bodies free to move under their mutual Newtonian attraction. The two body interaction problem and the effect of their relative motion on Kepler's law for the orbital period is studied in the next chapter.

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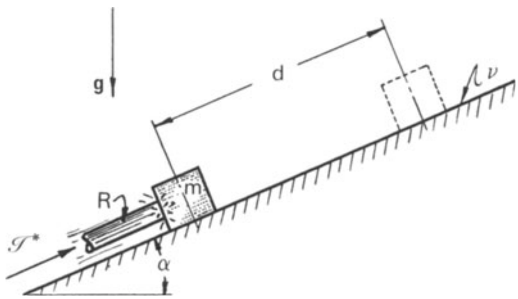
Problems

7.1. A small ball of mass m strikes a horizontal surface with speed v_1 at an angle θ from the surface, and it bounces off with speed v_2 at an angle ϕ from the surface. Show that the magnitude $|\mathscr{T}^*|$ and direction ψ of the impulse exerted on the ball by the wall are determined by

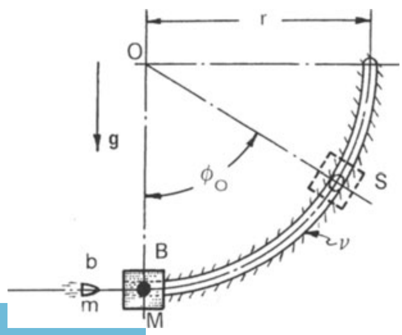
$$|\mathscr{T}^*| = m\left[v_1^2 + v_2^2 - 2v_1 v_2 \cos(\theta + \phi)\right]^{1/2}, \qquad \tan\psi = \frac{v_2 \sin\phi + v_1 \sin\theta}{v_2 \cos\phi - v_1 \cos\theta}.$$

7.2. A rigid rod R strikes a small block of mass m initially at rest on a rough plane inclined at an angle α shown in the figure. The block moves up the incline a distance d where it comes to rest. The dynamic coefficient of friction is ν. Find the initial impulse of the force exerted by the rod on the block.



Problem 7.2.

7.3. A ballistic pendulum consists of a heavy block B of mass M initially at rest but free to slide in a rough, circular guide slot of radius r. A bullet b of mass m is fired into the block which then swings from its initial vertical position shown in the figure through an angle ϕ_0 where the system $S = \{b, B\}$ comes to rest. The dynamic coefficient of friction is ν. (a) Derive the Newton–Euler differential equation giving the squared speed of S as a function of its angular placement, and thus find $v^2(\phi)$ exactly. (b) Determine the initial velocity $\mathbf{v}_0$ of S. (c) What relations will determine exactly the impulse of the force $\mathscr{T}^*$ exerted by the bullet on the block and the bullet's impact speed β? (d) Find $\mathscr{T}^*$ and β when ϕ_0 is a small angle.



Problem 7.3.

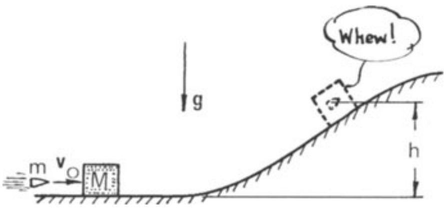
7.4. A force $\mathbf{F} = (3x^2 + 4y - 6z^3)\mathbf{i} + (x - 2y^2 - 3xz)\mathbf{j} + xy\mathbf{k}$ moves a particle P along the path $y = 4x - 2x^2$ from the origin (0, 0, 0) to the point (2, 0, 0). Find the work done by $\mathbf{F}$. Compare this with the work done by $\mathbf{F}$ in moving P along a straight line joining these points.

7.5. (a) What work is done by the force $\mathbf{F}(\mathbf{x}) = xy\mathbf{i} + y^2\mathbf{j}$ in moving a particle from the point (0, 0) to the point (1, 2) along (i) the parabola $y = ax^2$, (ii) the orthogonal paths $y = 0$ and $x = 1$, and (iii) the straight line $y = kx$? Is the force $\mathbf{F}(\mathbf{x})$ conservative? (b) The force $\mathbf{F}(\mathbf{x}) = y^3\mathbf{i} + x\mathbf{j}$ moves a particle on a plane path defined by the time-parametric equations $x = at^2,\ y = bt^3$. Find the work done during the period $t = 0$ to $t = 1$.

7.6. A hockey puck of mass m is driven over a frozen lake with an initial speed of 6 m/sec. Its speed 3 seconds later is 5 m/sec. Apply the work–energy principle to find the dynamic coefficient of friction, and determine the distance traveled by the puck during this time.

7.7. An electron E of mass m initially at rest at (0, 0) is acted upon by a plane propulsive force $\mathbf{P}$ of constant magnitude. (a) What work is done by $\mathbf{P}$ in moving E along an arbitrary simple path from (0, 0) to (2, 2)? (b) Find the work done when E moves between the same end points on (i) a circle centered at (0,2) and (ii) on a straight line. (c) If all other forces acting on E are workless, determine its speed at the point (2, 2) on each of the three paths. (d) What can be said about the motion, if $\mathbf{P}$ were the only force acting on the electron?

7.8. A bullet of mass m is fired with muzzle velocity $\mathbf{v}_0$ into a block of mass M initially at rest on a smooth horizontal surface, as shown in the figure. After the impact, the block and imbedded bullet move on a smooth curve in the vertical plane. The system ultimately comes to rest at the distance h above the plane. Find the muzzle speed.



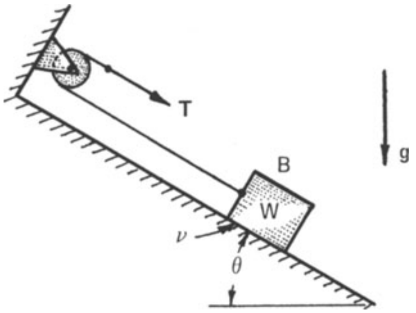
Problem 7.8.

7.9. A particle P of unit mass is acted upon by a force equal to twice its velocity. The initial velocity is $\mathbf{v}_0 = \mathbf{u}$ at the place $\mathbf{x}_0 = \frac{1}{2}\mathbf{u}$, where $\mathbf{u}$ is a unit vector. (a) Determine the change of the kinetic energy of P and the mechanical power expended during a time t. (b) Describe the trajectory of P.

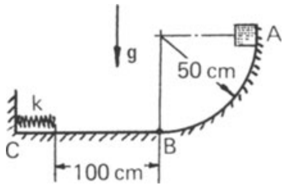
7.10. A cable under constant tension T passes over a smooth pulley and is attached to a block B of weight W initially at rest on a rough inclined plane shown in the figure. Suppose that $T = W$ and the dynamic coefficient of friction is ν. Neglect the mass of the cable and pulley. (a) What is the speed v_B of B after being dragged a distance d? (b) At the moment when B reaches d_0, the cable snaps. Find the additional distance $\hat{d}$ traveled by B. (c) Find equations for v_B and $\hat{d}$ for the motion of B on a horizontal surface.

7.11. A 40 N weight is released from rest at A on a smooth circular surface shown in the diagram. At B, it continues to move on a rough horizontal surface BC with dynamic coefficient of friction $\nu = 0.4$, and it subsequently strikes a spring with stiffness $k = 10$ N/cm at C. Find the deflection of the spring.

7.12. Apply the work–energy equation to determine the angular speed $\dot{\theta}(t)$ following impact of the ballistic pendulum described in Fig. 7.3, page 226. Then (i) show that the result is equivalent to one of the two scalar equations of motion for the load, and (ii) find the rope tension as a function of $\theta(t)$.



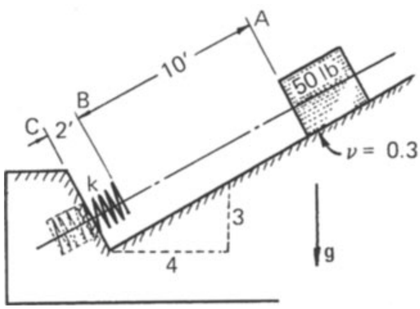
Problem 7.10.



Problem 7.11.

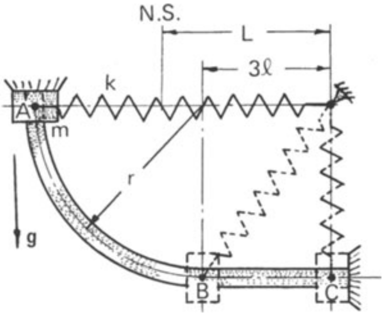
7.13. The motion of a particle P falling from rest with air resistance given by Stokes's law is described in Example 6.11, page 120. (a) Find the total mechanical power of the forces acting on P as a function of its speed v. Then show that the terminal speed v_∞ is the particle speed at which the power generated by gravity is balanced by the power dissipated by air resistance, and hence v_∞ is the speed at which the total power vanishes. (b) Determine as a function of time the total work done on P, and thus show that as $t \to \infty$, $\mathscr{W} \to \frac{1}{2}mv_\infty^2$, the kinetic energy of P at its terminal speed.

7.14. A 50 lb crate is released from rest at A on an inclined surface shown in the figure. It ultimately strikes and fully compresses a spring of modulus k before coming to rest at C. The dynamic coefficient of friction is 0.30. Find the modulus k.



Problem 7.14.

7.15. A linear spring of modulus k and unstretched length $L = 4\ell$ is attached to a slider block of mass m shown in the figure. The block experiences a negligible disturbance from rest at A and slides in the vertical plane on a smooth circular rod of radius $r = L$. Find in terms of L the speed of the block at B and C.



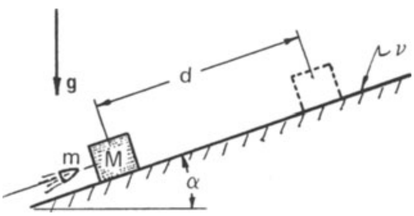
Problem 7.15.

7.16. It appears from (7.34) that the work–energy principle (7.36) can be used only when the particle mass is constant. (a) Show from (5.34) that if $m = m(v)$ is a function of the particle speed $v(t)$, the work done by a force $\mathbf{F}(\mathbf{x})$ acting over the particle path from time t_0 to time t is determined by

$$\mathscr{W} = m(v)v^2 - m(v_0)v_0^2 - \frac{1}{2}\int_{v_0}^{v} m(v)dv^2, \tag{P7.16}$$

wherein $v_0 = v(t_0)$. Let $m(v) = m$, a constant, and *derive* the work–energy principle (7.36). (b) Now let $m(v)$ be the relativistic mass (6.9), and suppose that the work–energy relation (7.36) is *postulated* as a fundamental principle of mechanics. Use (P7.16) to show that the change in the relativistic kinetic energy is $\Delta K = [m(v) - m(v_0)]c^2$. With $m_o \equiv m(0)$ and $K \equiv E$, it is seen that this reduces to (7.43e), obtained somewhat differently in the text. (c) Verify that when $v/c \ll 1$, $\Delta K = \frac{1}{2}m_o(v^2 - v_0^2)$.

7.17. A bullet of mass m is fired into a block of mass M initially at rest on a rough inclined plane, as shown. After the impact, the system moves up the plane a distance d where it comes to rest. Apply the energy principle to find the impulse of the force exerted by the bullet on the block, and determine its impact velocity.



Problem 7.17.

7.18. Show that the force $\mathbf{F}(\mathbf{x}) = 2xy\mathbf{i} + (x^2 + ay)\mathbf{j}$ is conservative, and determine the work done on an arbitrary path from the origin to the point (0, 2, 0). What is the work done by $\mathbf{F}$ along a straight line through the points (2, 2, 0) and (0, 2, 0) when $a = 1$?

7.19. (a) Find the potential energy function for the conservative force

$$\mathbf{F}(\mathbf{x}) = (2z^2 \cosh x - y^2)\mathbf{i} + 2y(z - x)\mathbf{j} + (4z \sinh x + y^2)\mathbf{k}.$$

What is the work done by $\mathbf{F}$ in moving a particle from the origin along the path $y = \sin(\pi x/2)$ to the point (2, 0, 0)? (b) Is the following force conservative?

$$\mathbf{F}(\mathbf{x}) = (2z^2 + 5\cos y)\mathbf{i} + (z - 5x \sin y)\mathbf{j} + (4zx - y)\mathbf{k}.$$

7.20. (a) Establish that the force $\mathbf{F}(\mathbf{x}) = -xy^2\mathbf{i} - yx^2\mathbf{j}$ is conservative, and find the potential energy function. (b) Write down the scalar equations for the plane motion of a particle of unit mass moving under this force alone, and find a first integral of this system of equations.

7.21. Show that the force $\mathbf{F}(\mathbf{x}) = (y\mathbf{i} - x\mathbf{j})/r^2$, with $r^2 = x^2 + y^2 \neq 0$, is conservative. Determine the potential energy function.

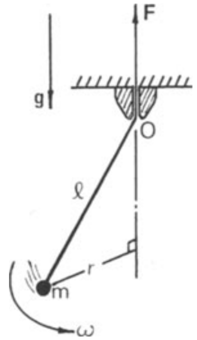
7.22. This problem illustrates the necessity for the conservative force to be defined over a simply connected region. The force $\mathbf{F}(\mathbf{x}) = \mathbf{k} \times \mathbf{x}/r^2$, where $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$ and $r = |\mathbf{x}|$, has the curious property that curl $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ while the work done by $\mathbf{F}$ in moving a particle around a circle $\mathscr{C}$ about the origin O in the plane region $\mathscr{R}$ does not vanish. (a) Does $\nabla \times \mathbf{F}(\mathbf{x}) = \mathbf{0}$ hold everywhere in $\mathscr{R}$? Is $\mathbf{F}(\mathbf{x})$ defined at all points of $\mathscr{R}$, specifically at $\mathbf{x} = \mathbf{0}$? (b) Show that on a circle $\mathscr{C}$ of radius R centered at O in the plane of the force, $\mathscr{W} = \oint_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = 2\pi$. Hence, by Stokes's theorem (7.57), $\int_{\mathscr{A}}$ curl $\mathbf{F}(\mathbf{x}) \cdot d\mathbf{A}$ over the area bounded by $\mathscr{C}$ equals 2π. This plainly implies that $\nabla \times \mathbf{F} \neq \mathbf{0}$ everywhere. (c) To construct a simply connected region that excludes point O, first draw a small circle c of radius ε around O. Next, remove tiny slices from c and $\mathscr{C}$ at their intersections with the x-axis, and join corresponding points on c and $\mathscr{C}$ by lines drawn parallel to the x-axis. This produces a single closed region $\mathscr{R}$ without O that looks like a split washer. Now, compute the work done by $\mathbf{F}$ in moving a particle around this new closed path and determine its value as $\varepsilon \to 0$.

7.23. Prove that the force

$$\mathbf{F}(\mathbf{x}) = (y + 5z \sin x)\mathbf{i} + (x + 4yz)\mathbf{j} + (2y^2 - 5 \cos x)\mathbf{k},$$

is conservative, and derive the potential energy function. What is the work done by $\mathbf{F}$ in moving a particle from the origin to the point (4, 3, 0) along (i) a circular path centered on the x-axis and joining the end points, and (ii) a straight line between the same end points?

7.24. A particle of mass m is suspended vertically by a light inextensible string of length ℓ and twirled with angular speed ω in a circular path of radius r, as indicated. The restraining force $\mathbf{F}$ is slowly increased so that the particle moves in a circle of radius $r/2$. Use the moment of momentum principle to find its new angular speed Ω.



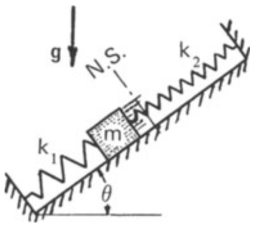
Problem 7.24.

7.25. Two projectiles of masses m_1 and m_2 are fired consecutively with the same initial speed v_0, but at different angles of elevation θ_1 and θ_2, respectively. The second shell is fired t_1 seconds after the first. The shells subsequently collide at the time t_2. Neglect air resistance, observe the principle of conservation of momentum, and find the angle θ_2 at which the second shell was fired. Is θ_2 smaller or larger than θ_1?

7.26. A particle of mass m, supported by a smooth horizontal surface, is fastened to a string of length 50 cm and twirled anticlockwise at 25 rad/sec about a fixed point O in the surface. If the string is pulled through a small hole at O with a constant speed of 150 cm/sec as the particle moves around the fixed surface, what is the absolute speed of the particle at 25 cm from O?

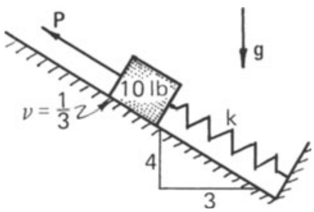
7.27. A particle of mass m, supported by a smooth horizontal surface, is attached to a string of length L and twirled about a fixed point O in the surface with constant counterclockwise angular velocity ω. The string strikes a nail inserted suddenly through the surface at a distance R from O. (a) Apply the energy principle to find the new angular velocity $\boldsymbol{\Omega}$ of m. (b) Apply the principle of conservation of moment of momentum to find $\boldsymbol{\Omega}$.

7.28. Two springs having moduli k_1 and k_2 are fastened to a mass m, as shown. The load is released from rest at the natural unstretched state of both springs to oscillate on the smooth, inclined plane. Determine by the energy method the maximum displacement of m, and find the period and the amplitude of the oscillations.



Problem 7.28.

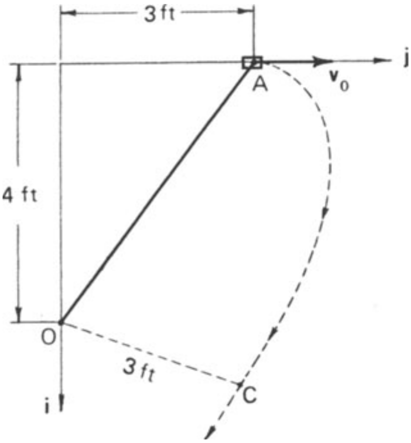
7.29. During an interval of interest, a constant force $\mathbf{P} = 20\mathbf{i}$ lb is applied to a 10 lb block attached to a spring of stiffness $k = 20$ lb/in. At the initial instant, the system is compressed an amount $\delta = 6$ in. from its natural state and released from rest on a rough surface inclined as shown in the figure. Assume that $g = 32$ ft/sec^2 and $\nu = \frac{1}{3}$. Find the speed of the block when it has moved 9 in.



Problem 7.29.

7.30. One end of a linearly elastic rubber string, having an unstretched length of 3 ft and a modulus $k = 9$ lb/ft, is fixed at O on the $\mathbf{i}$-axis. Its other end is fastened to a small block of mass 0.03 slug. The string is stretched to the point A shown in the figure, and the block is given a velocity $\mathbf{v}_0 = 20\mathbf{j}$ ft/sec at A. The block then slides on a smooth horizontal supporting surface. (a) Find the speed of the block at the instant the string becomes slack at the place C, and determine its subsequent closest approach to O. Describe the path of the block before the string loses its slack in the motion beyond C. (b) Determine the block's greatest distance from O on the path AC, and find its speed there.

7.31. The initial velocity $\mathbf{v}_0 = v_0\mathbf{e}_r$ that will enable a rocket to just escape from the Earth's gravitation is called the *escape velocity*. Consider a simple model of a rocket for space which has no propulsion system of its own after it has been projected vertically from the ground with an initial speed v_0. Account for the variation in the gravitational attraction with the distance from the center of the Earth, whose radius is 4000 miles, neglect air resistance and the Earth's rotation,



Problem 7.30

and assume that the speed vanishes at great distance from the Earth. Find the escape velocity in mph for this model.

7.32. A small block of mass m is attached to a linear spring of length l and stiffness k. The system is at rest in its natural state on a smooth horizontal plane when a suddenly applied horizontal force imparts to the block a velocity $\mathbf{v}_0 = v_0\mathbf{j}$ perpendicular to the spring axis $\mathbf{i}$. (a) Apply the moment of momentum and energy principles to determine the speed of the block as a function of the spring's extension δ. (b) Find the extensional rate $\dot{\delta}$ of the spring, and thus formulate an integral expression for the travel time t as a function of δ.

7.33. A slider block B of unknown mass is moving with a constant speed v_0 inside a smooth, horizontal straight tube when suddenly it strikes a linear spring of unknown stiffness. Its measured speed subsequently is reduced to v_1 when the observed spring compression is δ_1. Determine the deceleration of B at the maximum spring deflection δ_m. What is δ_m?

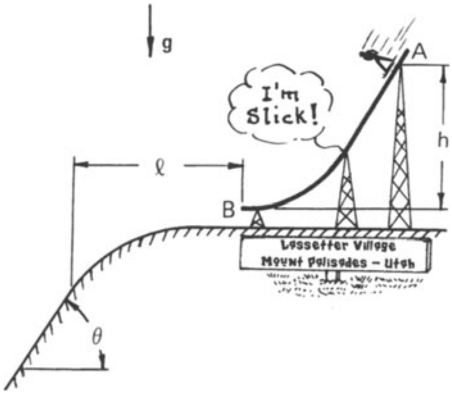
7.34. Solve Problem 6.57 by the energy method.

7.35. A rectangular steel plate AB weighing 100 lb is suspended by four identical springs of elasticity $k = 20$ lb/in. attached to its corner points. A 300 lb block is then placed centrally on the plate and released. (a) Apply the energy method to find the subsequent maximum displacement d of the system. (b) Derive the equation of motion, and compute the vibrational period of the total load.

7.36. A skier starts from rest at A, slides down a slick ski ramp of height h, and after leaving the ramp at B eventually lands smoothly on a steep slope inclined at the angle θ shown in the figure. The horizontal landing distance from B is $\ell = 2h$. Find the angle θ.

7.37. A particle of mass m slides in a smooth parabolic tube $y = kx^2$ in the vertical plane frame $\varphi = \{O; \mathbf{i}, \mathbf{j}\}$. In addition to the usual forces, a conservative force with potential $V^*(y)$ acts on P so that the horizontal component of its oscillatory motion within the tube is $x = A\cos\omega t$, A and ω being constants. (a) Find the potential energy $V^*(y)$. (b) Determine as a function of y the magnitude of the normal force exerted on P by the tube.

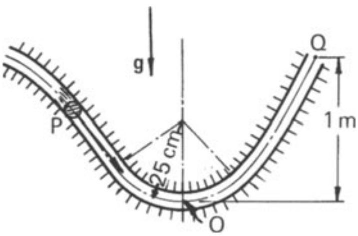
7.38. The force $\mathbf{F} = F_0[\sin(\pi y/a)\mathbf{i} + (\pi x/a)(1 + \cos(\pi y/a))\mathbf{j}]$, where F_0 is a constant, acts on a particle of mass m initially at rest at O in $\Phi = \{O; \mathbf{i}_k\}$. The particle moves in the vertical plane on a circle of radius a and center at $(a, 0)$. (a) Apply Stokes's theorem (7.57) to



Problem 7.36.

find the speed of the particle after one revolution. (b) Use the line integral (7.21) to calculate the work done by **F** in one revolution.

7.39. The figure shows a particle P of mass 2 kg moving in a smooth, curved tube in the vertical plane on a planet where the apparent acceleration of gravity is $g = 8.0$ m/sec^2. The particle has a speed of 5.0 m/sec when it passes the point O on the circular arc of radius 25 cm. (a) What force does the tube exert on the particle at O? (b) Determine the speed of P at the exit point Q. (c) If P started from rest initially, what was its initial location at h above O?

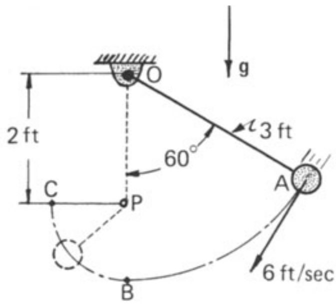


Problem 7.39.

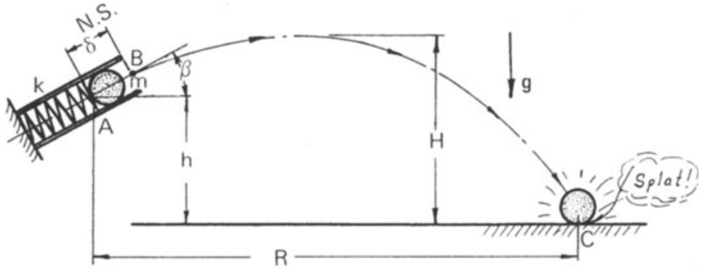
7.40. Apply the energy method to solve Problem 6.46. What is the static displacement of the load?

7.41. A pendulum bob is released from the position A with a speed of 6 ft/sec, as shown in the figure. In its vertical position OB, the cord strikes a fixed pin P, and the bob continues to swing on a smaller circular arc BC. (a) Apply the general energy principle to find the speed of the bob at its horizontal position at C. (b) What is the ratio of the angular speeds immediately after and before the string strikes P? (c) Derive equations for the angular speed $\dot{\theta}$ as a function of the angular placement θ of the bob measured from the vertical line OB, so that $\theta < 0$ on AB and $\theta > 0$ on BC. Plot $\dot{\theta}$ versus $\theta \in [-60°, 180°]$. Identify all important points of the plot. What is the jump in the angular speed at B?

7.42. A spring of stiffness k is compressed an amount δ from its natural state at B. When released, it projects a small mass m which lands at the point C located in the figure. Neglect friction and air resistance. (a) Find the speed of m when it strikes the ground at C. (b) What is the greatest height H attained in the motion?

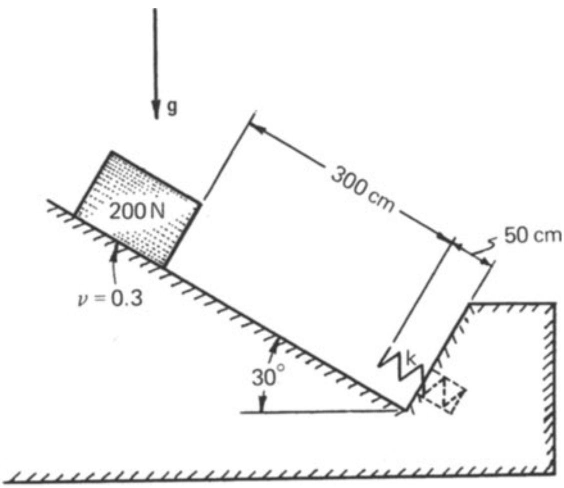


Problem 7.41.



Problem 7.42.

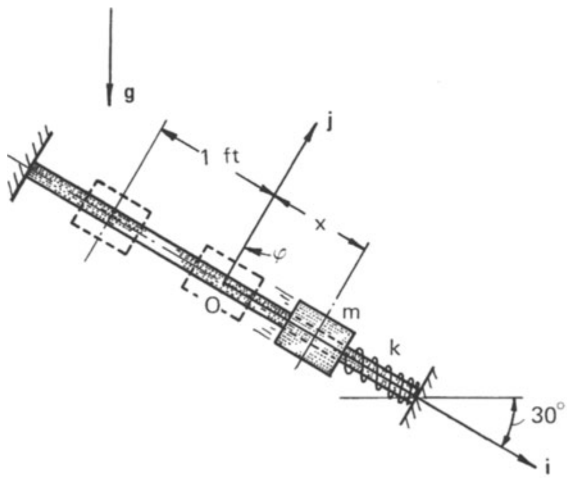
7.43. A block weighing 200 N is released from rest at the position shown on an inclined plane surface for which $\nu = 0.3$, and it ultimately contacts a spring of modulus $k = 20$ N/cm. Apply the general energy principle to find the maximum compression of the spring.



Problem 7.43.

7.44. The pilot of a cargo carrier is making an airdrop to a remote, tornado stricken area. At an appropriate time, an airman pushes a bundle of blankets from the rear of the aircraft with a constant speed v_B relative to the plane which has a ground speed v_P at an altitude h. (a) Apply conservation principles to determine the distance d from the recovery target area at which the bundle should be released, and find its trajectory. Find the drop time t^* to the target. Model the bundle as a particle and ignore environmental effects. (b) Evaluate the results for the case when $v_B = 5$ mph, $v_P = 185$ mph, and $h = 1610$ ft.

7.45. A slider block of mass $m = 0.5$ slug is attached to a spring of stiffness $k = 32$ lb/ft at a place where $g = 32$ ft/sec^2. The block is displaced, as shown, 1 ft from the natural state at O and released to move on a smooth, inclined supporting rod. (a) Apply the energy method to derive the equation of motion for m. (b) Find the equilibrium position of m, and determine the period and circular frequency of the oscillations. (c) Find the motion as a function of time, determine its amplitude, and sketch its graph for one period.



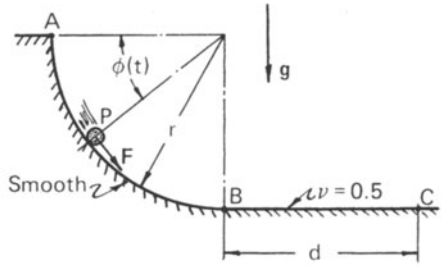
Problem 7.45.

7.46. The slider block described in Problem 6.53 is given an initial displacement $x(0) = x_0$ and released from rest relative to the table. Apply only the general energy and moment of momentum principles to derive its equation of motion. Interpret the energy equation for a stable equilibrium state at the origin, if initially $x_0 = 0$.

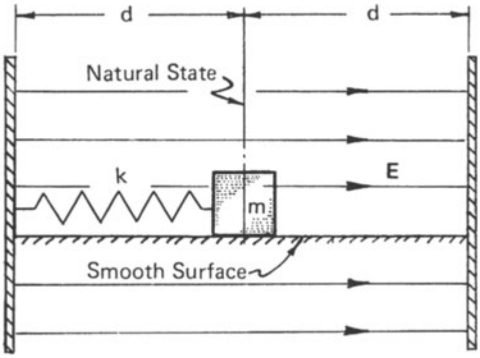
7.47. A particle P of weight W starts from rest at A, shown in the figure, and is driven along a smooth circular track of radius r by a tangential propulsive force of variable intensity $F(t) = 2W \cos\phi(t)$ for $\phi \in [0, \pi/2]$. At B, it transfers to a rough horizontal surface BC on which $\nu = 0.5$. (a) What is the horizontal distance d traveled by P before coming to rest at C? (b) Determine the surface force exerted on P at the instant it reaches B on the circular arc and immediately afterward.

7.48. A body of mass m, initially at rest on a smooth, electrically insulated, horizontal surface, is attached to an insulated linear spring. The assembly, in its natural state, is placed in a constant electric field of strength $\mathbf{E}$, directed as shown. The body is suddenly charged an amount q and oscillates within the field. What is the work done by the electrical force? Apply the work–energy principle to find the motion of m and describe its physical characteristics. Sketch the solution function.

7.49. The spring and pulley suspension system shown in the figure for Problem 6.46 is modified to introduce a spring of modulus k connected at the end of the rigid supporting rod OA and attached to the load M. Neglect friction, ignore the mass of the pulley and the support system, and suppose that the pulley belt is inextensible. (a) Find the static displacement of the load M and



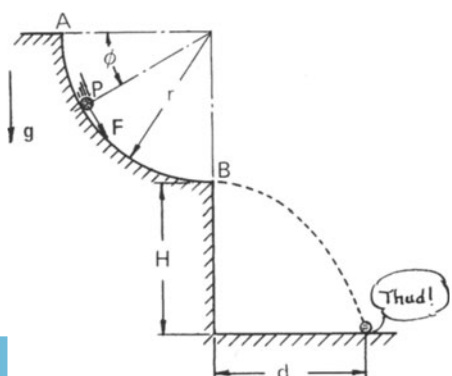
Problem 7.47.



Problem 7.48.

derive its equation of motion about the static equilibrium state. (b) What is the stiffness of an equivalent simple spring-mass system having the same frequency for the same load? (c) Describe the major vibrational characteristics of the motion. (d) How many degrees of freedom does this system have?

7.50. A particle P of mass m is moved along a smooth circular track by a tangential propulsive force whose magnitude $F = F_0 \cos\phi(t)$ varies with the position angle $\phi(t) \in [0, \pi/2]$ shown in the figure. The particle starts from rest at A and projects from the track at point B. Find the subsequent trajectory of P, and determine the horizontal distance d at which P strikes the horizontal plane at the vertical distance H below B.



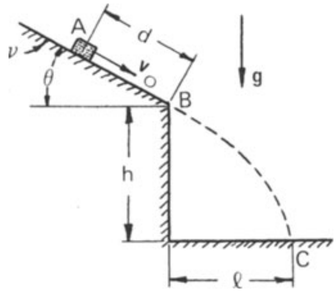
Problem 7.50.

7.51. The rod tension exerted on a simple pendulum bob is given by $\mathbf{T}(\mathbf{x}) = T(\theta)\mathbf{n}(\theta)$, where $T(\theta)$ is defined by (7.86c) and $\mathbf{n}(\theta)$ is the unit normal vector shown in Fig. 6.15, page 138. (a) Write $\mathbf{T}(\mathbf{x})$ as a function of the x- and y-position coordinates of the bob, and show that $\mathbf{T}(\mathbf{x})$ is not conservative. (b) The curl of a vector field $\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z$ in cylindrical coordinates is defined by

$$\nabla \times \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\mathbf{e}_z. \tag{P7.51}$$

Write $\mathbf{T}(\mathbf{x})$ in cylindrical coordinates and apply (P7.51) to show that $\mathbf{T}(\mathbf{x})$ is not conservative. Fortunately, $\mathbf{T}$ does no work in the motion.

7.52. A sled of mass m is driven at a constant velocity $\mathbf{v}_0$ down a rough plane with $\nu = 0.5$ and inclined at an angle $\theta = 45°$. The power is suddenly lost at point A, shown in the figure, but the sled continues to slide down the plane to point B where it leaves the surface and later impacts a horizontal plane at the point C, a distance $h = 2\ell$ below B. Find in terms of $v_0 = |\mathbf{v}_0|$ and ℓ the distance d from B at which the power was lost.



Problem 7.52.

7.53. (a) A bullet of mass m is fired with velocity β and passes through a block of mass M initially at ease on a smooth horizontal surface S. The block travels the distance d shown in the figure, projects from S at A, and ultimately lands at C at a distance D from B on a horizontal plane at H below A. Determine the exit speed α of the bullet. Ignore frictional effects and energy and small mass losses from permanent deformation and tearing of the block and the bullet. (b) Now suppose that the surface S is rough with coefficient of friction ν. Find the exit speed of the bullet. What is the largest initial velocity that M may have and remain on S?

7.54. Two other Jacobian elliptic functions related to the elliptic sine function $\operatorname{sn} u$ in (7.88f) are defined by

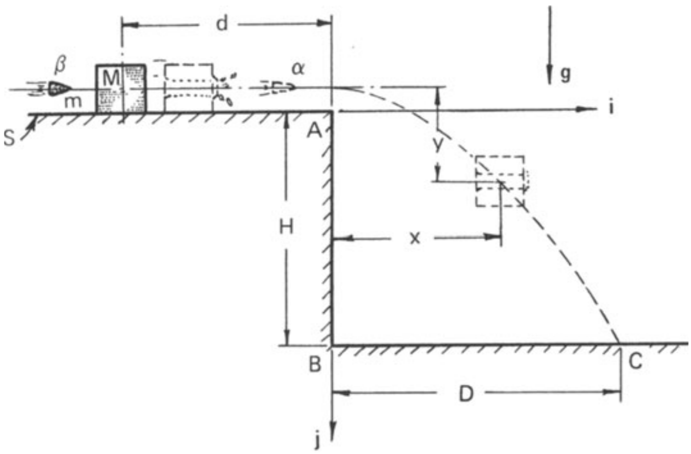
$$\operatorname{cn} u = \cos\phi, \qquad \operatorname{dn} u = \sqrt{1 - k^2 \sin^2\phi}, \tag{P7.54a}$$

in which ϕ is the argument of the elliptic integral in (7.87d), or equivalently (7.88d): $u = F(\phi; k)$. (a) Show that

$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1, \qquad k^2 \operatorname{sn}^2 u + \operatorname{dn}^2 u = 1. \tag{P7.54b}$$

(b) Prove that the *elliptic cosine function* $\operatorname{cn} u$, read as *"see—en—u"*, is an even periodic function of period $4K(k)$ and $-1 \le \operatorname{cn} u \le 1$. (c) Show that when $k = 0$,

$$\operatorname{sn} u = \sin u, \qquad \operatorname{cn} u = \cos u, \qquad \operatorname{dn} u = 1, \qquad K(0) = \frac{\pi}{2}, \tag{P7.54c}$$



Problem 7.53.

and hence the Jacobian elliptic functions reduce to the trigonometric functions. In all, there are twelve Jacobian elliptic functions; the others are defined in terms of the three basic functions described above. For details, see the Byrd and Friedman Handbook cited in footnote ‡, page 266.

7.55. (a) Recall (7.88f), (7.88g), and note that $u = F(\phi; k)$ in (7.87d). Show that these relations and those of the previous problem yield the following derivatives of the Jacobian elliptic functions:

$$\frac{d}{du}(\mathrm{sn}u) = \mathrm{cn}u \ \mathrm{dn}u, \tag{P7.55a}$$

$$\frac{d}{du}(\mathrm{cn}u) = -\mathrm{sn}u \ \mathrm{dn}u, \tag{P7.55b}$$

$$\frac{d}{du}(\mathrm{dn}u) = -k^2\mathrm{sn}u \ \mathrm{cn}u. \tag{P7.55c}$$

Verify that when $k = 0$, these reduce to the familiar trigonometric rules. (b) Recall (7.88h) and (7.88i). On the same plot, sketch graphs of the three basic Jacobian elliptic functions, and thus show that the third elliptic function $\mathrm{dn}u$, read as *"dee—en—u"*, is a positive-valued, even periodic function of period $2K(k)$ with values in the interval $k' \le \mathrm{dn}u \le 1$, where *the complementary modulus* $k' \equiv (1 - k^2)^{1/2}$.

7.56. The exact relation for the period of a simple pendulum is given by (7.87f). (a) Use the ratio τ^*/τ to determine the percentage error in the period that occurs when the small amplitude solution τ is used for large amplitudes $\alpha = 30°,\ 60°,$ and $90°$. (b) Compute the percentage error based on the second order approximate solution in (7.90d) in comparison with (7.87f). How does the error in this case compare with that in part (a) for the same amplitudes?

7.57. A bead B of mass m is constrained to slide in the vertical plane on a smooth, fixed circular wire of radius a and vertical diameter AC. The bead is projected counterclockwise from the lowest point A with initial speed $v_0 = \alpha v$, where v is the smallest initial speed that will drive B to its highest point at C and α is a constant. (a) Determine the speed v. (b) Let ϕ denote the angle at C between the diameter CA and the chord CB at time t. Find the time required for the bead to describe the arc AB subtended by ϕ. Then analyze three cases: (i) $\alpha = 1$, (ii) $\alpha > 1$, and (iii) $\alpha < 1$, and interpret their physical nature.

7.58. The pendulum for the data described in Problem 6.47 is a scaled model for a certain low speed vibration control device. The preliminary design requires that the period of the finite amplitude oscillations of the pendulum must be two seconds. The bob design mass $m = 0.01$ kg. Derive the exact equations of motion for the pendulum, and find as functions of the amplitude angle β_0 exact relations for the angular speed of the table and the maximum tension in the string. The project leader wants you to present the results to the technical management team who may not recall the mathematics used to express the solution. The results for all values of $\beta_0 \leq 60°$ must be discussed, but it is anticipated that questions concerning the effects of variations in the period and the potential influence of larger amplitudes may arise. Provide your supervisor with a brief preliminary report that will convey clearly all of the desired information.

7.59. (a) The small amplitude period of a simple pendulum is 2 sec. What is its period for an amplitude $\alpha = \pm\pi/2$ rad? (b) Suppose the same pendulum has just adequate initial velocity to complete a full revolution. Find the time required for the bob to advance 90° from its lowest position.

7.60. A spring-mass system similar to that in Fig. 6.13, page 134, consists of a mass m attached to two concentric springs. The inner spring has linear response with stiffness k_1. The other is a nonlinear conical spring with stiffness k_2, whose spring force is proportional to the cube of its extension x from the natural state. The mass is given an initial displacement x_0 from the natural state and released to oscillate on the smooth horizontal surface. (a) Derive the equation of motion for m, and solve it to obtain an integral for the travel time $t = t(x)$. (b) Introduce the change of variable $x = x_0 \cos\phi$ to obtain $t = t(\phi)$, and derive exactly the motion $x(m, t)$ in terms of a Jacobian elliptic function. (c) What is the period of the finite amplitude oscillations of m?

7.61. A bead of mass m slides on a smooth wire in the vertical plane. Find its small amplitude frequency when the wire is (a) a parabola $y = ax^2$ and (b) a catenary $y = a\cosh x$, where a is the same positive constant. On which curve is the frequency greater?

7.62. Solve the last problem for (a) an ellipse $x^2/a^2 + y^2/b^2 = 1$ and (b) a hyperbola $y^2/b^2 - x^2/a^2 = 1$, where a and b are the usual constants. Confirm the solution for the ellipse in its special application to a circular wire. What is the particle's small amplitude frequency on an equilateral hyperbola? What is interesting about these results?

7.63. A particle is given a small displacement from a stable equilibrium state on a smooth Archimedean spiral $r = a\phi$ in the vertical plane. Here a is a positive constant. Find the frequency of the oscillation about the first stable equilibrium state. Is the frequency about other stable equilibrium states larger or smaller? Explain this and support your answer with an example. See Problem 4.51 in Volume 1.

7.64. A particle of mass m, initially at rest, slides in the vertical plane on a smooth cycloid shown in Fig. 7.16, page 273, and described by (7.94a). Let $\gamma(s)$ denote the slope angle of the curve at s, and γ_0 its value at the particle's initial position. (a) Find as a function of γ and γ_0 the time to reach a lower point on the curve. Do this two ways: (i) apply the energy integral (7.92a) and (ii) use the general solution of (7.94e). (b) Hence, show that regardless of its initial position, the particle will always reach the minimum point on the cycloid in the time $t = \tau/4$.

7.65. A bead of mass m slides on a smooth wire in the vertical plane. If the distance it travels in time t is $s(t) = a\sinh(nt)$, where a and n are constants, determine the shape of the wire and the initial conditions.

7.66. A particle of mass m moves on a smooth convex curve in the vertical plane. If its speed is proportional to the distance traveled from the highest point on the curve, determine the path.

7.67. A particle of mass m is at rest at the vertex of a smooth, inverted cycloid in the vertical plane. When slightly disturbed, it slides down the cycloidal surface. (a) Find the vertical distance below the vertex at which the particle leaves the surface. (b) Determine the distance traveled and the speed at that instant.

7.68. Consider an arbitrary point P on a smooth cycloid (7.94a) in the vertical plane frame $\Phi = \{O; \mathbf{i}, \mathbf{j}\}$ in Fig. 7.16, page 273. A particle of mass m is released from rest to slide on a smooth, straight wire from the point P to the origin O. (a) Derive an equation for the normalized time of descent $T_l \equiv t_l/\sqrt{a/g}$ along the wire, as a function of the angle β at the initial point P on the cycloid. (b) Show that for all values of $\beta \in [0, \pi]$ the normalized time of descent T_l exceeds the normalized time of descent $T_c \equiv t_c/\sqrt{a/g}$ of a particle sliding on the cycloid from the same point P. In particular, show that for $\beta \in [0, \pi]$, $T_l \in [4, (\pi^2 + 4)^{1/2}]$, which is everywhere greater than T_c. Hence, no matter where the motion starts, a particle that slides on the cycloid from point P always is the first to reach O; moreover, the result is independent of the particle's mass.

7.69. The orbit of a boomerang B thrown from point O is a petal of a lemniscate described by the polar coordinate equation $r^2 = a^2 \cos 2\theta$, where a is its greatest distance from O. Find the total distance L traveled by B in its return to O.

8

Dynamics of a System of Particles

8.1. Introduction

The principles of mechanics for a particle are extended here to a system of n discrete material points. We begin with Newton's second law for a system of particles and formulate the momentum, impulse–momentum, moment of momentum, work–energy, conservation, and general energy principles for a system of particles. Several of the concepts introduced here are especially useful in the study (in Chapter 11) of Lagrange's general equations for arbitrary dynamical systems, and the development of the moment of momentum principle for a system of particles provides a foundation for the independent presentation (in Chapter 10) of parallel results for the moment of momentum of a rigid body.

8.2. Equation of Motion for the Center of Mass

The total force $\mathbf{F}_k = \mathbf{F}(P_k, t)$ that acts on the k^{th} particle of a system $\beta = \{P_i\}$ of n particles consists of a *total external force* $\mathbf{f}_k = \mathbf{f}(P_k, t)$ exerted by bodies outside of β and a *total internal force* $\mathbf{b}_k = \mathbf{b}(P_k, t)$ due to the mutual interaction between P_k and all other particles in β. Let $\mathbf{b}_{kj}$ denote the mutual internal force exerted on the particle P_k by the particle P_j. Then the total internal force on P_k is

$$\mathbf{b}_k = \sum_{\substack{j=1 \\ j \neq k}}^{n} \mathbf{b}_{kj}. \tag{8.1}$$

Thus, the total force $\mathbf{F}(\beta, t) = \sum_{k=1}^{n} \mathbf{F}_k = \sum_{k=1}^{n}(\mathbf{f}_k + \mathbf{b}_k)$ acting on the system is

$$\mathbf{F}(\beta, t) = \sum_{k=1}^{n} \mathbf{f}_k + \sum_{k=1}^{n} \sum_{\substack{j=1 \\ j \neq k}}^{n} \mathbf{b}_{kj}. \tag{8.2}$$

In accordance with the third law, the internal forces occur in equal, oppositely directed pairs so that

$$\mathbf{b}_{jk} = -\mathbf{b}_{kj}; \tag{8.3}$$

and hence the total internal force, the last sum in (8.2), vanishes. *Therefore, the total force that acts on a system of particles is equal to the total external force*:

$$\mathbf{F}(\beta, t) = \sum_{k=1}^{n} \mathbf{f}_k. \tag{8.4}$$

We recall from (5.7) that the total momentum of a system of particles is equal to the momentum of its center of mass, and use of this result in (5.40) leads to the familiar classical form (5.41) of Newton's second law of motion for a system of particles in which only external forces (8.4) arise.

Newton's principle of motion for a system of particles: *The total external force on a system of particles is equal to the time rate of change of the momentum of the center of mass relative to an inertial frame* Φ, *and is thus equal to the product of the total mass of the system and the acceleration of the center of mass in* Φ:

$$\mathbf{F}(\beta, t) = \dot{\mathbf{p}}^{*}(\beta, t) = m(\beta)\mathbf{a}^{*}(\beta, t). \tag{8.5}$$

This equation aids in determination of the motion of the center of mass of the system and the external forces that act on it. In applications, however, the auxiliary center of mass relations (5.5) through (5.8), as well as the separate equations of motion of the particles, often are needed in problem solutions. Of course, the motion of an individual particle is governed by (5.39), which depends on the action of *all* forces that act on the particle, including internal forces that do not appear in (8.5). Without these auxiliary equations, (8.5) alone may not be very helpful. Plainly, all of the principles of mechanics for a single particle apply directly to the unique center of mass particle of a system of particles subjected to only the total external force (8.4). The familiar principle of conservation of momentum (7.69), for example, may be read immediately from (8.5), as follows below. Afterwards, an important application of the principle illustrates the need for the aforementioned auxiliary equations for the center of mass and Newton's law for a particle.

The principle of conservation of momentum of a system of particles: *The total external force component in a fixed direction* $\mathbf{e}$ *vanishes for all time if and only if the corresponding component of the momentum of the center of mass is* *constant*:

$$\mathbf{F}(\beta, t) \cdot \mathbf{e} = 0 \Leftrightarrow \mathbf{p}^*(\beta, t) \cdot \mathbf{e} = \text{const.} \tag{8.6}$$

Thus, the momentum of the center of mass is a constant vector if and only if the total external force vanishes for all time, in which case the center of mass moves uniformly on a straight line, or, if at rest initially, it remains so.

8.3. The Two Body Problem

Our study in the last chapter of the central gravitational attraction of a body in its orbital motion about another body assumed *fixed* in an inertial reference frame led to proof of Kepler's empirical laws on the elliptical path and orbital period of the attracted body. Here we study the related classical problem of the *relative motions* of two celestial bodies due only to their mutual gravitational attraction. This so-called *two body problem* models the relative motions of two astronomical bodies like the Earth and the Moon, the Sun and a planet, or a double star, for example, all of whose mutual distances of separation are so great that a pair of these huge celestial objects may be treated as two particles remote from the gravitational influence of all other bodies.

Consider two bodies of mass m_1 and m_2 subject only to their mutual gravitational force. These are internal forces for the system. Therefore, the total external force on the system vanishes in (8.5). It follows that the center of mass must have a uniform motion or be at rest in the astronomical inertial frame. (Although this is useful information, it does not address the relative motion of the particles.) As a consequence, we may choose an inertial frame $\Psi = \{C; \mathbf{I}_k\}$ with its origin at the center of mass, and now introduce the relative position vectors $\boldsymbol{\rho}_k$ of the two particles in Ψ. Then by (5.6), relative to the center of mass, we have

$$m_1 \boldsymbol{\rho}_1 + m_2 \boldsymbol{\rho}_2 = \mathbf{0}. \tag{8.7}$$

This shows that the particles must be situated along a line through C. Consider the motion of m_2 relative to m_1 described by

$$\mathbf{r} \equiv \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1. \tag{8.8}$$

The motion of each particle relative to the center of mass is then obtained from (8.7) and (8.8) in terms of the relative motion vector $\mathbf{r}$ by

$$\boldsymbol{\rho}_1 = -\frac{m_2}{m_1 + m_2}\mathbf{r}, \qquad \boldsymbol{\rho}_2 = \frac{m_1}{m_1 + m_2}\mathbf{r}. \tag{8.9}$$

The equations of motion of the separate bodies in the inertial frame Ψ are

$$m_1 \ddot{\boldsymbol{\rho}}_1 = \frac{G m_1 m_2}{r^2}\mathbf{e}, \qquad m_2 \ddot{\boldsymbol{\rho}}_2 = -\frac{G m_1 m_2}{r^2}\mathbf{e}, \tag{8.10}$$

wherein

$$\mathbf{e} \equiv \frac{\mathbf{r}}{r} = \frac{\rho_2 - \rho_1}{|\rho_2 - \rho_1|}, \tag{8.11}$$

is a unit vector directed from m_1 toward m_2. The equation for the motion (8.8) of m_2 relative to m_1 is now provided by $\ddot{\mathbf{r}} = \ddot{\rho}_2 - \ddot{\rho}_1$ and (8.10); namely,

$$m_r \ddot{\mathbf{r}} = -\frac{G m_1 m_2}{r^2}\mathbf{e} = -\frac{G(m_1 + m_2) m_r}{r^2}\mathbf{e}. \tag{8.12}$$

This is the equation of motion of a single "particle" having the *reduced mass m_r* defined by

$$m_r \equiv \frac{m_1 m_2}{m_1 + m_2}. \tag{8.13}$$

The problem of the relative motion of two bodies is thus transformed to an equivalent single body problem for which we need determine only the motion of a fictitious particle of mass m_r under the same central force experienced by the two bodies separated a distance r.

We recall (6.4) in which $\mathbf{a} = \ddot{\mathbf{r}}$ and $\mathbf{e}_r = \mathbf{e}$ to obtain from (8.12) the scalar equations of motion in cylindrical coordinates:

$$\ddot{r} - r\dot{\phi}^2 = -\frac{G(m_1 + m_2)}{r^2}, \qquad r^2\dot{\phi} = \gamma, \text{ a constant.} \tag{8.14}$$

Use of the second equation in the first and integration in r leads to an energy equation of the same form as (7.96a) in which now $\mu = G(m_1 + m_2)$ and $m = m_r$. The orbit analysis is then similar to what is presented at the end of the last chapter. With the solution of (8.14) in hand, the individual motions of the two bodies relative to the center of mass in Ψ may be obtained from (8.9). The solution of (8.14) thus suffices to determine their relative orbital motion. We shall omit these details.

An immediate effect of our accounting for the relative motions of the two bodies is that Kepler's third law (7.98e) for the orbital periodic time τ is changed by the modified factor μ:

$$\tau = 2\pi\sqrt{\frac{a^3}{\mu}} = 2\pi\sqrt{\frac{a^3}{G(m_1 + m_2)}}. \tag{8.15}$$

Therefore, the orbital period for the two body problem with a moving attractor is smaller than the Kepler period (7.98e) *for the single body problem with a fixed attractor; so, the orbital period is **not** the same for all orbits having the same semi-major axis.* The period varies from planet to planet due to the presence in (8.15) of the mass m_2 of the orbiting body. Of course, the same thing follows symmetrically for the body of mass m_1 in its motion relative to m_2. If the mass of either of the bodies is much greater than the other, like the mass of the Sun compared with that of the Earth and other planets in our solar system, for example, the deviations of the periodic times of the different planets discovered in the two body problem are small, and only then does Kepler's third law prevail. On the other hand, because the mass of Jupiter is roughly $1/1000$ of the mass of the Sun, the two body departure from Kepler's approximate law for the period of Jupiter is observable.

8.4. The Impulse–Momentum Equation

Integration of (8.5) with respect to time yields

$$\mathbf{p}^*(\beta, t) - \mathbf{p}^*(\beta, t_0) = \int_{t_0}^{t} \mathbf{F}(\beta, t)dt \equiv \mathscr{I}(t; t_0), \tag{8.16}$$

in which $\mathscr{I}(t; t_0)$ is called the *impulse of the external force* $\mathbf{F}(\beta, t)$ on the system. Thus, (8.16) provides in the usual notation the following rule.

The impulse–momentum equation for a system of particles: *The impulse of the external force on a system of particles over the time interval* $[t_0, t]$ *is equal to the change in the momentum of the center of mass during that time*:

$$\Delta\mathbf{p}^* = \mathscr{I}(t; t_0). \tag{8.17}$$

The change of momentum is always in the direction of the impulse. Since (8.17) is similar to rule (7.2), rules similar to (7.7) and (7.8) characterizing an instantaneous impulse hold also for the center of mass. Of course, any finite external force will contribute nothing to the instantaneous impulse; but other impulsive external reaction forces that might act on the system must be accounted for. Now, if the instantaneous impulse is due to equal, oppositely directed and collinear internal impulsive forces only and all external forces are finite, then at the impulsive instant $\lim_{\Delta t \to 0} \mathscr{I}(t; t_0) = \mathbf{0}$ and (8.17) yields $\Delta\mathbf{p}^* = \mathbf{0}$. *Therefore, in this case, for a system of particles subject to finite external forces, the instantaneous momentum of the center of mass is constant during the internal impulsive interval.* This is shown differently in (7.12) for a system of two particles on which the mutual instantaneous impulsive forces are equal, oppositely directed internal forces.

Example 8.1. A gun of mass M fires a shell S of mass m with a muzzle velocity $\mathbf{v}_{SG} = \mathbf{v}_0$ relative to the gun barrel G, at an elevation angle α in the ground frame $\Phi = \{F; \mathbf{I}_k\}$ in Fig. 8.1. The gun carriage C is mounted on a greased horizontal track. (a) Find the instantaneous recoil velocity $\mathbf{v}_{GF}$ of the gun (i.e. the center of mass of the gun assembly). (b) Compare the magnitude v of the instantaneous, absolute muzzle velocity $\mathbf{v}_{SF}$ of the shell in Φ with its relative value v_0. (c) What is the instantaneous impulsive reaction exerted by the track on the gun carriage? □

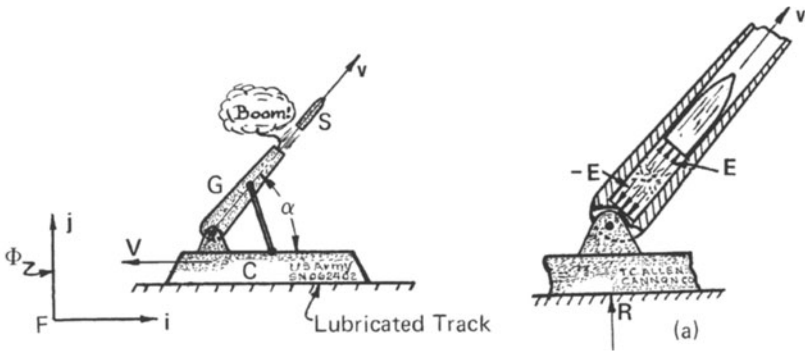


Figure 8.1. Momentum and impulse reaction in firing a gun.

Solution of (a). The gun and shell are modeled as center of mass objects—"a system of two particles." Then the instantaneous, absolute muzzle velocity of the shell S relative to the ground frame $\Phi = \{F; \mathbf{I}_k\}$ is $\mathbf{v}_{SF} = \mathbf{v}_{SG} + \mathbf{v}_{GF}$; that is,

$$\mathbf{v}_{SF} = \mathbf{v}_0 + \mathbf{v}_{GF} = (v_0 \cos\alpha - V)\mathbf{i} + v_0 \sin\alpha\mathbf{j}, \tag{8.18a}$$

wherein the instantaneous recoil velocity of the gun is $\mathbf{v}_{GF} = -V\mathbf{i}$. To find $\mathbf{v}_{GF}$, we observe that the only external forces that act on the system at the impulsive instant are vertical forces—the total weight of the system, the equipollent static track reaction force on the carriage, and the additional impulsive vertical reaction force $\mathbf{R}$ exerted on the gun carriage by the lubricated track. Therefore, in accordance with (8.6), the component $\mathbf{p}^* \cdot \mathbf{i}$ of the linear momentum of the center of mass, hence the system, is conserved. Initially, $\mathbf{p}^* = \mathbf{0}$; hence, after the impulse, the component $\mathbf{p}^* \cdot \mathbf{i}$ of the instantaneous momentum of the system must vanish in Φ:

$$\mathbf{p}^* \cdot \mathbf{i} = (M\mathbf{v}_{GF} + m\mathbf{v}_{SF}) \cdot \mathbf{i} = (-MV\mathbf{i} + m\mathbf{v}_{SF}) \cdot \mathbf{i} = 0. \tag{8.18b}$$

Equations (8.18a) and (8.18b) yield $-MV + m(v_0 \cos\alpha - V) = 0$; and hence the instantaneous recoil velocity of the gun is

$$\mathbf{v}_{GF} = -V\mathbf{i} = -\frac{m}{m+M} v_0 \cos\alpha\mathbf{i}. \tag{8.18c}$$

After the impulsive instant, additional forces exerted by a recoil spring and viscous damper, not shown in the diagram, retard the subsequent motion of the gun and restore it to its firing station. We shall not explore this motion.

Solution of (b). The instantaneous, absolute muzzle velocity $\mathbf{v}_{SF}$ of the shell follows from (8.18a):

$$\mathbf{v}_{SF} = v_0 \left(\frac{M}{m+M} \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j} \right); \tag{8.18d}$$

and hence the absolute muzzle speed v is related to the relative speed v_0 by

$$v = v_0 \left(1 - \frac{m(m+2M)}{(m+M)^2}\cos^2\alpha\right)^{1/2}. \tag{8.18e}$$

Inasmuch as $m \ll M$, to the first order in $\mu \equiv m/M$, $v = v_0(1 - \mu\cos^2\alpha) < v_0$; the instantaneous, absolute muzzle speed of the shell in Φ is somewhat less than its muzzle speed v_0 relative to the gun.

Solution of (c). The instantaneous, external normal reaction impulse $\mathscr{I}^*_R$ exerted by the smooth track on the system is obtained from (8.17):

$$\mathscr{I}^*_R \equiv \lim_{t\to t_0}\int_{t_0}^{t} \mathbf{R}dt = \Delta\mathbf{p}^* = M\mathbf{v}_{GF} + m\mathbf{v}_{SF}. \tag{8.18f}$$

Since $\mathbf{R}\cdot\mathbf{i} = 0$, (8.18f) requires $\Delta\mathbf{p}^*\cdot\mathbf{i} = 0$, which is the same as (8.18b); and hence, by (8.18c) and (8.18d), the external impulsive reaction of the track on the gun is

$$\mathscr{I}^*_R = (\Delta\mathbf{p}^*\cdot\mathbf{j})\mathbf{j} = m v_0 \sin\alpha\mathbf{j}. \tag{8.18g}$$

8.5. Moment of Momentum of a System of Particles

In preparation for the study of the moment of momentum principle for a system of particles, we next express the moment of momentum for the system in terms of the motion of its center of mass C. By (5.32), the moment about C of the momenta of the system relative to frame $\Phi = \{F; \mathbf{I}_k\}$ in Fig. 8.2 is given by

$$\mathbf{h}_C(\beta, t) = \sum_{k=1}^{n} \boldsymbol{\rho}_k \times m_k\dot{\mathbf{X}}_k, \tag{8.19}$$

wherein $\boldsymbol{\rho}_k$ is the position vector from C to the particle P_k and $m_k\dot{\mathbf{X}}_k$ is its momentum relative to the origin in Φ. It is useful to define the moment about C of the momenta of the system of particles relative to C in Φ, called *the moment of momentum relative to* C, in accordance with

$$\mathbf{h}_{rC}(\beta, t) \equiv \sum_{k=1}^{n} \boldsymbol{\rho}_k \times m_k\dot{\boldsymbol{\rho}}_k. \tag{8.20}$$

Here and in similar relations below, the subscript r is used for moments of momenta relative to points in Φ, but not with respect to the origin F in Φ, as emphasized in (8.19) and (8.20).

To relate $\mathbf{h}_{rC}$ to $\mathbf{h}_C$, introduce $\mathbf{X}_k = \mathbf{X}^* + \boldsymbol{\rho}_k$ in (8.19) to obtain

$$\mathbf{h}_C(\beta, t) = \sum_{k=1}^{n} m_k\boldsymbol{\rho}_k \times \dot{\mathbf{X}}^* + \sum_{k=1}^{n} \boldsymbol{\rho}_k \times m_k\dot{\boldsymbol{\rho}}_k. \tag{8.21}$$

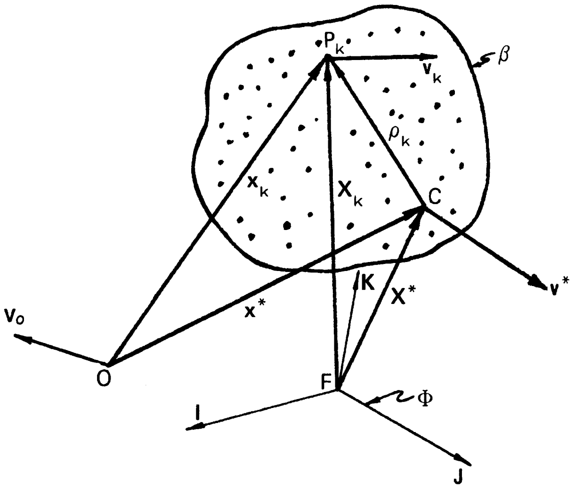


Figure 8.2. Schema for the moment of momentum of a system of particles.

By (5.6), however, the first product term vanishes, and we have with (8.20),

$$\mathbf{h}_C(\beta, t) = \sum_{k=1}^{n} \boldsymbol{\rho}_k \times m_k \dot{\boldsymbol{\rho}}_k = \mathbf{h}_{rC}(\beta, t). \tag{8.22}$$

In sum, *the moment about C of the momentum of the system relative to the origin in* Φ *is equal to the moment about C of the momentum of the system relative to C in* Φ.

The importance of this interesting property of the moment of momentum of a system of particles is revealed in the next section. The following exercises and subsequent example prepare the reader for development of principles presented there.

Exercise 8.1. The *moment of momentum relative to O in* Φ is defined by

$$\mathbf{h}_{rO}(\beta, t) \equiv \sum_{k=1}^{n} \mathbf{x}_k \times m_k \dot{\mathbf{x}}_k, \tag{8.23}$$

where $\mathbf{x}_k$ is the position vector of particle P_k from O in Fig. 8.2. This is the moment about a point O in Φ of the momenta of the system relative to O. Recall (5.32) for the moment of momentum $\mathbf{h}_O(\beta, t)$ of a system of particles about point O in Φ, introduce a velocity transformation, and show that

$$\mathbf{h}_O(\beta, t) = \mathbf{h}_{rO}(\beta, t) + m(\beta)\mathbf{x}_O^* \times \mathbf{v}_O, \tag{8.24}$$

in which $\mathbf{v}_O$ is the velocity of O in Φ and $\mathbf{x}_O^* = \mathbf{x}^*(\beta, t)$ is the position vector of C from O. Describe three cases for which $\mathbf{h}_O(\beta, t) = \mathbf{h}_{rO}(\beta, t)$. □

Exercise 8.2. Use the point transformation $\mathbf{x}_k = \mathbf{x}^* + \rho_k$ in Fig. 8.2 and show that the moment of momentum relative to a point O in Φ is related to the moment of momentum relative to the center of mass C in accordance with

$$\mathbf{h}_{rO}(\beta, t) = \mathbf{h}^*_{rO}(\beta, t) + \mathbf{h}_{rC}(\beta, t), \tag{8.25}$$

in which, by definition,

$$\mathbf{h}^*_{rO}(\beta, t) \equiv \mathbf{x}^*_O \times \mathbf{p}^*_O, \tag{8.26}$$

and $\mathbf{p}^*_O = m(\beta)\dot{\mathbf{x}}^*(\beta, t)$ is the momentum of the center of mass relative to O. The vector $\mathbf{h}^*_{rO}$, therefore, is the *moment of momentum of the center of mass relative to O*. Describe the content of (8.25) in words. □

Exercise 8.3. Introduce a velocity transformation in (5.32) to show that *the moment about point O of the momentum in Φ of a system of particles is equal to the moment about O of the momentum of the center of mass relative to the origin in Φ plus the moment of momentum relative to C in Φ*:

$$\mathbf{h}_O(\beta, t) = \mathbf{h}^*_O(\beta, t) + \mathbf{h}_{rC}(\beta, t), \tag{8.27}$$

where $\mathbf{h}^*_O \equiv \mathbf{x}^* \times m(\beta)\mathbf{v}^*$ with $\mathbf{v}^* = \dot{\mathbf{X}}^*$. Discuss the major difference between the moment of momentum vector $\mathbf{h}^*_O(\beta, t)$ and $\mathbf{h}^*_{rO}(\beta, t)$ in (8.26). Replacing O with C in (8.27) or (8.24), we recover (8.22). □

We conclude this extended review with an example demonstrating some calculations that include the application of (8.25).

Example 8.2. Two particles of mass $m_1 = m$ and $m_2 = 3m$ are moving with respective velocities $\mathbf{v}_1 = (4v, -7v, 0)$ and $\mathbf{v}_2 = (0, v, 4v)$ relative to frame $\Phi = \{F; \mathbf{i}_k\}$. (a) Find the velocity of the center of mass in Φ. (b) At a certain instant the respective particles are at $\mathbf{x}_1 = (0, -1, 3)$ and $\mathbf{x}_2 = (8, -1, 3)$ from a fixed point O in Φ. What is the moment about O of the momentum of the center of mass of the system in Φ? (c) What is the moment of momentum of C relative to O?

Solution of (a). To find $\mathbf{v}^*$, consider the momentum of the center of mass in Φ : $\mathbf{p}^* = m(\beta)\mathbf{v}^*$, where $m(\beta) = m_1 + m_2 = 4m$. Then, by (5.7), the total momentum of the system is $\mathbf{p}^* = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m(4v, -7v, 0) + 3m(0, v, 4v)$, i.e.

$$\mathbf{p}^* = 4m\mathbf{v}^* = 4mv(1, -1, 3); \tag{8.28a}$$

and hence the center of mass has velocity

$$\mathbf{v}^* = v(\mathbf{i} - \mathbf{j} + 3\mathbf{k}). \tag{8.28b}$$

Solution of (b). To determine $\mathbf{h}^*_O = \mathbf{x}^* \times \mathbf{p}^*$, the moment about point O of the momentum of the center of mass in Φ, we need to determine the position vector

$\mathbf{x}^*$ of the center of mass from O at the instant of interest. From (5.5), the reader will find that the center of mass is at the place from O given by

$$\mathbf{x}^* = 6\mathbf{i} - \mathbf{j} + 3\mathbf{k}; \tag{8.28c}$$

and with (8.28a) the moment about O of the momentum of C in Φ is

$$\mathbf{h}_O^* = \mathbf{x}^* \times \mathbf{p}^* = 4mv \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -1 & 3 \\ 1 & -1 & 3 \end{vmatrix} = -20mv(3\mathbf{j} + \mathbf{k}). \tag{8.28d}$$

Solution of (c). The moment about O of the momentum of C relative to the origin in Φ is given by $\mathbf{h}_O^* = \mathbf{x}^* \times m(\beta)\mathbf{v}^*$, in which $\mathbf{v}^* \equiv \dot{\mathbf{X}}^* = \mathbf{v}_O + \dot{\mathbf{x}}^*$ in Fig. 8.2. Because point O is fixed in Φ, $\mathbf{v}_O = \mathbf{0}$ and hence $\mathbf{p}^* = \mathbf{p}_O^*$. (In general, of course, $\mathbf{p}^* \neq \mathbf{p}_O^*$.) Then, by (8.26) and (8.28d), $\mathbf{h}_{rO}^* = \mathbf{h}_O^* = -20mv(3\mathbf{j} + \mathbf{k})$.

More generally, (8.24) holds for a system consisting of only a single particle, hence for the center of mass particle alone,

$$\mathbf{h}_O^* = \mathbf{h}_{rO}^* + m(\beta)\mathbf{x}_O^* \times \mathbf{v}_O. \tag{8.28e}$$

Therefore, when $\mathbf{v}_O = \mathbf{0}$, we have the special result $\mathbf{h}_O^* = \mathbf{h}_{rO}^*$ found in the example. Characterization of other situations for which this holds is left for the reader.

8.6. The Moment of Momentum Principle

We shall now prove that the moment of momentum principle (6.79) for a particle extends to a system of particles. First, differentiate (5.32) with respect to time in Φ to obtain

$$\frac{d\mathbf{h}_O(\beta, t)}{dt} = \sum_{k=1}^{n} (\dot{\mathbf{x}}_{Ok} \times \mathbf{p}_k + \mathbf{x}_{Ok} \times \dot{\mathbf{p}}_k), \tag{8.29}$$

in which* $\mathbf{p}_k = m_k\dot{\mathbf{X}}_k = m_k\mathbf{v}_k$ is the momentum relative to the origin in Φ of the particle P_k at the place $\mathbf{x}_{Ok} = \mathbf{x}_k$ from an arbitrary point O in Fig. 8.2. If O is fixed in Φ, then $\dot{\mathbf{x}}_{Ok} = \dot{\mathbf{X}}_k = \mathbf{v}_k$, and the first product term in (8.29) vanishes. The total force on P_k is $\mathbf{F}_k = \mathbf{f}_k + \mathbf{b}_k = \dot{\mathbf{p}}_k$; hence, by (8.29), *for O fixed in Φ*,

$$\frac{d\mathbf{h}_O(\beta, t)}{dt} = \sum_{k=1}^{n} \mathbf{x}_{Ok} \times \mathbf{F}_k. \tag{8.30}$$

The right-hand side of (8.30) is the total moment about a fixed point O of all forces that act on the system of particles. We accept that all internal forces (8.1)

* Unless explicitly stated otherwise, the summation convention on repeated indices is suspended throughout this chapter. Summation is explicitly indicated by a summation sign.

occur in equal, oppositely directed and collinear pairs; hence, by (8.3), the total moment of the internal forces about any point O whatsoever is zero:

$$\sum_{k=1}^{n} \mathbf{x}_{Ok} \times \mathbf{b}_k = \sum_{\substack{j=1 \\ j\neq k}}^{n} \sum_{k=1}^{n} \mathbf{x}_{Ok} \times \mathbf{b}_{kj} = \mathbf{0}. \tag{8.31}$$

The right-hand side of (8.30), therefore, reduces to the total moment about a fixed point O of only the external forces that act on the system, written as

$$\mathbf{M}_O(\beta, t) \equiv \sum_{k=1}^{n} \mathbf{x}_{Ok} \times \mathbf{f}_k. \tag{8.32}$$

From (8.30) and (8.32), we obtain the following important principle.

The moment of momentum principle for a system of particles: *The total moment of the external forces about a fixed point O in an inertial frame Φ is equal to the time rate of change of the moment of momentum of the system about O:*

$$\mathbf{M}_O(\beta, t) = \frac{d\mathbf{h}_O(\beta, t)}{dt}. \tag{8.33}$$

This yields the following easy supplementary rule.

The principle of conservation of moment of momentum of a system of particles: *The total torque of external forces about a fixed line with direction* $\mathbf{e}$ *through O may vanish in Φ if and only if the corresponding component of the moment of momentum of the system about O is constant*:

$$\mathbf{M}_O(\beta, t) \cdot \mathbf{e} = 0 \Leftrightarrow \mathbf{h}_O(\beta, t) \cdot \mathbf{e} = \text{const.} \tag{8.34}$$

Moreover, the moment about O of the external forces vanishes if and only if the moment of momentum of the system about O is a constant vector.

Integration of (8.33) with respect to time leads to the torque–impulse, moment of momentum relation for the moment about a fixed point O in the inertial frame Φ. The result is similar to (7.15), and its instantaneous form is similar to (7.17); so, these equations are not repeated here.

8.6.1. Moment of Momentum Principle for a Moving Reference Point

The moment of momentum law (8.33) holds only for an arbitrary *fixed point* O in an inertial frame Φ; but use of a moving reference point often is essential and more practicable. Therefore, we seek those circumstances for which the moment of momentum principle in the form (8.33) may hold for a moving reference point. First, consider the moment about a moving point Q of all forces that act on the system, and recall from (8.31) that the torque of the mutual internal forces about

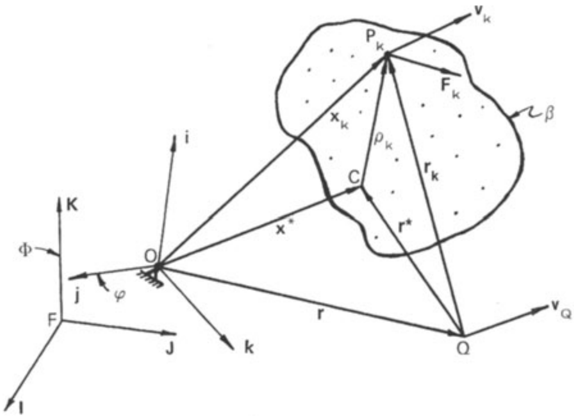


Figure 8.3. Schema for the moment of momentum about a moving reference point.

any point in Φ vanishes. Then using notation introduced in Fig. 8.3, we have

$$\mathbf{M}_Q = \sum_{k=1}^{n} \mathbf{r}_k \times \mathbf{F}_k = \sum_{k=1}^{n} \mathbf{r}_k \times \mathbf{f}_k = \sum_{k=1}^{n} \mathbf{r}_k \times \dot{\mathbf{p}}_k, \tag{8.35}$$

wherein[†] $\mathbf{F}_k = \mathbf{b}_k + \mathbf{f}_k = \dot{\mathbf{p}}_k$ and $\mathbf{r}_k$ is the vector of P_k from Q.

The moment about point Q of the momenta $\mathbf{p}_k = m_k \dot{\mathbf{x}}_k$ in Φ of all particles of the system is given by

$$\mathbf{h}_Q(\beta, t) = \sum_{k=1}^{n} \mathbf{r}_k \times \mathbf{p}_k, \tag{8.36}$$

and hence

$$\dot{\mathbf{h}}_Q = \sum_{k=1}^{n} \dot{\mathbf{r}}_k \times \mathbf{p}_k + \sum_{k=1}^{n} \mathbf{r}_k \times \dot{\mathbf{p}}_k. \tag{8.37}$$

Since $\mathbf{r}_k = \mathbf{x}_k - \mathbf{r}$ and $\dot{\mathbf{r}} = \mathbf{v}_Q$ in Fig. 8.3, the first product term yields

$$\sum_{k=1}^{n} \dot{\mathbf{r}}_k \times \mathbf{p}_k = -\mathbf{v}_Q \times \sum_{k=1}^{n} \mathbf{p}_k = -\mathbf{v}_Q \times \mathbf{p}^*.$$

Thus, with this relation in (8.37) and recalling (8.35), we obtain *the first form of the moment of momentum principle for a moving reference point* Q:

$$\mathbf{M}_Q(\beta, t) = \frac{d\mathbf{h}_Q(\beta, t)}{dt} + \mathbf{v}_Q \times \mathbf{p}^*. \tag{8.38}$$

Consequently, there exist moving points Q with respect to which $\mathbf{M}_Q(\beta, t) = \dot{\mathbf{h}}_Q(\beta, t)$ has the same form as (8.33) for a fixed point O, if and only if $\mathbf{v}_Q \times \mathbf{p}^* = \mathbf{0}$.

[†] Note that $\mathbf{X}_k = \mathbf{x}_k + \mathbf{R}$ is the vector of P_k in Φ and $\mathbf{R}$, not shown here, is the constant vector of the fixed point O from the origin F in Fig. 8.3. Hence, $\dot{\mathbf{X}}_k = \dot{\mathbf{x}}_k$, $\ddot{\mathbf{X}}_k = \ddot{\mathbf{x}}_k$ throughout these results. See also Fig. 8.2.

This holds when (i) trivially, either Q or the center of mass C is at rest in Φ, or (ii) when the velocity of Q is parallel to the velocity of the center of mass. In particular, this is so when Q is the center of mass. *Therefore, the moment about the center of mass of the external forces acting on a system of particles is equal to the time rate of change of the moment of momentum about the center of mass, which may be either at rest or moving arbitrarily in* Φ:

$$\mathbf{M}_C(\beta, t) = \frac{d\mathbf{h}_C(\beta, t)}{dt}. \tag{8.39}$$

This is the *first center of mass form of the moment of momentum principle.*

8.6.2. Second Form of the Moment of Momentum Principle for a Moving Point

Another formulation for the torque about a moving point Q, suggested by (8.22) and other results sketched in the previous section, is to express (8.38) in terms of *the moment of momentum relative to* Q, namely,

$$\mathbf{h}_{rQ}(\beta, t) = \sum_{k=1}^{n} \mathbf{r}_k \times m_k \dot{\mathbf{r}}_k, \tag{8.40}$$

in accordance with (8.23). Here $\mathbf{r}_k$ is the position vector of P_k from Q in Fig. 8.3. To relate (8.36) and (8.40), we recall (8.24) in which O is replaced by Q to obtain

$$\mathbf{h}_Q(\beta, t) = \mathbf{h}_{rQ}(\beta, t) + m(\beta)\mathbf{r}^* \times \mathbf{v}_Q, \tag{8.41}$$

wherein $\mathbf{r}^* = \mathbf{x}_Q^*$ is the position of the center of mass from Q.

Differentiation of (8.41) with respect to time in Φ gives

$$\dot{\mathbf{h}}_Q = \dot{\mathbf{h}}_{rQ} + m(\beta)\dot{\mathbf{r}}^* \times \mathbf{v}_Q + m(\beta)\mathbf{r}^* \times \mathbf{a}_Q,$$

in which $\mathbf{a}_Q = \dot{\mathbf{v}}_Q$ is the acceleration of Q in Φ. With $m(\beta)\dot{\mathbf{r}}^* = \mathbf{p}^* - m(\beta)\mathbf{v}_Q$ from Fig. 8.3, the last equation may be written as

$$\dot{\mathbf{h}}_Q + \mathbf{v}_Q \times \mathbf{p}^* = \dot{\mathbf{h}}_{rQ} + \mathbf{r}^* \times m(\beta)\mathbf{a}_Q. \tag{8.42}$$

Therefore, in place of (8.38), we find the *second form of the moment of momentum principle for a moving reference point* Q:

$$\mathbf{M}_Q(\beta, t) = \frac{d\mathbf{h}_{rQ}}{dt} + \mathbf{r}^* \times m(\beta)\mathbf{a}_Q. \tag{8.43}$$

Consequently, there exist points Q with respect to which $\mathbf{M}_Q(\beta, t) = \dot{\mathbf{h}}_{rQ}(\beta, t)$ has the same basic form (8.33) for a fixed point, if and only if $\mathbf{r}^* \times m(\beta)\mathbf{a}_Q = \mathbf{0}$. This occurs when (i) trivially, Q is either at rest or in uniform motion in Φ so that $\mathbf{a}_Q = \mathbf{0}$, in which case Q may be chosen as the origin of an inertial frame, (ii) the acceleration of Q is along a line passing through the center of mass so that $\mathbf{r}^*$ and $\mathbf{a}_Q$ are parallel vectors, or (iii) Q is the center of mass so that $\mathbf{r}^* = \mathbf{0}$, this being the most general of these situations for a moving reference point. *Therefore, the moment about the center of mass of the external* *forces acting on a system of particles is equal to the time rate of change of the moment of momentum relative to the center of mass, which may be either at ease, in uniform motion, or moving arbitrarily in* Φ:

$$\mathbf{M}_C(\beta, t) = \frac{d\mathbf{h}_{rC}(\beta, t)}{dt}. \tag{8.44}$$

This is the *second center of mass form of the moment of momentum principle*.

8.6.3. Summary: The Moment of Momentum Principle for a System of Particles

For an arbitrary moving reference point Q, the moment of momentum principle in (8.38) or (8.43) must be used. In view of (8.22), however, equations (8.39) and (8.44) are equivalent, and hence *the simplest formulation of the moment of momentum principle for a moving reference point is provided by the equation for the moving center of mass*:

$$\mathbf{M}_C(\beta, t) = \frac{d\mathbf{h}_C(\beta, t)}{dt} = \frac{d\mathbf{h}_{rC}(\beta, t)}{dt}. \tag{8.45}$$

Otherwise, for any point Q that either is fixed or has a uniform motion in an inertial frame Φ,

$$\mathbf{M}_Q(\beta, t) = d\mathbf{h}_{rQ}(\beta, t)/dt. \tag{8.46}$$

In the latter case, Q may be chosen as a fixed point at the origin of a new inertial frame Φ' with respect to which, trivially, $\dot{\mathbf{h}}_{rQ} = \dot{\mathbf{h}}_Q$.

The first law of motion (8.5) for a system of particles essentially determines the motion of the center of mass of the system, and the second law of motion (8.45) determines the motion of the system relative to the center of mass. In addition, however, we must bear in mind in applications that the moment of momentum about C may be referred to a moving frame φ having an angular velocity $\boldsymbol{\omega}_f$ relative to the inertial frame Φ. In this case (see (4.11) in Volume 1) $\mathbf{h}_C = \mathbf{h}_{rC}$ is a vector referred to a moving reference frame, and (8.45) is written as

$$\mathbf{M}_C(\beta, t) = \frac{\delta\mathbf{h}_C(\beta, t)}{\delta t} + \boldsymbol{\omega}_f \times \mathbf{h}_C(\beta, t). \tag{8.47}$$

Two examples that illustrate use of the results (8.22), (8.45), and (8.47) follow.

Example 8.3. A communications van has an antenna system modeled in Fig. 8.4 as two coils of equal mass m that move radially along a rigid control shaft that rotates with angular speed ω about the vertical antenna axis. At an instant of interest, each coil is at a distance d from the center C and is moving with center directed variable speed v relative to the shaft frame $\varphi = \{C; \mathbf{i}_k\}$. The van moves with speed v_O in the ground frame $\Phi = \{F; \mathbf{I}_k\}$. (i) What is the total momentum of the system? (ii) What is the moment of momentum of the system relative to the

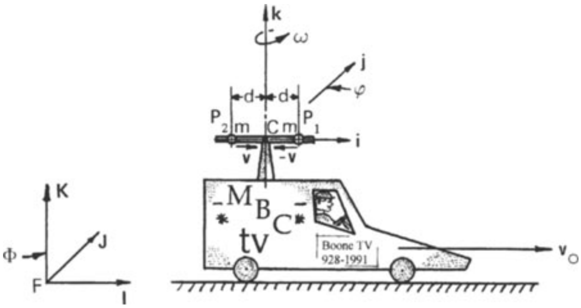


Figure 8.4. A two particle system model of moving antenna coils.

center of mass? (iii) Apply (8.19) to determine the moment of momentum about the center of mass. Ignore the mass of the shaft and model the coils as particles.

Solution of (i). The total momentum in Φ of the system of coils P_1 and P_2 is equal to the momentum of its center of mass whose velocity in Φ is $\mathbf{v}^* = \mathbf{v}_O = v_O \mathbf{I}$, the velocity of the van. Hence, $\mathbf{p}^* = m(\mathscr{B})\mathbf{v}^* = 2mv_O\mathbf{I}$.

Solution of (ii). The moment of momentum of the system relative to the center of mass is determined by (8.22). The respective relative position vectors of P_1 and P_2 from C are

$$\boldsymbol{\rho}_1 = -\boldsymbol{\rho}_2 = d\mathbf{i}. \tag{8.48a}$$

The angular velocity of the shaft frame φ is $\boldsymbol{\omega}_f = \omega\mathbf{k}$, and hence their velocity vectors relative to the center of mass at the instant of interest are

$$\dot{\boldsymbol{\rho}}_1 = -\dot{\boldsymbol{\rho}}_2 = -\mathbf{v} + \boldsymbol{\omega}_f \times \mathbf{d} = -v\mathbf{i} + \omega d\mathbf{j}. \tag{8.48b}$$

Use of (8.48a) and (8.48b) in (8.22) yields

$$\mathbf{h}_{rC} = \boldsymbol{\rho}_1 \times m_1\dot{\boldsymbol{\rho}}_1 + \boldsymbol{\rho}_2 \times m_2\dot{\boldsymbol{\rho}}_2 = 2m\boldsymbol{\rho}_1 \times \dot{\boldsymbol{\rho}}_1 = 2md\mathbf{i} \times (\omega d\mathbf{j} - v\mathbf{i}).$$

That is,

$$\mathbf{h}_{rC} = 2md^2\omega\mathbf{k}. \tag{8.48c}$$

Solution of (iii). To find the moment about C of the momentum relative to the origin in Φ, we shall need the total velocity of each particle in the inertial frame Φ, namely, $\dot{\mathbf{X}}_k = \mathbf{v}_O + \dot{\boldsymbol{\rho}}_k$. Then, with (8.48a) and (8.48b), (8.19) gives

$$\mathbf{h}_C = \boldsymbol{\rho}_1 \times m_1(\mathbf{v}_O + \dot{\boldsymbol{\rho}}_1) + \boldsymbol{\rho}_2 \times m_2(\mathbf{v}_O + \dot{\boldsymbol{\rho}}_2) = 2\boldsymbol{\rho}_1 \times m_1\dot{\boldsymbol{\rho}}_1 = \mathbf{h}_{rC},$$

in agreement with the general rule (8.22): $\mathbf{h}_C = \mathbf{h}_{rC} = 2md^2\omega\mathbf{k}$. Notice in passing that the velocity of the center of mass $\mathbf{v}^* = \mathbf{v}_O$ does not affect the final result. In view of (5.6), the term $(m_1\boldsymbol{\rho}_1 + m_2\boldsymbol{\rho}_2) \times \mathbf{v}_O \equiv \mathbf{0}$. □

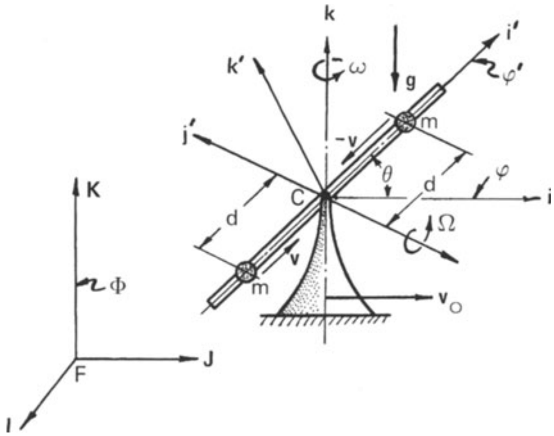


Figure 8.5. Moment of momentum of a system of antenna coils referred to a moving frame.

Example 8.4. The antenna coil system in the previous example has an additional angular velocity $\boldsymbol{\Omega}$ normal to the plane of $\mathbf{i}'$ and $\mathbf{k}$ in Fig. 8.5, and relative to its initially oriented shaft frame $1 = \varphi = \{C; \mathbf{i}_k\}$, which is turning with angular velocity $\boldsymbol{\omega}$ relative to the ground frame $0 = \Phi = \{F; \mathbf{I}_k\}$ at the instant shown. Find the applied torque about the center of mass required to sustain the motion of the system referred to the shaft frame $2 = \varphi' = \{C; \mathbf{i}'_k\}$. Ignore the mass of the control shaft.

Solution. The torque about C is given by (8.45), so we must first find $\mathbf{h}_C$ in (8.22) referred to the moving frame. The total angular velocity of the moving frame $2 = \varphi' = \left\{C; \mathbf{i}'_k\right\}$ fixed in the control shaft is $\boldsymbol{\omega}_f \equiv \boldsymbol{\omega}_{20} = \boldsymbol{\omega}_{21} + \boldsymbol{\omega}_{10} = \boldsymbol{\Omega} + \boldsymbol{\omega}$. Hence, with reference to Fig. 8.5, referred to φ',

$$\boldsymbol{\omega}_f = -\Omega\mathbf{j}' + \omega(\sin\theta\mathbf{i}' + \cos\theta\mathbf{k}'). \tag{8.49a}$$

The velocity of each coil relative to C is $\dot{\boldsymbol{\rho}}_k = (-1)^k\mathbf{v} + \boldsymbol{\omega}_f \times \boldsymbol{\rho}_k$, where we recall (8.48a) in which $\mathbf{i} \to \mathbf{i}'$. Specifically,

$$\dot{\boldsymbol{\rho}}_1 = -\dot{\boldsymbol{\rho}}_2 = -v\mathbf{i}' + \omega d\cos\theta\mathbf{j}' + \Omega d\mathbf{k}'. \tag{8.49b}$$

Then (8.22) yields

$$\mathbf{h}_C = 2\boldsymbol{\rho}_1 \times m\dot{\boldsymbol{\rho}}_1 = 2md^2(-\Omega\mathbf{j}' + \omega\cos\theta\mathbf{k}'). \tag{8.49c}$$

When $\theta = 0$ and $\Omega = 0$, we recover (8.48c).

The total torque about C required to sustain the motion of the system is determined by (8.45) for $\mathbf{h}_C$ given in (8.49c). But $\mathbf{h}_C$ is a vector referred to a moving reference frame, so we shall need to apply (8.47). With the aid of (8.49a), (8.49c), and noting that $\dot{d}(t) = -v(t)$ and $\dot{\theta} = \Omega$, we determine

$$\frac{\delta \mathbf{h}_C}{\delta t} = 2md[(2v\Omega - d\dot{\Omega})\mathbf{j}' + (d\dot{\omega}\cos\theta - d\omega\Omega\sin\theta - 2v\omega\cos\theta)\mathbf{k}'],$$

$$\boldsymbol{\omega}_f \times \mathbf{h}_C = 2md^2 \begin{vmatrix} \mathbf{i}' & \mathbf{j}' & \mathbf{k}' \\ \omega\sin\theta & -\Omega & \omega\cos\theta \\ 0 & -\Omega & \omega\cos\theta \end{vmatrix}$$
$$= 2md^2\,(-\omega^2\sin\theta\cos\theta\,\mathbf{j}' - \Omega\omega\sin\theta\,\mathbf{k}').$$

Thus, by (8.47), the total moment about C of all external forces exerted on the coil system by the control shaft, by gravity, and by the drive mechanism is

$$\mathbf{M}_C = 2md[(2v\Omega - d\dot{\Omega} - d\omega^2\sin\theta\cos\theta)\mathbf{j}' + (d\dot{\omega}\cos\theta - 2d\Omega\omega\sin\theta - 2v\omega\cos\theta)\mathbf{k}']. \tag{8.49d}$$

When $\Omega = 0$ and $\theta = 0$, the applied torque required to sustain the motion of the system considered initially is $\mathbf{M}_C = 2md(d\dot{\omega} - 2v\omega)\mathbf{k}$, which also follows easily from (8.45) and (8.48c) wherein now $\mathbf{k}' = \mathbf{k}$ is fixed in Φ. □

8.7. Kinetic Energy of a System of Particles

The kinetic energy $K(\beta, t)$ of a system of particles $\beta = \{P_k\}$ in frame $\varphi = \{O; \mathbf{i}_k\}$ of Fig. 8.3 is defined as the sum of the kinetic energies $K_k(t) \equiv K(P_k, t)$ of particles P_k:

$$K(\beta, t) \equiv \sum_{k=1}^{n} K_k(t) = \sum_{k=1}^{n} \frac{1}{2} m_k \mathbf{v}_k \cdot \mathbf{v}_k, \tag{8.50}$$

where $\mathbf{v}_k = \dot{\mathbf{x}}_k$. With $m(\beta)$ defined by (5.3), *the kinetic energy $K^*(\beta, t)$ of the center of mass* is defined by

$$K^*(\beta, t) \equiv \frac{1}{2} m(\beta)\mathbf{v}^* \cdot \mathbf{v}^*, \tag{8.51}$$

wherein $\mathbf{v}^*(\beta, t) = \dot{\mathbf{x}}^*(\beta, t)$ is the velocity of the center of mass of the system in φ.

To relate (8.50) and (8.51), with reference to Fig. 8.3, substitute the relation $\mathbf{v}_k = \mathbf{v}^* + \dot{\boldsymbol{\rho}}_k$ in (8.50) and expand the result to obtain

$$K(\beta, t) = \frac{1}{2} m(\beta)\mathbf{v}^* \cdot \mathbf{v}^* + \mathbf{v}^* \cdot \sum_{k=1}^{n} m_k \dot{\boldsymbol{\rho}}_k + \sum_{k=1}^{n} \frac{1}{2} m_k \dot{\boldsymbol{\rho}}_k \cdot \dot{\boldsymbol{\rho}}_k.$$

Then, by (5.8), the second term vanishes and the last term is *the kinetic energy of the system relative to the center of mass C*, defined by

$$K_{rC}(\beta, t) \equiv \sum_{k=1}^{n} \frac{1}{2} m_k \dot{\boldsymbol{\rho}}_k \cdot \dot{\boldsymbol{\rho}}_k. \tag{8.52}$$

Hence, with (8.51), we obtain

$$K(\beta, t) = K^*(\beta, t) + K_{rC}(\beta, t). \tag{8.53}$$

That is, *the kinetic energy of a system of particles is equal to the kinetic energy of the center of mass plus the kinetic energy of the system relative to the center mass.*

Equations (8.51) and (8.52), therefore, are two independent kinetic energy relations for a system of particles, and (8.53) is the decomposition of the total kinetic energy (8.50) into these independent parts. In the two body problem, for example, the kinetic energy of the center of mass in the inertial frame Ψ at C is zero, and the reader will find from (8.52) that the kinetic energy relative to C is given by $K_{rC} = \frac{1}{2}m_r\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{1}{2}m_r(\dot{r}^2 + \gamma^2/r^2)$, which by (8.53) also is the total kinetic energy of the system in Ψ. This differs from the kinetic energy in (7.96a) for the one body problem. The results are approximately the same only when the mass of one body is much greater than that of the other, say $m_1 \gg m_2 = m$, so that by (8.13) $m_r \approx m$.

Example 8.5. Find the kinetic energy of the antenna system in Example 8.3, page 314.

Solution. The center of mass C of the two coil system has velocity $\mathbf{v}^* = \mathbf{v}_O$, and $m(\beta) = 2m$; so, by (8.51), the kinetic energy of the center of mass is

$$K^*(\beta, t) = mv_O^2. \tag{8.54a}$$

The velocity of each coil relative to C is given in (8.48b); therefore, by (8.52), the kinetic energy of the system relative to C is

$$K_{rC} = \frac{1}{2}m(\dot{\boldsymbol{\rho}}_1 \cdot \dot{\boldsymbol{\rho}}_1 + \dot{\boldsymbol{\rho}}_2 \cdot \dot{\boldsymbol{\rho}}_2) = m(v^2 + \omega^2 d^2). \tag{8.54b}$$

Finally, (8.53) yields the kinetic energy of the antenna coil system:

$$K(\beta, t) = m\big(v_O^2 + v^2 + \omega^2 d^2\big). \tag{8.54c}$$

The reader will find the same result on starting from (8.50). □

Exercise 8.4. What is the kinetic energy of the antenna coil system in Example 8.4? □

8.8. Work–Energy Equations for a System of Particles

The total external force acting on a system of particles may be considered to act on the center of mass particle whose motion is governed by (8.5) and from which a work–energy equation follows as a first integral. Let us think of the total external force $\mathbf{F}(\beta, t) = \mathbf{F}(\mathbf{x}^*)$ as varying only with the position of the center of

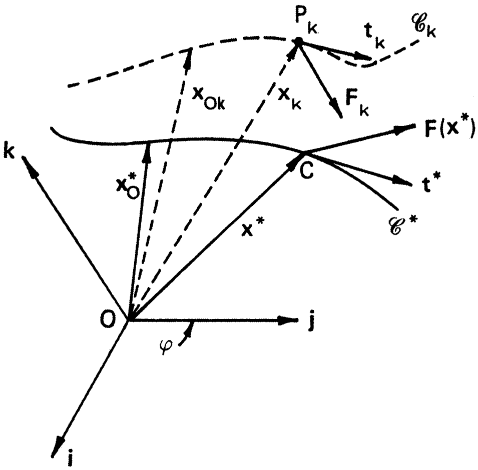


Figure 8.6. Schema for the work done by forces acting on a system of particles.

mass along its path $\mathscr{C}^*$ at time t in the inertial frame φ in Fig. 8.6. Then by (7.21) this force does work

$$\mathscr{W}^* \equiv \int_{\mathscr{C}^*} \mathbf{F}(\mathbf{x}^*) \cdot d\mathbf{x}^* = \int_{t_0}^{t} \mathbf{F}(\mathbf{x}^*) \cdot \mathbf{v}^* dt, \tag{8.55}$$

where t_0 and t are the instants when the center of mass is at its respective end states $\mathbf{x}_0^*$ and $\mathbf{x}^*$ on $\mathscr{C}^*$, $d\mathbf{x}^*$ is the elemental displacement vector tangent to $\mathscr{C}^*$, and $\mathbf{v}^* = \dot{\mathbf{x}}^*$. We recall (7.34) applied to the center of mass particle, use (8.5), and thus obtain from (8.55) the *work–energy equation for the center of mass*:

$$\mathscr{W}^* = \Delta K^*, \tag{8.56}$$

where $\Delta K^* = K^*(\beta, t) - K^*(\beta, t_0)$ is the change in the kinetic energy (8.51) of the center of mass during the time $[t_0, t]$. In sum, formally, *the work done by the total external force that acts at the center of mass of a system of particles is equal to the change in the kinetic energy of the center of mass.*

Moreover, similarly, by (7.37), (7.38), and (8.56), formally, *the mechanical power $\mathscr{P}^*$ expended by the total external force acting at the center of mass of a system of particles is equal to the time rate of change of the kinetic energy of the center of mass*:

$$\mathscr{P}^* \equiv \frac{d\mathscr{W}^*}{dt} = \frac{dK^*}{dt}. \tag{8.57}$$

The results (8.56) and (8.57) hinge on our writing the total external force as a function of the motion $\mathbf{x}^*$ of the center of mass particle in (8.55). Generally, however, this cannot be done. Nevertheless, in view of (8.5) and because these results are expressed in terms of the kinetic energy of the center of mass, they are meaningful—the work $\mathscr{W}^*$ and the power $\mathscr{P}^*$ are determined by the kinetic energy K^* of the unique center of mass particle. The work $\mathscr{W}^*$, however, is *not* the total work done on the system. Because the internal forces $\mathbf{b}_k$ act over paths $\mathscr{C}_k$ traversed by the individual particles P_k, these forces generally contribute to the work done by the total force $\mathbf{F}_k = \mathbf{f}_k + \mathbf{b}_k$ on P_k in Fig. 8.6. It is assumed that each of the forces depends on only the position $\mathbf{x}_k$ of the respective particle P_k. Hence, by (7.21), the work $\mathscr{W}_k$ done on the particle P_k in φ is

$$\mathscr{W}_k \equiv \int_{\mathscr{C}_k} \mathbf{F}_k \cdot d\mathbf{x}_k = \int_{t_0}^{t} (\mathbf{f}_k + \mathbf{b}_k) \cdot d\mathbf{x}_k = \Delta K_k, \tag{8.58}$$

wherein t_0 and t, respectively, are the instants when the particle P_k is at the end points $\mathbf{x}_{0k}$ and $\mathbf{x}_k$ on its path $\mathscr{C}_k$. Of course, all of the individual particle paths $\mathscr{C}_k$ and the path $\mathscr{C}^*$ in Fig. 8.6 generally are different; but the interval $[t_0, t]$ applies to the motion of every particle and to the center of mass of the system. Therefore, the total work $\mathscr{W} \equiv \Sigma_{k=1}^{n} \mathscr{W}_k$ done by all forces that act on the system is given by

$$\mathscr{W} \equiv \sum_{k=1}^{n} \int_{\mathscr{C}_k} \mathbf{F}_k \cdot d\mathbf{x}_k = \sum_{k=1}^{n} \int_{t_0}^{t} \mathbf{f}_k \cdot d\mathbf{x}_k + \sum_{k=1}^{n} \int_{t_0}^{t} \mathbf{b}_k \cdot d\mathbf{x}_k. \tag{8.59}$$

Recalling (8.50) for the total kinetic energy, with (8.58), we have *the work–energy equation for the system*:

$$\mathscr{W} = \Delta K, \tag{8.60}$$

where $\Delta K = \Delta \Sigma_{k=1}^{n} K_k$. *Hence, the total work done by all forces acting on a system of particles is equal to the change in the total kinetic energy of the system.*

Introduce $d\mathbf{x}_k = \mathbf{v}_k dt = (\mathbf{v}^* + \dot{\boldsymbol{\rho}}_k)dt$ in (8.59) to write the total work done as

$$\mathscr{W} = \int_{t_0}^{t} \sum_{k=1}^{n} \mathbf{F}_k \cdot d\mathbf{x}_k = \int_{t_0}^{t} \left(\sum_{k=1}^{n} \mathbf{F}_k \cdot \mathbf{v}^* + \sum_{k=1}^{n} \mathbf{F}_k \cdot \dot{\boldsymbol{\rho}}_k \right) dt.$$

In view of (8.4), the first term on the far right-hand side of this expression is equivalent to (8.55) for the total external force; therefore, with (8.56), (8.60) may be written as

$$\mathscr{W} - \mathscr{W}^* = \Delta(K - K^*) = \int_{t_0}^{t} \sum_{k=1}^{n} \mathbf{F}_k \cdot \dot{\boldsymbol{\rho}}_k dt. \tag{8.61}$$

Then, with (8.53) and K_{rC} in (8.52), *the work–energy equation relative to the center of mass* is

$$\mathscr{W}_{rC} \equiv \sum_{k=1}^{n} \int_{\mathscr{C}_k} \mathbf{F}_k \cdot d\boldsymbol{\rho}_k = \int_{t_0}^{t} \sum_{k=1}^{n} \mathbf{F}_k \cdot \dot{\boldsymbol{\rho}}_k dt = \Delta K_{rC}. \tag{8.62}$$

Therefore, the work $\mathscr{W}_{rC}$ done by all forces acting on a system of particles in motion relative to the center of mass is equal to the change in the total kinetic energy of the system relative to the center of mass.

To conclude, *the total work done by all forces acting on a system of particles is equal to the total of the work done by external forces acting at the center of mass* *and the work done by all forces in the motion relative to the center of mass*:

$$\mathscr{W} = \mathscr{W}^* + \mathscr{W}_{rC}. \tag{8.63}$$

Of course, $\mathscr{P}^*$ in (8.57) is not the total mechanical power expended; rather, the total power $\mathscr{P} = d\mathscr{W}/dt = dK/dt$ is easily seen to be $\mathscr{P} = \mathscr{P}^* + \mathscr{P}_{rC}$, in which $\mathscr{P}_{rC} = d\mathscr{W}_{rC}/dt = dK_{rC}/dt$.

Exercise 8.5. (a) Derive (8.56) as a formal first integral of (8.5). (b) Introduce the equation of motion for the k^{th} particle in the inertial frame φ, observe from Fig. 8.6 that $\mathbf{x}_k = \mathbf{x}^* + \boldsymbol{\rho}_k$, and thus confirm (8.62). □

Equations (8.56) and (8.62) are two independent work–energy equations for a system of particles; the first is related to the motion of the center of mass and is influenced by external forces only, whereas the second is related to the motion of the system relative to the center of mass and is influenced by both external and internal forces. The decomposition (8.63) of the total work (8.60) into these independent parts parallels the decomposition (8.53) of the total kinetic energy into corresponding independent parts; and, of course, the decomposition of the total mechanical power is similar.

A rigid system of particles is an important special case for which the distances between all pairs of particles are constant. Moreover, for a rigid system of particles, $\mathbf{v}_k = \mathbf{v}^* + \boldsymbol{\omega} \times \boldsymbol{\rho}_k$; and hence for mutual internal forces for which $\mathbf{b}_{ij} = -\mathbf{b}_{ji}$, use of (8.1) and (8.31), which holds for any point O and hence for C, yield

$$\sum_{k=1}^{n} \int_{\mathscr{C}_k} \mathbf{b}_k \cdot d\mathbf{x}_k = \int_{t_0}^{t} \left(\sum_{k=1}^{n} \mathbf{b}_k \cdot \mathbf{v}^* + \boldsymbol{\omega} \cdot \sum_{k=1}^{n} \boldsymbol{\rho}_k \times \mathbf{b}_k \right) dt = 0. \tag{8.64}$$

Therefore, mutual internal forces do no total work in any motion of a rigid system. *Consequently, for a rigid system of particles only the external forces contribute to the total work done.* Now, with (8.59), (8.60) may be written as

$$\mathscr{W} = \sum_{k=1}^{n} \int_{\mathscr{C}_k} \mathbf{f}_k \cdot d\mathbf{x}_k = \int_{t_0}^{t} \sum_{k=1}^{n} \mathbf{f}_k \cdot \dot{\mathbf{x}}_k dt = \Delta K. \tag{8.65}$$

Hence, the total work done by external forces acting on a rigid system of particles is equal to the change in the total kinetic energy of the system. Moreover, it follows from (8.62) that

$$\mathscr{W}_{rC} = \sum_{k=1}^{n} \int_{\mathscr{C}_k} \mathbf{f}_k \cdot d\boldsymbol{\rho}_k = \int_{t_0}^{t} \sum_{k=1}^{n} \mathbf{f}_k \cdot \dot{\boldsymbol{\rho}}_k dt = \Delta K_{rC}. \tag{8.66}$$

That is, *the work done by external forces acting on a rigid system of particles in motion relative to the center of mass is equal to the change in the total kinetic energy of the system relative to the center of mass.* Plainly, for a rigid system of

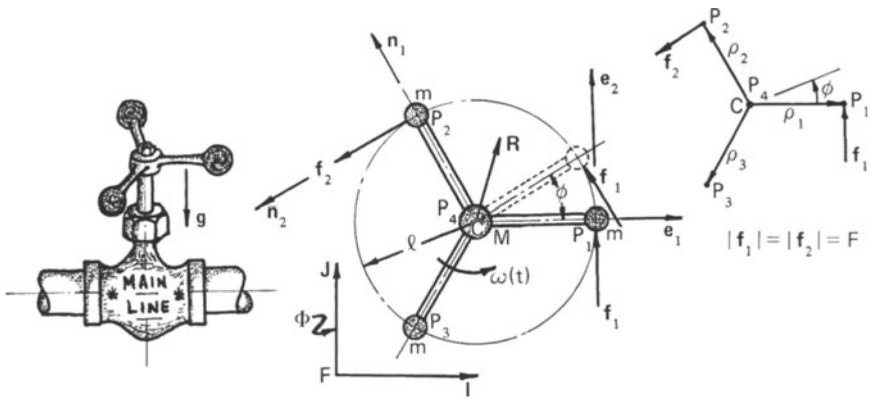


Figure 8.7. Forces acting on a rigid system of three particles.

particles, the total work done by external forces and the total kinetic energy in (8.65) may be decomposed in accordance with (8.63) and (8.53), respectively; and the total mechanical power expended, the rate of working of the external forces only, may be similarly decomposed.

Exercise 8.6. Show that (8.66) follows from (8.62). □

Example 8.6. A pipeline valve handle consists of three equally spaced handle grips of equal mass m attached to the valve body of mass M by thin rigid torque bars of equal length ℓ and negligible mass. The handle, initially at rest, is turned by forces $\mathbf{f}_1$ and $\mathbf{f}_2$ of equal constant magnitude F applied at two grips in the plane of, and perpendicular to the torque bars, as shown in Fig. 8.7. The handle turns freely without friction. (a) Apply the work–energy principle to find as a function of time the angular speed $\omega(t)$ of the handle. (b) Derive the same result by use of the moment of momentum principle.

Solution of (a). We model the handle as a rigid system of three particles of equal mass m attached by massless rigid rods to the valve body, which is a particle of mass M at C, the center of mass of the system. Since the handle turns freely without friction at C, the reaction force of the valve body is equipollent to a single force $\mathbf{R}$ at C that does no work in the motion; and, of course, the normal gravitational force also is workless. The work done by the remaining applied external forces, relative to the center of mass, is determined by the first equation in (8.66), and hence, with reference to Fig. 8.7,

$$\mathscr{W}_{rC} = \int_{\mathscr{C}_1} \mathbf{f}_1 \cdot d\boldsymbol{\rho}_1 + \int_{\mathscr{C}_2} \mathbf{f}_2 \cdot d\boldsymbol{\rho}_2 = \int_0^{\phi} F\mathbf{e}_2 \cdot \ell d\phi \mathbf{e}_2 + \int_0^{\phi} F\mathbf{n}_2 \cdot \ell d\phi \mathbf{n}_2.$$

Hence,

$$\mathscr{W}_{rC} = 2F\ell\phi. \tag{8.67a}$$

Notice that since C is fixed in Φ, $\mathscr{W}^* = 0$ in (8.55); hence, by (8.63), $\mathscr{W} = \mathscr{W}_{rC}$ also is the total work done on the rigid system.

Since each of the three handle particles has the same speed $|\dot{\rho}_1| = \ell\dot{\phi}$ relative to C, and because the system is at rest initially, the change in kinetic energy (8.52) relative to the center of mass is

$$\Delta K_{rC} = \frac{3}{2}m\ell^2\dot{\phi}^2. \tag{8.67b}$$

Use of (8.67a) and (8.67b) in the work–energy equation (8.66) thus yields

$$\omega(\phi) = \dot{\phi} = \sqrt{\frac{4F\phi}{3m\ell}}. \tag{8.67c}$$

To determine $\omega(t)$ as a function of time, we integrate this equation with $\phi(0) = 0$ initially to obtain the angular placement

$$\phi(t) = \alpha t^2, \qquad \text{with} \quad \alpha \equiv \frac{F}{3m\ell}. \tag{8.67d}$$

Now (8.67c) yields the desired result:

$$\omega(t) = 2\alpha t. \tag{8.67e}$$

Solution of (b). The same result may be obtained from the principle of moment of momentum relative to the center of mass in (8.45). First, we note that the applied torque about C is $\mathbf{M}_C = \boldsymbol{\rho}_1 \times \mathbf{f}_1 + \boldsymbol{\rho}_2 \times \mathbf{f}_2 = 2F\ell\mathbf{k}$. The moment about C of the momenta relative to C is $\mathbf{h}_{rC} = \boldsymbol{\rho}_1 \times m_1\dot{\boldsymbol{\rho}}_1 + \boldsymbol{\rho}_2 \times m_2\dot{\boldsymbol{\rho}}_2 + \boldsymbol{\rho}_3 \times m_3\dot{\boldsymbol{\rho}}_3 = 3m\ell^2\dot{\phi}\mathbf{k}$, and $d\mathbf{h}_{rC}/dt = 3m\ell^2\,\ddot{\phi}\,\mathbf{k}$. Hence, (8.45) yields $2F\ell\mathbf{k} = 3m\ell^2\,\ddot{\phi}\,\mathbf{k}$; that is,

$$\ddot{\phi} = \frac{2F}{3m\ell} = 2\alpha. \tag{8.67f}$$

Integration with respect to time, with $\dot{\phi}(0) = 0$ initially, returns (8.67e). □

Exercise 8.7. Determine the valve body reaction force $\mathbf{R}$ in the plane frame $\{\mathbf{P}_1; \mathbf{e}_k\}$. What is its magnitude? □

8.9. The Principle of Conservation of Energy

Suppose that for a system of particles the total external force (8.4) is conservative with potential energy $V^* = V^*(\mathbf{x}^*)$ depending only on the position $\mathbf{x}^*$ of the center of mass. Then the total external force is given by

$$\mathbf{F}(\beta, t) = -\boldsymbol{\nabla} V^*, \tag{8.68}$$

where $\boldsymbol{\nabla} \equiv \partial/\partial \mathbf{x}^*$, and hence (8.55) may be integrated to yield $\mathscr{W}^* = -\Delta V^*$. It then follows from (8.56) that

$$K^* + V^* = E^*, \text{ a constant.} \tag{8.69}$$

But this is a very weak and superficial principle of conservation of energy. In the first place, it applies only to a conservative **total** external force expressed as a function of $\mathbf{x}^*$, and it is unlikely that (8.4) will admit such a representation. Moreover, E^* is only the total energy of the center of mass, not the total energy of the system. Internal forces are not involved, and (8.69) suggests that the total energy of the center of mass may be conserved even in the presence of dissipative or other nonconservative internal forces. Of course, both workless external and internal forces also might be present. So, we discard (8.69) and seek a more substantial and meaningful principle of conservation of energy. If both the external and internal forces that act on every particle of a system are conservative, the system is called *conservative*; otherwise, it is called *nonconservative*. With this in mind, a useful general energy principle for a system of particles is developed.

8.9.1. External Potential Energy

Let $\phi_k(\mathbf{x}_k)$ be the *external potential energy* of the particle P_k due to the conservative external force $\mathbf{f}_k$. This potential function depends only on the position $\mathbf{x}_k$ of the particle P_k on which the force acts;

$$\mathbf{f}_k = -\boldsymbol{\nabla}_k \phi_k(\mathbf{x}_k), \qquad k = 1, 2, \ldots, n, \tag{8.70}$$

wherein $\boldsymbol{\nabla}_k \equiv \partial/\partial \mathbf{x}_k$. Thus, the total work done by a conservative system of external forces acting on all n particles of the system is

$$\sum_{k=1}^{n} \int_{t_0}^{t} \mathbf{f}_k \cdot d\mathbf{x}_k = -\sum_{k=1}^{n} \int_{t_0}^{t} \boldsymbol{\nabla}_k \phi_k \cdot d\mathbf{x}_k = -\Delta \Phi(\beta), \tag{8.71}$$

wherein each integrand is an exact differential $d\phi_k(\mathbf{x}_k)$, and by definition,

$$\Phi(\beta) \equiv \sum_{k=1}^{n} \phi_k(\mathbf{x}_k), \tag{8.72}$$

is the *total external potential energy of the system.* The function $\Phi(\beta) \equiv \Phi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ depends on the positions of all particles of the system on which external forces act along diverse particle paths having end points $\mathbf{x}_{Ok}$ and $\mathbf{x}_k$, the time interval $[t_0, t]$ being the same for all paths. Also, in (8.71), $\Delta\Phi(\beta) \equiv \sum_{k=1}^{n} \phi_k(\mathbf{x}_k) - \phi_k(\mathbf{x}_{Ok}) = \sum_{k=1}^{n} \Delta\phi_k$.

8.9.2. Internal Potential Energy

Let $\beta_{jk} = \psi(\mathbf{x}_j, \mathbf{x}_k)$ denote the *internal potential energy of a particle P_j due to a conservative mutual internal force $\mathbf{b}_{jk}$ exerted on P_j by P_k*. This function depends on the positions of both particles as emphasized by the subscripts, and $\psi(\mathbf{x}_j, \mathbf{x}_k)$ generally may be a different function for each pair of particles. In general, then, for a fixed k, the conservative internal force on P_j is given by

$$\mathbf{b}_{jk} = -\boldsymbol{\nabla}_j \beta_{jk}, \tag{8.73}$$

where for each position $\mathbf{x}_j$, in Cartesian coordinates,

$$\boldsymbol{\nabla}_j \equiv \partial/\partial \mathbf{x}_j = \sum_{p=1}^{3} \mathbf{i}_p \partial/\partial x_p^j. \tag{8.74}$$

It was proved in Section 5.18.3, page 81, that the mutual internal force $\mathbf{b}_{jk}$ between any pair of discrete material points P_j and P_k, which depends only on their distinct positions $\mathbf{x}_j$ and $\mathbf{x}_k$, has a magnitude that depends only on the distance $r = |\mathbf{r}_{jk}|$ between them and is directed along their common straight line. Therefore, for a conservative internal force, the mutual internal potential energy $\psi(\mathbf{x}_j, \mathbf{x}_k)$ for an arbitrary pair of interacting particles must be a function of only the magnitude of the vector $\mathbf{r}_{jk} \equiv \mathbf{x}_j - \mathbf{x}_k$ joining the two particles, namely, $\psi(\mathbf{x}_j, \mathbf{x}_k) = \psi(|\mathbf{r}_{jk}|)$. Bearing in mind that the function $\psi(|\mathbf{r}_{jk}|)$ generally may be a different function for each pair of particles, we have[‡]

$$\beta_{jk} = \psi(|\mathbf{r}_{jk}|). \tag{8.75}$$

Conversely, the reader will find that when the internal potential energy (8.75) of a pair of particles P_j and P_k is a function only of the distance $|\mathbf{r}_{jk}|$ between them, the mutual force $\mathbf{b}_{jk}$ must be parallel to $\mathbf{r}_{jk}$, hence directed along their mutual line. *Therefore, the internal potential energy for each pair of particles is a function of only the distance between the particles if and only if their mutual internal force is directed along their common line.* This kind of mutual action occurs in most interactions in nature, but not all. The general validity of this rule fails for molecular, atomic, electron, proton, or other elementary particle force interactions for which the internal potential energy does *not* depend on r alone.

Exercise 8.8. Show that $\boldsymbol{\nabla}(\mathbf{x} \cdot \mathbf{x}) = 2\mathbf{x}$, and hence demonstrate that $\boldsymbol{\nabla}(\sqrt{\mathbf{x} \cdot \mathbf{x}})$ is a unit vector parallel to $\mathbf{x}$. Apply this rule to prove the converse result stated above and confirm (8.77) and (8.78) below. Recall that $\boldsymbol{\nabla} \equiv \partial/\partial \mathbf{x} = \sum_{p=1}^{3} \mathbf{i}_p \partial/\partial x_p$. □

It is evident that two particles P_j and P_k share the same internal potential energy, because the internal potential energy function is symmetric with respect

[‡] An alternative but weaker proof of (8.75) that does not appeal to frame indifference is given in Appendix C.

to an interchange of the particles, namely, $\psi(|\mathbf{r}_{jk}|) = \psi(|\mathbf{r}_{kj}|)$, and hence

$$\beta_{jk} = \beta_{kj}. \tag{8.76}$$

By (8.73), $\mathbf{b}_{jk} = -(\partial\beta_{jk}/\partial r)(\mathbf{x}_j - \mathbf{x}_k)/r$, where $r = |\mathbf{r}_{jk}|$; and with (8.76) it follows that

$$\mathbf{b}_{jk} = -\frac{\partial\beta_{jk}}{\partial\mathbf{x}_j} = \frac{\partial\beta_{kj}}{\partial\mathbf{x}_k} = -\mathbf{b}_{kj}, \tag{8.77}$$

which is a statement of the third law in (8.3). It is also useful to observe that

$$\frac{\partial\beta_{jk}}{\partial\mathbf{x}_j} = \frac{\partial\beta_{jk}}{\partial\mathbf{r}_{jk}}. \tag{8.78}$$

The *total internal potential energy* $B(\beta)$ *of the system* is defined by

$$B(\beta) = \frac{1}{2}\sum_{\substack{j=1\\ j\neq k}}^{n}\sum_{k=1}^{n} \beta_{jk}(|\mathbf{r}_{jk}|), \tag{8.79}$$

in which $B(\beta) \equiv B(|\mathbf{r}_{12}|, |\mathbf{r}_{13}|, \ldots, |\mathbf{r}_{1n}|, |\mathbf{r}_{23}|, \ldots, |\mathbf{r}_{2n}|, |\mathbf{r}_{34}|, \ldots, |\mathbf{r}_{(n-1)n}|)$ is a function of the mutual distances between all pairs of particles, and the factor $1/2$ reflects the symmetry in (8.76). Consider, for example, a system of two particles. With (8.76), (8.79) yields the total internal potential energy $B(\beta) = \frac{1}{2}(\beta_{12} + \beta_{21}) = \beta_{12} \overset{\text{or}}{=} \beta_{21}$. And for a system of three particles, $B(\beta) = \beta_{12} + \beta_{13} + \beta_{23}$, each term of which is a function of the distance between the corresponding particles so that, for example, $\beta_{12} = \beta_{12}(|\mathbf{r}_{12}|)$. Therefore, (8.79) is the total of all of the internal potential energy functions.

We wish to relate the total internal potential energy to the work done by all of the conservative internal forces. By (8.1) and (8.73), the total conservative internal force due to the other $n-1$ particles acting on P_j is

$$\mathbf{b}_j = \sum_{\substack{k=1\\ k\neq j}}^{n} \mathbf{b}_{jk} = -\sum_{\substack{k=1\\ k\neq j}}^{n} \nabla_j \beta_{jk}(|\mathbf{r}_{jk}|). \tag{8.80}$$

Hence, with the aid of (8.77), the total work done by the conservative internal forces is determined by

$$\sum_{j=1}^{n}\int_{t_0}^{t} \mathbf{b}_j \cdot d\mathbf{x}_j = \sum_{\substack{j=1\\ j\neq k}}^{n}\sum_{k=1}^{n}\int_{t_0}^{t} \mathbf{b}_{jk}\cdot d\mathbf{x}_j = -\frac{1}{2}\sum_{\substack{j=1\\ j\neq k}}^{n}\sum_{k=1}^{n}\int_{t_0}^{t} \nabla_j\beta_{jk}\cdot d\mathbf{r}_{jk}. \tag{8.81}$$

To see this more clearly, it is best to write out several integrand terms of the sums in (8.81), and observe (8.1) and (8.77) to obtain

$$\begin{aligned}
\mathbf{b}_1\cdot d\mathbf{x}_1 + \mathbf{b}_2\cdot d\mathbf{x}_2 + \ldots &= (\mathbf{b}_{12} + \mathbf{b}_{13} + \ldots)\cdot d\mathbf{x}_1 + (\mathbf{b}_{21} + \mathbf{b}_{23} + \ldots)\cdot d\mathbf{x}_2 + \ldots \\
&= \mathbf{b}_{12}\cdot d(\mathbf{x}_1 - \mathbf{x}_2) + \mathbf{b}_{13}\cdot d(\mathbf{x}_1 - \mathbf{x}_3) \\
&\quad + \mathbf{b}_{23}\cdot d(\mathbf{x}_2 - \mathbf{x}_3) + \ldots \\
&= \mathbf{b}_{12}\cdot d\mathbf{r}_{12} + \mathbf{b}_{13}\cdot d\mathbf{r}_{13} + \mathbf{b}_{23}\cdot d\mathbf{r}_{23} + \ldots,
\end{aligned}$$

in which $\mathbf{r}_{jk} = \mathbf{x}_j - \mathbf{x}_k$. Introducing the internal potential energy from (8.73), we have

$$\sum_{j=1}^{n} \mathbf{b}_j \cdot d\mathbf{x}_j = \mathbf{b}_1 \cdot d\mathbf{x}_1 + \mathbf{b}_2 \cdot d\mathbf{x}_2 + \ldots$$
$$= -\boldsymbol{\nabla}_1 \beta_{12} \cdot d\mathbf{r}_{12} - \boldsymbol{\nabla}_1 \beta_{13} \cdot d\mathbf{r}_{13} - \boldsymbol{\nabla}_2 \beta_{23} \cdot d\mathbf{r}_{23} - \ldots.$$

Bearing in mind (8.77), it is seen that this is equivalent to the last sum in (8.81); indeed, consider, for example, the two terms $-\frac{1}{2}\boldsymbol{\nabla}_1\beta_{12} \cdot d\mathbf{r}_{12} - \frac{1}{2}\boldsymbol{\nabla}_2\beta_{21} \cdot d\mathbf{r}_{21} = -\boldsymbol{\nabla}_1\beta_{12} \cdot d\mathbf{r}_{12}$. By (8.78), however, $\partial\beta_{12}/\partial\mathbf{x}_1 = \partial\beta_{12}/\partial\mathbf{r}_{12}$, $\partial\beta_{23}/\partial\mathbf{x}_2 = \partial\beta_{23}/\partial\mathbf{r}_{23}$, and so on. Since each $\beta_{jk} = \beta_{jk}(|\mathbf{r}_{jk}|)$ is a function of a single variable $\mathbf{r}_{jk}$, each term in the last sum above is an exact differential of the form $\nabla_1\beta_{12} \cdot d\mathbf{r}_{12} = \partial\beta_{12}/\partial\mathbf{r}_{12} \cdot d\mathbf{r}_{12} = d\beta_{12}$, and so forth; therefore,

$$\sum_{j=1}^{n} \mathbf{b}_j \cdot d\mathbf{x}_j = -d\beta_{12} - d\beta_{13} - d\beta_{23} - \ldots = -d(\beta_{12} + \beta_{13} + \beta_{23} + \ldots).$$

The term in parentheses, however, is just the total of all of the internal potential energy functions given by (8.79). We thus obtain

$$\sum_{j=1}^{n} \int_{t_0}^{t} \mathbf{b}_j \cdot d\mathbf{x}_j = -\int_{t_0}^{t} d\left(\frac{1}{2}\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} \beta_{jk}(|\mathbf{r}_{jk}|)\right) = -\int_{t_0}^{t} dB(\beta). \quad (8.82)$$

In sum, *the total work done by conservative internal forces is equal to the decrease in the total internal potential energy of the system*:

$$\sum_{k=1}^{n} \int_{t_0}^{t} \mathbf{b}_k \cdot d\mathbf{x}_k = -\,\Delta B(\beta). \quad (8.83)$$

The dummy summation index in (8.82) is here replaced by k for convenience below.

8.9.3. Total Energy of the System

Let $V(\beta) = V(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ denote *the total potential energy of the system* defined by

$$V(\beta) \equiv \Phi(\beta) + B(\beta). \quad (8.84)$$

Then, with the aid of (8.71) and (8.83) in (8.59), *the total work done by all conservative external and internal forces* is given by

$$\mathscr{W} = -\Delta\Phi(\beta) - \Delta B(\beta) = -\Delta V(\beta). \quad (8.85)$$

Hence, the total work done on a conservative system of particles is equal to the decrease in the total potential energy of the system.

Now, the work energy equation (8.60) holds for all kinds of force fields $\mathbf{F}_k$, conservative or not. Therefore, with (8.85), we have our final result.

The principle of conservation of energy for a system of particles: *If the external and internal forces that act on a system of particles are conservative, or otherwise do no work in a given motion of the system, the sum of the total kinetic energy and the total potential energy of the system of particles, or briefly the total energy of the system, is constant:*

$$K + V = E, \text{ a constant.} \tag{8.86}$$

If the system is rigid, the particles are constrained in their relative positions; so, the internal forces are workless, as shown in (8.64). *Therefore, the total potential energy for a rigid system of particles is equal to the total external potential energy of the system*: $V(\beta) = \Phi(\beta)$.

Finally, let us confirm that the conservative forces derive from their corresponding potential energy functions. We can show that the total force $\mathbf{F}_k$ acting on the k^{th} particle may be deduced from (8.84) in accordance with

$$\mathbf{F}_k = -\boldsymbol{\nabla}_k V(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = -\boldsymbol{\nabla}_k \Phi - \boldsymbol{\nabla}_k B. \tag{8.87}$$

From (8.72) and (8.70), we have

$$\boldsymbol{\nabla}_k \Phi = \boldsymbol{\nabla}_k(\phi_1(\mathbf{x}_1) + \ldots + \phi_k(\mathbf{x}_k) + \ldots + \phi_n(\mathbf{x}_n)) = \boldsymbol{\nabla}_k \phi_k(\mathbf{x}_k) = -\mathbf{f}_k. \tag{8.88}$$

Similarly, recalling (8.76), we find with (8.79) and (8.80)

$$\boldsymbol{\nabla}_k B = \sum_{\substack{j=1 \\ k \neq j}}^{n} \boldsymbol{\nabla}_k \beta_{kj} = -\mathbf{b}_k. \tag{8.89}$$

Hence, (8.87) becomes $\mathbf{F}_k = \mathbf{f}_k + \mathbf{b}_k$, the total force acting on P_k. Furthermore, because the total internal force vanishes, $\Sigma_{k=1}^{n} \boldsymbol{\nabla}_k B = -\Sigma_{k=1}^{n} \mathbf{b}_k = \mathbf{0}$; and hence, from (8.87) and in agreement with (8.4), the total force

$$\mathbf{F}(\beta, t) = \sum_{k=1}^{n} \mathbf{F}_k = -\sum_{k=1}^{n} \boldsymbol{\nabla}_k V(\beta) = -\sum_{k=1}^{n} \boldsymbol{\nabla}_k \Phi = \sum_{k=1}^{n} \mathbf{f}_k. \tag{8.90}$$

Finally, it follows from (8.87) and Newton's law for a particle that *the separate equations of motion of the particles of a conservative system may be expressed in terms of the total potential energy function* (8.84) *for the system*:

$$m_k \ddot{\mathbf{x}}_k = -\boldsymbol{\nabla}_k V(\beta), \qquad k = 1, 2, \ldots, n. \tag{8.91}$$

8.9.4. The General Energy Principle for a System of Particles

Let $\mathbf{F}_{Nk} = \mathbf{f}_{Nk} + \mathbf{b}_{Nk}$ denote the total of the nonconservative external and internal forces acting on the k^{th} particle of a system. Then *the total work done by* *all nonconservative forces* is defined by

$$\mathscr{W}_N \equiv \sum_{k=1}^{n} \int_{\mathscr{C}_k} \mathbf{F}_{Nk} \cdot d\mathbf{x}_k. \tag{8.92}$$

Hence, the total work done by all conservative and nonconservative forces is given by $\mathscr{W} = -\Delta V(\beta) + \mathscr{W}_N$, with total potential energy (8.84). Now the general work–energy principle (8.60) may be written in the following alternative form.

General energy principle for a system of particles: *The change in the total energy $\mathscr{E} \equiv K + V$ of a system of particles is equal to the total work done by the nonconservative forces that act on the system:*

$$\Delta\mathscr{E} = \mathscr{W}_N. \tag{8.93}$$

Consequently, the total energy is conserved when and only when the nonconservative forces do no total work in the motion, or they are absent.

For a nonconservative system, the total force on the k^{th} particle may be written as $\mathbf{F}_k = -\nabla_k V(\beta) + \mathbf{F}_{Nk}$, in terms of its total conservative and total nonconservative parts. *Therefore, the separate equations of motion of the particles of a nonconservative system may be derived from*

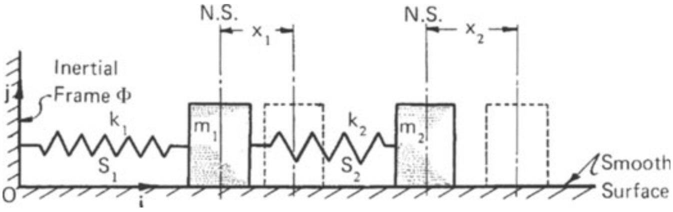
$$m_k \ddot{\mathbf{x}}_k = -\nabla_k V(\beta) + \mathbf{F}_{Nk}, \qquad k = 1, 2, \ldots, n. \tag{8.94}$$

Mutual nonconservative internal forces between a pair of particles must be equal but oppositely directed forces.

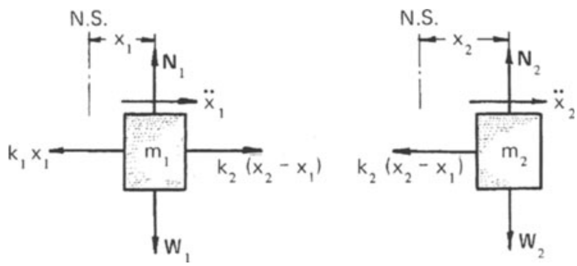
The following example illustrates some of the fine points and concepts encountered above. Afterwards, however, it will not be necessary to trace details of the foregoing construction leading to the general energy principle.

Example 8.7. A system shown in Fig. 8.8a consists of two blocks of mass m_1 and m_2 connected by a spring S_2 of stiffness k_2, while m_1 is fastened to a rigid wall by a spring S_1 of stiffness k_1. The system is displaced arbitrarily along its axis from its natural state and released to perform oscillations on a smooth horizontal surface. (i) Find the total energy of the system. (ii) Derive the differential equations of motion for the system. (iii) With reference to the free body diagram of each particle, derive each of the horizontal forces and their totals from their potential functions.

Solution of (i). We model the physical system in Fig. 8.8a as a system of two particles (center of mass objects) with mass m_1 and m_2, respectively. Their separate free body diagrams are shown in Fig. 8.8b. To find the total energy of the system $\beta = \{m_1, m_2\}$, we first note that the weights $\mathbf{W}_1$ and $\mathbf{W}_2$, and the normal, smooth surface reaction forces $\mathbf{N}_1$ and $\mathbf{N}_2$ are external forces that do no work in the motion, and the elastic spring forces are conservative. As usual, the infinitesimal



(a) The Physical System



(b) Free Body Diagrams

Figure 8.8. A spring-mass system modeled as a system of particles.

internal, mutual gravitational force between the particles is ignored. The system $\beta = \{m_1, m_2\}$, therefore, is conservative.

Let x_1 and x_2 denote the respective displacements of m_1 and m_2 from the natural state of the springs S_1 and S_2 in Fig. 8.8a. There are no constraints relating these variables, so the system has two degrees of freedom. The spring S_1 exerts an external force on m_1 with external potential energy ϕ_1; but no relevant external forces act on m_2, so $\phi_2 = 0$. The total external potential energy (8.72) is thus given by

$$\Phi = \phi_1 = \frac{1}{2} k_1 x_1^2. \tag{8.95a}$$

The mutual internal force on m_1 and m_2 is due to the spring S_2. These forces, shown in Fig. 8.8b, are equal but oppositely directed, so the total internal force is zero; but the total internal potential energy is not. The internal potential energy arising from the elastic force exerted on m_1 by m_2 is $\beta_{12} = \frac{1}{2} k_2 (x_2 - x_1)^2$, and the internal potential energy arising from the equal but oppositely directed elastic force exerted on m_2 by m_1 is $\beta_{21} = \frac{1}{2} k_2 (x_2 - x_1)^2$. Clearly, the mutual internal potential energy $\beta_{12} = \beta_{21}$ is a symmetric function of the change in distance, hence, also the current distance between the particles, as indicated in (8.75) and (8.76). The total internal potential energy (8.79) is thus determined by $B = \frac{1}{2}(\beta_{12} + \beta_{21})$, that is,

$$B = \beta_{12} = \frac{1}{2}k_2(x_2 - x_1)^2. \tag{8.95b}$$

Consequently, the total internal potential energy of the system in Fig. 8.8a is simply the elastic potential energy due to the spring S_2. Therefore, with (8.95a) and (8.95b), the total potential energy (8.84) of the system is

$$V(\beta) = \Phi + B = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2. \tag{8.95c}$$

It is evident that the total potential energy function $V(\beta)$ may be written down immediately by inspection of the system in Fig. 8.8a. In fact, this is the usual procedure to follow.

The total kinetic energy (8.50) of the system in Fig. 8.8a is

$$K(\beta, t) = \frac{1}{2}m_1 \dot{x}_1^2 + \frac{1}{2}m_2 \dot{x}_2^2\,. \tag{8.95d}$$

The total energy of the conservative system $\beta = \{m_1, m_2\}$ now follows from (8.86):

$$K + V = \frac{1}{2}m_1 \dot{x}_1^2 + \frac{1}{2}m_2 \dot{x}_2^2 + \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 = E, \text{ a constant.} \tag{8.95e}$$

Solution of (ii). Application of the law of motion to each particle shown separately in the free body diagrams of Fig. 8.8b yields the following equations of motion for the system:

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1), \qquad m_2 \ddot{x}_2 = -k_2(x_2 - x_1). \tag{8.95f}$$

Notice that the sum of these equations is $m_1\ddot{x}_1 + m_2\ddot{x}_2 = -k_1 x_1$, or $m(\beta)\ddot{x}^* = F(\beta)$, which is the equation of motion (8.5) of the system with total external force $F(\beta) = -k_1 x_1$. This system equation is not useful. The motions x_1 and x_2 of the particles are coupled, but independent—the motion of one particle influences but does not determine the motion of the other. As a consequence, these equations cannot be separately integrated. The solution of coupled equations of this kind is considered in Chapter 11.

Solution of (iii). We next determine the horizontal forces and their totals from their potential functions. The total external force acting on each particle is determined by use of (8.95a) in (8.70):

$$\mathbf{f}_1 = -\boldsymbol{\nabla}_1\phi_1 = -\frac{\partial \phi_1}{\partial x_1}\mathbf{i} = -k_1 x_1 \mathbf{i}, \qquad \mathbf{f}_2 = -\boldsymbol{\nabla}_2\phi_2 = -\frac{\partial \phi_2}{\partial x_2}\mathbf{i} = \mathbf{0}. \tag{8.95g}$$

Hence, the total external force on the system, by (8.4), is $\mathbf{F}(\beta, t) = \mathbf{f}_1 + \mathbf{f}_2 = -k_1 x_1 \mathbf{i}$, noted earlier. All other forces that act must be the equal, oppositely directed external and internal contributions. Specifically, the equal and oppositely directed external forces are $\mathbf{N}_1 = -\mathbf{W}_1$, $\mathbf{N}_2 = -\mathbf{W}_2$; these have no influence on the motion. The total internal force acting on each particle is obtained by use of (8.95b) in (8.80):

$$\mathbf{b}_1 = -\nabla_1 \beta_{12} = -\frac{\partial \beta_{12}}{\partial x_1}\mathbf{i} = k_2(x_2 - x_1)\mathbf{i},$$
$$\mathbf{b}_2 = -\nabla_2 \beta_{21} = -\frac{\partial \beta_{21}}{\partial x_2}\mathbf{i} = -k_2(x_2 - x_1)\mathbf{i}. \tag{8.95h}$$

Of course, the total internal force acting on the system is $\mathbf{b}_1 + \mathbf{b}_2 = \mathbf{0}$. From (8.95g) and (8.95h), the total force $\mathbf{F}_k = \mathbf{f}_k + \mathbf{b}_k$ acting on each particle separately is

$$\mathbf{F}_1 = [-k_1 x_1 + k_2(x_2 - x_1)]\mathbf{i}, \qquad \mathbf{F}_2 = -k_2(x_2 - x_1)\mathbf{i}. \tag{8.95i}$$

These are the totals of the forces on the right-hand side of the equations in (8.95f) and shown in the free body diagrams of Fig. 8.8b.

Finally, we apply equations (8.87), (8.89), and (8.90). Using (8.95c) in the first relation of (8.87), we have

$$\mathbf{F}_1 = -\nabla_1 V = [-k_1 x_1 + k_2(x_2 - x_1)]\mathbf{i}, \qquad \mathbf{F}_2 = -\nabla_2 V = -k_2(x_2 - x_1)\mathbf{i}, \tag{8.95j}$$

in agreement with (8.95i). It is seen immediately that use of these relations in (8.91) returns the separate equations on motion (8.95f). With (8.95b) in (8.89), we find

$$\mathbf{b}_1 = -\nabla_1 B = k_2(x_2 - x_1)\mathbf{i}, \qquad \mathbf{b}_2 = -\nabla_2 B = -k_2(x_2 - x_1)\mathbf{i}, \tag{8.95k}$$

in accord with (8.95h). And, finally, substitution of (8.95j) into (8.90) yields

$$\mathbf{F}(\beta, t) = -\nabla_1 V - \nabla_2 V = -k_1 x_1 \mathbf{i}, \tag{8.95l}$$

the total external force on the system. □

This example illustrates the notation and several concepts introduced earlier in the development of the basic energy principles. Fortunately, in applications of the energy principles, it is not necessary to retrace these fine points. Consider the system in Fig. 8.8a and suppose, for example, that an additional mass m_3 is attached to the mass m_2 by a linear spring S_3 with stiffness k_3. We then have a new system of three collinear particles, and this new element introduces an additional internal potential energy function, which by inspection is given immediately by $\beta_{23} = \beta_{32} = \frac{1}{2}k_3(x_3 - x_2)^2$. This extra internal energy contribution is then added to (8.95c) to obtain the total potential energy for the system, and the total kinetic energy is increased by $\frac{1}{2}m_3\dot{x}_3^2$. Alternatively, suppose that the same spring is attached to m_2 at one end and to a rigid wall at the other. We now have a new system of two collinear particles with three spring elements. The new element, by inspection, introduces an additional external potential energy $\phi_3 = \frac{1}{2}k_3 x_2^2$ in contribution

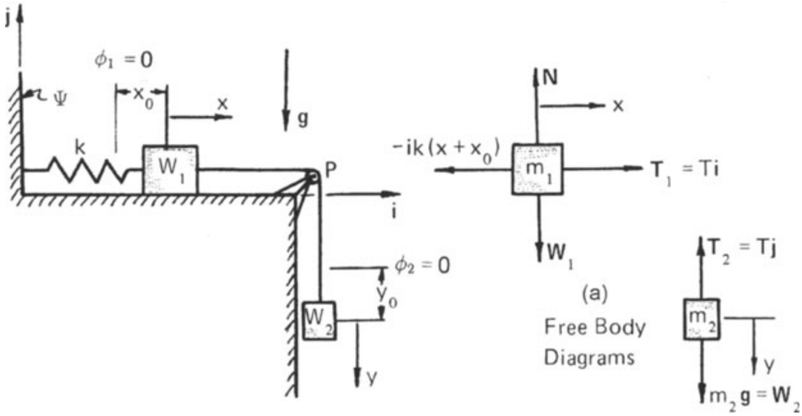


Figure 8.9. Gravity induced free vibrations of a system of two particles.

to (8.95c) for the total potential energy of our new system, and the total kinetic energy is given by (8.95d). Let the reader consider the forces that act on the free bodies involved in these additional models and write down the separate equations of motion for the particles of the respective systems. Try to derive the additional forces from their potential energy functions. Now let us turn to another example and apply the theory directly.

Example 8.8. Two small blocks of weight W_1 and W_2 are connected by a perfectly flexible and inextensible cable of negligible mass, as shown in Fig. 8.9. The weight W_1 rests on a smooth horizontal surface and is attached to a linear spring of stiffness k fastened to a rigid wall. The cable is free to slide over a smooth pulley at P and suspends the weight W_2. The system is at rest initially when W_2 is displaced vertically and released. (i) Find the total energy of the system. (ii) Derive the equation of motion and determine the frequency of the vibration. (iii) Describe alternative formulations of these issues.

Solution of (i). The physical system is modeled as a system of two particles of masses m_1 and m_2 for each of which the free body diagram is shown in Fig. 8.9a. The weight W_1 and the normal, smooth surface reactions at W_1 and at P do no work in the motion. The cable has negligible mass, so its motion around the smooth pulley may be ignored, and hence the oppositely directed, internal cable tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ have equal magnitude T, say. The cable is inextensible and perfectly flexible, so the total internal potential energy of the system is zero: $B = 0$, and W_1 and W_2 share the same displacement so that $x = y$. In the equilibrium state, $x_0 = y_0$ is the static displacement of W_1 and W_2 so that $kx_0 = m_2 g$. The relevant external forces W_2 and the elastic spring force are conservative. The system, therefore, is conservative with total potential energy (8.84) equal to the total external potential energy $\Phi(\beta) = \phi_1 + \phi_2$ from (8.72), namely,

$$V(\beta) = \frac{1}{2}k(x + x_0)^2 - m_2 g(y + y_0), \tag{8.96a}$$

obtained directly by inspection of the system diagram. The total kinetic energy of the system, with the inextensibility constraint $x = y$ in mind, is

$$K(\beta, t) = K_1 + K_2 = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 = \frac{1}{2}m(\beta)\dot{x}^2, \tag{8.96b}$$

wherein $m(\beta) \equiv m_1 + m_2$. Hence, the principle of conservation of energy gives the constant total energy of the system:

$$E = K + V = \frac{1}{2}m(\beta)\dot{x}^2 + \frac{1}{2}k(x + x_0)^2 - m_2 g(x + x_0). \tag{8.96c}$$

Solution of (ii). Because the system has only one degree of freedom, the equation of motion may be found by differentiation of (8.96c) with respect to the path variable x (or the time t) to obtain

$$\ddot{x} + \frac{k}{m(\beta)}x = \frac{m_2 g - kx_0}{m(\beta)}. \tag{8.96d}$$

The system is in its static equilibrium state at $x = 0$ where $m_2 g - kx_0 = 0$, and hence the equation of motion for the system is

$$\ddot{x} + p^2 x = 0, \qquad p = \sqrt{\frac{k}{m_1 + m_2}}, \tag{8.96e}$$

in which p is the circular frequency.

Solution of (iii). Other methods will lead to the same results. Because the spring is linear, one alternative approach is to consider the motion from the static equilibrium position directly. The kinetic energy is unchanged in (8.96b), while the total potential energy may be written as

$$V(\beta) = \frac{1}{2}kx^2. \tag{8.96f}$$

This procedure, however, cannot be used when the spring is nonlinear, whereas the earlier method leading to (8.96a) can. The principle of conservation of energy (8.86) yields

$$\frac{1}{2}m(\beta)\dot{x}^2 + \frac{1}{2}kx^2 = E. \tag{8.96g}$$

Differentiation of this equation with respect to x (or t) returns (8.96e).

Another approach starts with the separate equations of motion for each particle. With reference to the free body diagrams in Fig. 8.9a, we find easily

$$m_1\ddot{x} = T - k(x + x_0), \qquad m_2\ddot{y} = m_2 g - T, \qquad W_1 = N. \tag{8.96h}$$

Eliminating T and introducing the inextensibility constraint $y = x$ and the equilibrium condition $m_2 g = kx_0$, we recover (8.96e). The equations of motion also may be formulated relative to the static state: $m_1\ddot{x} = T - kx$, $m_2\ddot{y} = -T$, which again lead to (8.96e).

8.10. An Application of the General Energy Principle

Suppose the cable in the previous example has elasticity characterized by a force-elongation equation $S = k_2\delta + k_3\delta^3$, in which k_2 and k_3 are constants and δ is the cable elongation measured from its natural unstretched state. To model this case, the vertical portion of the cable in Fig. 8.9 is replaced with a nonlinear spring characterized by S, the remaining part of the cable being inextensible and perfectly flexible. Of course, now $x \neq y$. In view of the nonlinear character of the cable spring, it is best to measure the respective particle displacements $X \equiv x + x_0$ and $Y \equiv y + y_0$ from the natural state of the springs, then the cable elongation $\delta = Y - X$. We wish to derive the energy equation of the system for $\delta \geq 0$. Afterwards, the equations of motion for the linear system with $k_3 = 0$ are described.

The total external potential energy is unchanged: $\Phi(\beta) = \frac{1}{2}kX^2 - m_2 gY$, and the total kinetic energy of the system is $K = \frac{1}{2}m_1\dot{X}^2 + \frac{1}{2}m_2\dot{Y}^2$. However, the total energy must include the internal elastic energy of the cable. We anticipate that the cable restoring force $\mathbf{F}_S = -S\mathbf{j}$ is conservative; but in the absence of its potential energy function, we may use the general energy principle (8.93) in which, with $\mathbf{F}_N = \mathbf{F}_S$ and $d\mathbf{x} = d\delta\mathbf{j}$,

$$\mathscr{W}_N = \int_{\mathscr{C}} \mathbf{F}_N \cdot d\mathbf{x} = \int_0^{Y-X} -(k_2\delta + k_3\delta^3)d\delta = -\frac{1}{2}k_2(Y - X)^2 - \frac{1}{4}k_3(Y - X)^4. \tag{8.97a}$$

Since the work done by the nonlinear spring force is path independent, this confirms that $\mathbf{F}_S$ is indeed conservative, and the negative of (8.97a) is the internal potential energy function of the nonlinear spring: $B(\beta) = \frac{1}{2}k_2(Y - X)^2 + \frac{1}{4}k_3(Y - X)^4$. With reference to the aforementioned modified model for which the system initially is at rest in its natural state, the general energy principle (8.93) yields

$$\frac{1}{2}m_1\dot{X}^2 + \frac{1}{2}m_2\dot{Y}^2 + \frac{1}{2}kX^2 + \frac{1}{2}k_2(Y - X)^2 + \frac{1}{4}k_3(Y - X)^4 - m_2 gY = 0. \tag{8.97b}$$

Because the system is conservative with total potential energy

$$V(\beta) = \frac{1}{2}kX^2 + \frac{1}{2}k_2(Y - X)^2 + \frac{1}{4}k_3(Y - X)^4 - m_2 gY, \tag{8.97c}$$

the separate equations of motion of m_1 and m_2 may be easily derived from (8.91). The reader will then find the static equilibrium relation $kx_0 = m_2 g = k_2\delta_0 + k_3\delta_0^3$,

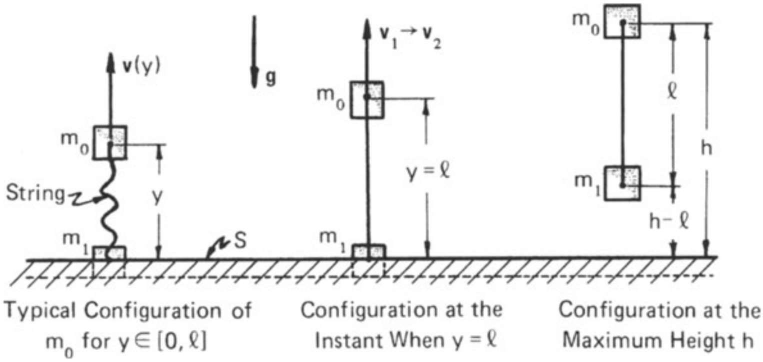


Figure 8.10. Application of the impulse–momentum and energy principles.

where $\delta_0 = y_0 - x_0$; but this offers no simplification of the equations by a coordinate shift to the static state. Now set $k_3 = 0$, recall that $X \equiv x + x_0$, $Y \equiv y + y_0$, and show that relative to the equilibrium state, the motion of the system is governed by the following coupled linear system of equations:

$$m_1\ddot{x} + (k + k_2)x - k_2 y = 0, \qquad m_2\ddot{y} + k_2 y - k_2 x = 0. \tag{8.97d}$$

8.11. An Application of the Energy and Impulse–Momentum Principles

Two blocks of masses m_0 and m_1 shown in Fig. 8.10 are connected by a flexible, inextensible string of length ℓ and negligible mass. The mass m_0 is projected vertically upward from the horizontal surface S with initial velocity $\mathbf{v}_0$. Apply the conservation of energy and instantaneous impulse–momentum principles to determine the maximum height attained by m_0 when its initial speed $v_0 > \sqrt{2g\ell}$.

Only the conservative gravitational force acts on m_0 prior to the impending impulse at $y = \ell$, while m_1 remains at ease at $y = 0$. Let v_1 denote the speed of m_0 at the instant just prior to the impulsive string reaction at $y = \ell$ (see Fig. 8.10); then the energy principle (8.86) gives

$$v_1^2 = v_0^2 - 2g\ell, \tag{8.98a}$$

where v_0 is the initial speed of m_0 at $y = 0$, the zero datum for the gravitational potential energy $V_g = m_0 g y$.

Henceforward, we suppose that $v_1 \neq 0$, i.e. $v_0 > \sqrt{2g\ell}$, and that the massless inextensible string does not break. The fully extended string at $y = \ell$ experiences an instantaneous internal impulse, so the instantaneous impulse of the total external force $\mathscr{I}^*(\beta) = \mathbf{0}$, finite forces contributing nothing. Therefore, by (8.17), $\Delta\mathbf{p}^* = \mathbf{0}$, i.e. the momentum of the system is constant during the impulsive instant: $m_0 v_1 \mathbf{i} = (m_0 + m_1) v_2 \mathbf{i}$. The impulse–momentum principle, with (8.98a), thus yields

$$v_2 = \frac{m_0}{m_0 + m_1} v_1 = \frac{m_0}{m_0 + m_1} \sqrt{v_0^2 - 2g\ell}. \tag{8.98b}$$

Now, after the impulsive instant, only the external conservative gravitational force acts on the system; therefore, the total energy of the system is conserved. Initially, at $y = \ell$, $E = \frac{1}{2}(m_0 + m_1) v_2^2 + m_0 g \ell$, and at the maximum height h where the system comes to rest, as shown in Fig. 8.10, $E = m_0 g h + m_1 g (h - \ell)$. Therefore, the principle of conservation of energy yields $h = \ell + v_2^2/2g$. Finally, use of (8.98b) leads to

$$h = \ell \left[1 - \left(\frac{m_0}{m_0 + m_1} \right)^2 \right] + \left(\frac{m_0}{m_0 + m_1} \right)^2 \frac{v_0^2}{2g}, \tag{8.98c}$$

for the greatest height attained by m_0, for $v_1 \neq 0$, i.e. $v_0 > \sqrt{2g\ell}$; otherwise, $h = \ell$.

8.12. Motion of a Chain on a Smooth Curve by the Energy Method

Let us consider the motion of a simple "deformable" body, a perfectly flexible and inextensible uniform chain, modeled as a contiguous system of particles. The chain has length $2l$ and mass ρ per unit length, and slides under gravity along a smooth, plane curved track $\mathscr{C}$ in the vertical plane, as shown in Fig. 8.11. The energy principle is applied to find the speed of the chain along the track. Then its motion along a cycloid is described.

Let s denote the arc length coordinate along $\mathscr{C}$ of the midpoint A of the chain from the origin O. Since $\mathscr{C}$ is smooth, only the gravitational force does work on the chain. The potential energy of the element of mass $dm = \rho d\sigma$ at the position $y(\sigma)$ in Fig. 8.11 is $dV = gy(\sigma)dm = \rho g y(\sigma) d\sigma$, wherein σ is the variable arc

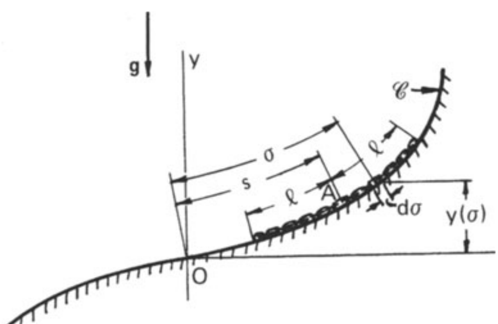


Figure 8.11. Motion of a chain on a smooth plane curve.

length parameter along the chain from O. Thus, the total potential energy of the contiguous system of chain elements is

$$V(s) = \rho g \int_{s-\ell}^{s+\ell} y(\sigma) d\sigma. \tag{8.99a}$$

The chain is inextensible, so all particles move with the same speed $\dot{s}$ along $\mathscr{C}$; hence, the total kinetic energy of the chain is

$$K = \int_{s-\ell}^{s+\ell} \frac{1}{2} \rho \dot{s}^2 d\sigma = \rho \ell \dot{s}^2. \tag{8.99b}$$

With (8.99a) and (8.99b), the energy principle (8.86) gives

$$\rho \ell \dot{s}^2 + \rho g \int_{s-\ell}^{s+\ell} y(\sigma) d\sigma = E. \tag{8.99c}$$

The constant E is determined from the assigned initial speed and position of A. The speed of the chain at any position s is thus determined by (8.99c) when the equation of the smooth track is specified.

The system's only degree of freedom is described by s. Hence, the equation of motion of the uniform chain on an arbitrary smooth curve in the vertical plane is obtained by differentiation of (8.99c) with respect to s. (See Problem 6.28, equation (P6.28c).) We thus find

$$\ddot{s} + \frac{g}{2\ell} (y(\sigma)|_{s+\ell} - y(\sigma)|_{s-\ell}) = 0. \tag{8.99d}$$

Now, suppose the track $\mathscr{C}$ is a cycloid defined by the parametric equations (7.94a), so that $y = 2a \sin^2(\beta/2)$, and recall that $\gamma = \beta/2$ in (7.94d). Then, in terms of the arc length coordinate σ of an arbitrary particle of the chain, we have

$$y(\sigma) = 2a \sin^2(\beta/2) = \frac{\sigma^2}{8a}. \tag{8.99e}$$

Hence, (8.99d) yields the equation of motion of a uniform chain on a smooth cycloid:

$$\ddot{s} + \frac{g}{4a} s = 0. \tag{8.99f}$$

Comparison of (8.99f) with (7.94e) reveals that the flexible, inextensible chain, regardless of its length and its mass, has the same motion as that of a single particle at its midpoint sliding under gravity along a smooth cycloid. The chain will perform simple harmonic oscillations with circular frequency $p = (g/4a)^{1/2}$ independent of the extent of its displacement from O. Consequently, the motion of the chain is governed by the motion of its midpoint, which from any initial position along the cycloid will reach the lowest point in the same quarter period time $t^* = \tau/4 = \pi \sqrt{a/g}$.

8.13. Law of Restitution

Consider the internal impulsive interaction of two bodies modeled as particles. We shall assume that all other forces remain finite during the impulsive instant and thus contribute nothing to the impulse. Then the total momentum of the center of mass, and hence of the system, is constant during the impulsive interval: $\Delta \mathbf{p}^* = \mathbf{0}$. Given their velocities $\mathbf{v}_k$ or momenta $\mathbf{p}_k$ immediately prior to the impulse, the problem of interest is to find the unknown velocities $\mathbf{v}'_k$ or momenta $\mathbf{p}'_k$ of each particle immediately after the impulse, altogether six unknowns. With this objective in mind, let $\mathbf{n}$ denote the direction of the instantaneous, mutual internal impulse, and let $\mathbf{i}_\alpha$ denote two suitably chosen orthogonal directions in the plane normal to $\mathbf{n}$. Then the momentum of the system in the direction of the impulse is conserved: $\Delta \mathbf{p}^* \cdot \mathbf{n} = (\mathbf{p}'^* - \mathbf{p}^*) \cdot \mathbf{n} = 0$; that is,

$$(\mathbf{p}'_1 + \mathbf{p}'_2) \cdot \mathbf{n} = (\mathbf{p}_1 + \mathbf{p}_2) \cdot \mathbf{n}, \tag{8.100}$$

and in the absence of all other forces during the impulsive instant, the components of the momentum of each particle and of the center of mass in the plane perpendicular to $\mathbf{n}$ must be conserved:

$$\mathbf{p}'_1 \cdot \mathbf{i}_\alpha = \mathbf{p}_1 \cdot \mathbf{i}_\alpha, \qquad \mathbf{p}'_2 \cdot \mathbf{i}_\alpha = \mathbf{p}_2 \cdot \mathbf{i}_\alpha, \qquad \alpha = 1, 2. \tag{8.101}$$

The addition of these components yields $(\mathbf{p}'_1 + \mathbf{p}'_2) \cdot \mathbf{i}_\alpha = (\mathbf{p}_1 + \mathbf{p}_2) \cdot \mathbf{i}_\alpha$ or $\mathbf{p}'^* \cdot \mathbf{i}_\alpha = \mathbf{p}^* \cdot \mathbf{i}_\alpha$, which together with (8.100) confirms that the momentum vector $\mathbf{p}^*$ of the center of mass is unchanged during an internal impulse, all other forces remaining finite. We now have a total of five equations for the six unknown momentum or velocity components. An additional equation relating the two unknown components $\mathbf{v}'_k \cdot \mathbf{n}$ to $\mathbf{v}_k \cdot \mathbf{n}$ is provided by the law of restitution introduced below. First, let us briefly consider some well-known effects of impact between real bodies.

When a ball strikes an essentially rigid wall, the ball actually becomes distorted during the impact, an effect that we have ignored for the sake of simplicity. If the ball is a highly elastic rubber ball, it will quickly recover its shape as it rebounds from the wall with almost no loss of energy. So this impact is perceived as a perfectly elastic collision. On the other hand, when two bodies collide, they generally suffer considerable distortion that is only partially recovered as the two bodies either separate or become so entangled that subsequent to the impact they continue in motion together. These are complex inelastic situations during which some portion of the distortional energy is expended in permanent deformation of the bodies, some is lost through heat and internal dissipation in creating their permanent distortion, and some through generation of substantial acoustic energy. Thus, during the impulsive instant the internal forces of deformation have done considerable work, some of which is recovered, most of which is not, at the expense of the total kinetic energy of the system. This loss of energy depends on the nature of the impact situation.

To model the impact of two colliding bodies, various kinds of impact situations, all ideal, are identified. Let P denote the point of contact of two colliding bodies. The line through the point P and perpendicular to the plane of contact tangent to both bodies at P is called the *line of action*. When the centers of mass of both bodies are situated on the line of action, the impact is called a *central impact*; otherwise, it is called an *eccentric impact*. In addition, if the initial velocities at the impulsive instant are parallel vectors, not necessarily collinear, the impact is called a *direct impact*; otherwise, the impact is called *oblique*. There are two kinds of direct impact. A direct impact for which the initial velocity vectors are directed along the line of action is called *collinear*; otherwise, it is called *noncollinear*. As a consequence, there are five general classes of impact: collinear and noncollinear direct central impact, an oblique central impact, a direct eccentric impact, and an oblique eccentric impact. Only the simplest models of direct central and oblique impact are studied here. Of course, in an impact situation one of the bodies may be at rest initially, and one may be so considerably more massive than the other that it suffers virtually no change in its state of rest.

The rapid deformation process in a real impact is just too complicated to describe in any analytical detail, and in any event this falls beyond the scope of our studies here. So, to avoid our getting into an awkward discussion of deformation of bodies and the time periods of their stress and recovery, we shall ignore all rotational, distortional, thermal, and acoustic phenomena that may occur in a real collision of bodies of finite size. For our purposes here, it is sufficient to suppose that the colliding bodies may be modeled as center of mass objects—two particles, pictured as small circular or spherical bodies, say, subject to an instantaneous internal impulse arising from a direct or oblique central impact only. To account for the energy lost in the impact, and to get around the impasse of complex deformation analysis, an empirical rule is introduced to model the restitution of the impacting bodies. With this in mind, let $\mathbf{v}_k$ and $\mathbf{v}'_k$ denote the respective instantaneous velocities of their centers of mass before and after the impact. Then $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}'_2 - \mathbf{v}'_1$ are the respective instantaneous relative velocities of approach and separation of their centers of mass before and after the impact. Let $\mathbf{n}$ be a unit vector along the line of action perpendicular to the plane of contact between the bodies. We shall assume that the impacting bodies are smooth, so that the impulsive force always is directed along $\mathbf{n}$. Now, to account for the energy loss in a direct or oblique central impact of two smooth bodies, we adopt the following empirical rule attributed to Newton. (See Cajori in the References.)

Law of restitution: *The normal component of the instantaneous relative velocity of separation of the centers of mass of two bodies after impact is proportional to the normal component of their instantaneous relative velocity of approach prior to impact; namely,*

$$(\mathbf{v}'_2 - \mathbf{v}'_1) \cdot \mathbf{n} = -e(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{n} \tag{8.102}$$

in which the constant $e \in [0, 1]$ is called the coefficient of restitution.

A perfectly elastic impact for which the normal component of the relative velocity of separation is equal but oppositely directed to the normal component of the relative velocity of approach is described by $e = 1$. In the case when one body is fixed and the bodies are smooth, (8.101) and (8.102) show that the rebound velocity of the particle is the opposite of its approach velocity, so no kinetic energy is lost. A perfectly inelastic impact for which the relative velocity of separation vanishes because the bodies do not separate, and hence their normal velocities after impact are equal, is described by $e = 0$. Otherwise, the value of e, which must be the same for all observers, depends on the nature of the colliding bodies, i.e., their material, shape, size, surface roughness, and so on—it is a physical constant determined by experiment, a difficult task even for simple situations. More generally, however, it turns out that e may also depend on the impact velocities of the bodies, in which case the simple law of restitution (8.102) is no longer valid.

Consider a direct, collinear central impact (a head-on collision). Then the instantaneous velocities are along the line of action, so that $\mathbf{v}_k \cdot \mathbf{n} = v_k$, $\mathbf{v}'_k \cdot \mathbf{n} = v'_k$, and $\mathbf{v}_k \cdot \mathbf{i}_\alpha = \mathbf{v}'_k \cdot \mathbf{i}_\alpha = 0$. Hence, (8.101) is trivially satisfied, (8.100) becomes

$$m_1 v'_1 + m_2 v'_2 = m_1 v_1 + m_2 v_2 = (m_1 + m_2) v^*, \tag{8.103}$$

and by the law of restitution (8.102),

$$v'_2 - v'_1 = -e(v_2 - v_1). \tag{8.104}$$

In a perfectly inelastic head-on collision, $e = 0$ and hence the instantaneous velocities of the centers of mass following a direct, collinear central collision are the same: $v'_1 = v'_2$—i.e. the colliding bodies remain in contact following their impact. Moreover, (8.103) shows that their velocities are equal to the instantaneous velocity of the center of mass of the system: $v'_1 = v'_2 = v^* = (m_1 v_1 + m_2 v_2)/(m_1 + m_2)$. In a perfectly elastic impact, $e = 1$ and (8.104) becomes $v'_2 - v'_1 = -(v_2 - v_1)$; consequently, the instantaneous relative rebound velocity after the impact is equal but oppositely directed to the instantaneous relative velocity of approach.

More generally, solving (8.103) and (8.104) for the unknown instantaneous speeds v'_k in a direct, collinear central impact, we obtain

$$v'_1 = \frac{m_1 - e m_2}{m_1 + m_2} v_1 + (1 + e) \frac{m_2}{m_1 + m_2} v_2, \tag{8.105}$$

and

$$v'_2 = (1 + e) \frac{m_1}{m_1 + m_2} v_1 + \frac{m_2 - e m_1}{m_1 + m_2} v_2. \tag{8.106}$$

The direct, collinear central impact problem is now solved completely for all values of the coefficient of restitution. In particular, if the bodies have the same mass $m_1 = m_2 = m$, and the collision is perfectly elastic with $e = 1$, we obtain $v'_1 = v_2$ and $v'_2 = v_1$, which shows that in a perfectly elastic, direct collinear central impact, bodies having the same mass exchange their instantaneous impact velocities.

Exercise 8.9. Let K and K' denote the instantaneous total kinetic energy immediately before and after a direct collinear central impact of two center of mass objects. Consider the motion of the system relative to the center of mass so that (5.8) holds. Show that the respective kinetic energies relative to the center of mass are related by

$$K'_{rC} = e^2 K_{rC}. \tag{8.107}$$

Because $e^2 \leq 1$, $K'_{rC} \leq K_{rC}$, equality holding if and only if the impact is perfectly elastic with $e = 1$. In all other cases, kinetic energy is lost in the impact. The instantaneous loss of kinetic energy is given by

$$K - K' = (1 - e^2) K_{rC}. \tag{8.108}$$

There is no loss of kinetic energy when $e = 1$, and the greatest loss occurs when $e = 0$. The coefficient of restitution, therefore, is a measure of the loss of kinetic energy. □

References

1. Blanco, V. M., and Mccuskey, S. W., *Basic Physics of the Solar System,* Addison-Wesley, Reading Massachusetts, 1961. This is a concise treatment of the main physical and dynamical aspects of the solar system, including an introduction to the basic principles of celestial mechanics, written for scientists, engineers, and other nonspecialists with interests in space science. Celestial mechanics and the two body problem are introduced in Chapter 4. The three body problem and the general n-body problem are discussed in Chapter 5.
2. Cajori, F., *Newton's Principia.* English translation of *Mathematical Principles of Natural Philosophy* by Isaac Newton, 1687, University of California Press, Berkeley, 1947. The empirical principle of restitution is introduced in the Scholium (pp. 21–5) of Newton's laws of motion, as a consequence and in support of the third law. In primary work, Wallis, Wren, and Huygens, in the order of their priority according to Newton, "did severally determine the rules of the impact and reflection of hard bodies, and about the same time communicated their discoveries to the Royal Society, exactly agreeing among themselves as to those rules. But Sir Christopher Wren confirmed the truth of the thing before the Royal Society by the experiments on pendulums, ..." Newton addresses the effects of air resistance on impacting pendula, and further on states: "By the theory of Wren and Huygens, bodies absolutely hard return from one another with the same velocity with which they met. But this may be affirmed with more certainty of bodies perfectly elastic. In bodies imperfectly elastic the velocity of the return is to be diminished together with the elastic force; because that force is certain and determined, and makes the bodies to return one from the other with a relative velocity, which is in a given ratio to that relative velocity with which they met."
3. Housner, G. W., and Hudson, D. E., *Applied Mechanics,* Vol. II, *Dynamics,* 2nd Edition, Van-Nostrand, Princeton, New Jersey, 1959. Systems of particles are studied in Chapter 6 and the coefficient of restitution in Chapter 4. A few problems given below are modelled upon those provided in this text.
4. Marion, J. B., and Thornton, S. T., *Classical Dynamics of Particles and Systems,* 3rd Edition, Harcourt Brace Jovanovich, New York, 1988. Central force motion is investigated in Chapter 7. See also Chapter 10 of the earlier 2nd edition by Marion cited in the References to Chapter 6, page 197.

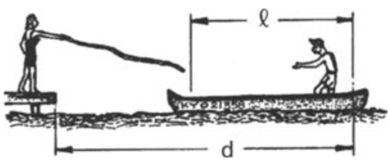
5. Meriam, J. L., and Kraige, L. G., *Engineering Mechanics*, Vol. 2 *Dynamics,* 3rd Edition, Wiley, New York, 1992. Direct and oblique central impact of smooth spheres are investigated in Chapter 3. Here the reader will find many additional examples and exercises for further study.
6. Shames, I. H., *Engineering Mechanics. Statics and Dynamics,* 4th Edition, Prentice-Hall, New Jersey, 1997. Additional examples and exercises on the central and oblique impact of particles are provided in Chapter 14, and problems on the eccentric impact of bodies by impulsive forces and torques (topics not treated herein) are discussed in Chapter 17.
7. Timoshenko, S., and Young, D. H., *Advanced Dynamics*, McGraw-Hill, New York, 1948. A classic, but non-vector treatment of applied topics in dynamics. A few problems provided in the present chapter are modelled upon examples found in Chapters 2 and 3 dealing with a system of particles.

Problems

8.1. Three particles with mass $m_1 = 5$ kg, $m_2 = 2$ kg, $m_3 = 4$ kg are initially located in $\Phi = \{F; \mathbf{i}_k\}$ at $\mathbf{X}_1 = 2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ m, $\mathbf{X}_2 = \mathbf{i} - 2\mathbf{k}$ m, $\mathbf{X}_3 = 2\mathbf{i} + \mathbf{k}$ m with initial velocities $\mathbf{v}_1 = 2\mathbf{i} + \mathbf{k}$ m/sec, $\mathbf{v}_2 = 3\mathbf{i} - \mathbf{j}$ m/sec, $\mathbf{v}_3 = -2\mathbf{j}$ m/sec. Determine for the initial instant (a) the location and velocity of the center of mass, (b) the momentum of the system, and (c) the total moment about F of the momenta of the system. (d) What is the total moment of momentum about a fixed point O at $\mathbf{X}_O = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ m in Φ? (e) Suppose the point O has velocity $\mathbf{v}_O = 2\mathbf{i} - 3\mathbf{j}$ m/sec. How does this affect the moment of momentum of the system about O?

8.2. The three particles described in the previous problem are acted upon by the respective external forces $\mathbf{F}_1 = 30\mathbf{i} - 6t\mathbf{j} + 45\mathbf{k}$ N, $\mathbf{F}_2 = 60t\mathbf{i} - 15\mathbf{k}$ N, $\mathbf{F}_3 = \mathbf{0}$. (a) Find the acceleration of the center of mass and determine its position in Φ after 2 sec. (b) What is the moment about F in Φ of the forces acting on the system at the initial instant?

8.3. A man of weight w is standing at the rear of a small boat of weight W. Initially, the man is adrift at a distance d from the pier, as shown. A friend tosses him a rope that is just long enough to reach the forward end of the boat, so the man moves forward a distance ℓ to improve his position. However, to their mutual surprise, the couple discovers that the rope does not reach the boat. What minimum additional length of rope is needed to reach the man at the forward end of the boat? In particular, what additional length is required when $W = w$? Neglect drag and current effects between the boat and the water.



Problem 8.3.

8.4. A bullet of mass m is fired with muzzle speed v_0 directed downward and inclined at an angle θ from a horizontal line perpendicular to the impact face of a large wooden block of mass M. The block is supported on smooth, rigid roller bearings at its base and by a series of recoil springs at the face opposite to the impact face. (a) Find the instantaneous impulsive force exerted on the bullet by the block. (b) What is the instantaneous impulsive reaction of the bearings on the block?

8.5. Two automobiles of masses m_A and m_B collide at an intersection in an oblique, direct central impact. The approach speed of car A was v_A directed at an angle θ north of east and that of car B was v_B directed south. After the collision, the entangled cars skidded together with unknown speed v directed at an angle ϕ north of east. (a) Determine the ratio v_B/v_A of their collision speeds. (b) It was found that $m_A = 5m_B/4$, $\theta = 30°$, and ϕ, though inaccurately measured, was smaller than θ. Which vehicle was traveling faster prior to the impact?

8.6. A system of four particles of equal mass m are situated at the ends of orthogonal cross-shaped rigid bars forming four spokes of length l and negligible mass. The system initially is rotating in the horizontal plane with constant angular velocity $\boldsymbol{\omega}_0 = \omega_0 \mathbf{k}$ when a constant torque $\mathbf{M} = M\mathbf{k}$ is suddenly applied. (a) Determine the new angular velocity of the system. (b) Suppose that $\mathbf{M}$ is reversed at $t = 0$. Determine the time required to bring the system to rest.

8.7. Two particles of equal mass m are symmetrically attached at a distance $L/2$ from the center O of a rigid rod of negligible mass and length $2L$. The rod is constrained by a bearing at O to rotate in the horizontal plane frame $\psi = \{O; \mathbf{i}, \mathbf{j}\}$ with a constant angular velocity $\boldsymbol{\omega} = \omega \mathbf{k}$. The particles are released simultaneously and move outward to the ends of the rod. (a) How is the angular speed of the system affected? (b) What will be the angular speed of the system if one of the particles fails to release? Find the bearing reaction torque required to sustain the plane motion.

8.8. A symmetrically balanced machine component consists of two particles of equal mass m attached to a massless rigid rod of length $l = 6r_0$ that rotates in the vertical plane about its central horizontal axle with constant counterclockwise angular velocity $\boldsymbol{\omega}$. The particles are released simultaneously at $r = r_0$ from the center of the rod at O, and a control mechanism then moves each particle toward its extreme end of the rod with the same speed $v(r)$ relative to the rod. (a) Determine the angular velocity and the angular acceleration of the system when each particle is at $r = 2r_0$ and $v(2r_0) = v_0$. (b) What is the angular speed when the particles reach their extreme positions at $3r_0$?

8.9. Two particles Q and S of equal mass m are attached to the ends of a rigid rod of negligible mass, and a third particle P of mass m is tied by an inextensible string to a point R of the rod at distances a and b from Q and S, respectively, with $a > b$. The system is at rest on a smooth horizontal surface when the particle P is projected in the horizontal plane with constant velocity $\mathbf{v}_0$ away from and perpendicular to the rod at R. Find the velocity of P immediately after the string becomes taut, and determine the angular speed of the rod. What are the results for the special case when $a = b$?

8.10. Two particles of mass m_1 and m_2 are connected by a massless rigid rod of length l initially situated on a smooth horizontal surface, along the $\mathbf{I}$-axis of the plane frame $\Phi = \{O; \mathbf{I}, \mathbf{J}\}$ with m_1 at O. The system is given an initial angular velocity $\boldsymbol{\omega}_0 = \omega_0 \mathbf{K}$ about O. (a) Find as functions of time the position $\mathbf{x}^*(t)$ of the center of mass C in Φ and the placement $\theta(t)$ of the rod. (b) Find as functions of time the moment of momentum about C and about O.

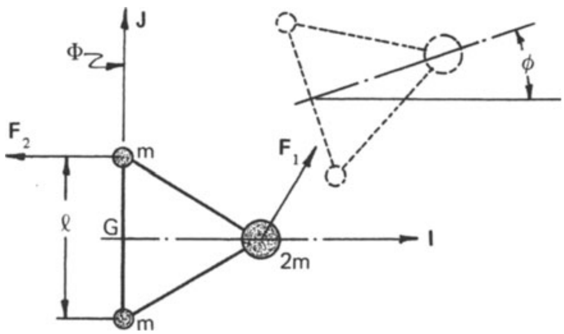
8.11. Two particles of mass m_1 and m_2 are connected by a massless rigid rod of length l initially situated along the $\mathbf{i}$-axis of the vertical plane frame $\varphi = \{O; \mathbf{i}, \mathbf{j}\}$ with m_1 at O. The system is given an initial angular velocity $\boldsymbol{\omega}_0 = \omega_0 \mathbf{k}$ about O. (a) Determine the position $\mathbf{x}^*(t)$ of the center of mass C in φ and the inclination $\theta(t)$ of the rod at time t. (b) Find as functions of time the moment of momentum about C and about O.

8.12. Three particles of mass m, $2m$, and $3m$ are moving with constant velocities $\mathbf{v}_1 = 2v\mathbf{i} + v\mathbf{j}$, $\mathbf{v}_2 = v\mathbf{k}$, and $\mathbf{v}_3 = v\mathbf{i} + 2v\mathbf{k}$, respectively, in $\Phi = \{O; \mathbf{i}_k\}$. Initially, the particles are at $\mathbf{X}_1 = \mathbf{j} + 2\mathbf{k}$, $\mathbf{X}_2 = 3\mathbf{i} + \mathbf{k}$, and $\mathbf{X}_3 = 2\mathbf{i} + 3\mathbf{j}$ in Φ. (a) Find the motion of the center of mass, and determine its initial location in Φ. (b) Determine the momentum and the kinetic energy of the system. What is the kinetic energy relative to the center of mass? (c) Find the initial values of (i) the moment about O of the momentum of the center of mass, (ii) the moment about O

of the momentum of the system, and (iii) the moment of momentum relative to the center of mass.

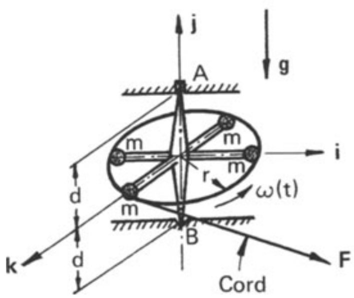
8.13. Two particles of equal mass m are connected by a rigid rod of length l and negligible mass, initially at rest along the X-axis on a smooth horizontal surface in frame $\Phi = \{O; \mathbf{i}, \mathbf{j}\}$. A constant force $\mathbf{F} = F\mathbf{j}$ acts on the particle at the right-hand end. Find the motion of the center of mass as a function of time; and find the angular speed of the rod as a function of its angular placement ϕ. Describe the motion of the rod.

8.14. A system of three particles of masses m, m, and $2m$ form the vertices of a massless rigid, equilateral triangular frame of side ℓ. The system, initially at rest in $\Phi = \{G; \mathbf{I}, \mathbf{J}\}$, moves in the horizontal plane under the action of follower forces $\mathbf{F}_1$ and $\mathbf{F}_2$ having equal constant magnitude F and directed always perpendicular to the sides of the triangle as shown in the figure. Find the angular speed $\dot{\phi}(t)$ of the triangular frame.



Problem 8.14.

8.15. A simple top consists of four particles of equal mass m spaced equally on a circular hoop of radius r. The rigid spokes, the axle, and the hoop have negligible mass, and the ball bearing at A and the ball-thrust bearing at B are frictionless. A cord of negligible mass is wound around the hoop as shown, and a constant tangential force $\mathbf{F}$ is applied for 2 sec. (a) Find the dynamic bearing reactions as functions of t. (b) Determine the angular velocity $\omega(t)$ of the system about the axle. (c) Find the angular speed when the cord is free.



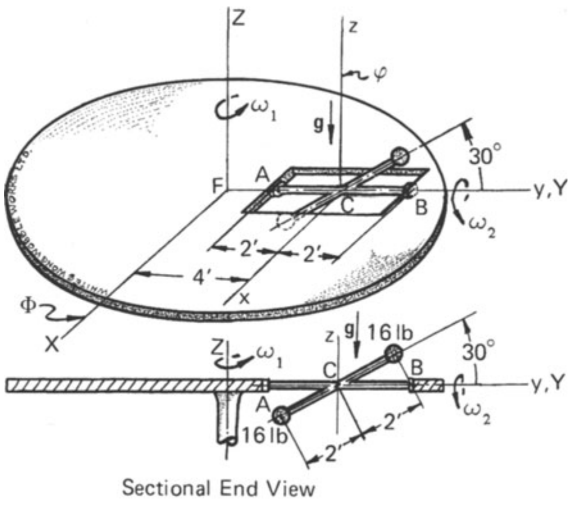
Problem 8.15.

8.16. Two particles of mass m_1 and m_2 are connected by an inextensible string of length l and negligible mass. The string passes through a small hole O in a smooth horizontal table. The mass m_1 moves on the table with cylindrical coordinates $(r(t), \phi(t))$, while m_2 hangs vertically below O. (a) Show that the differential equations of motion for the system are given by

$$\frac{d(m_1 r^2 \dot{\phi})}{dt} = 0, \qquad (m_1 + m_2)\,\ddot{r} - m_1 r \dot{\phi}^2 + m_2 g = 0. \tag{P8.16}$$

(b) Find the tension in the string as a function of r alone. (c) Equations (P8.16) show that a steady-state solution is possible for which m_2 is at rest while m_1 moves with a constant angular speed $\dot{\phi} = \omega$ on a circle of radius $r = a$, say. Find the steady-state values of a and ω. (d) Show that this dynamic equilibrium state is infinitesimally stable with respect to a small disturbance of m_1 for which $r = a + \eta$, where η and its derivatives are small quantities; in fact, establish that the system will execute small amplitude oscillations of frequency $f = (\omega/2\pi)[3m_1/(m_1 + m_2)]^{1/2}$.

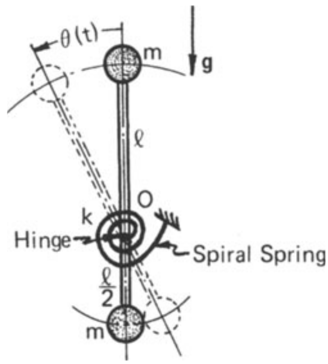
8.17. A wobble mechanism is modeled as a rigid system of two 16 lb balls attached symmetrically to the ends of a 4 ft rigid shaft welded at a 30° angle to a horizontal axle held in a roller bearing at A and restrained by an axial thrust bearing at B. The shaft–axle assembly rotates, as shown, with a constant angular speed $\omega_2 = 50$ rad/sec relative to a large circular control gear that has a constant angular speed $\omega_1 = 10$ rad/sec in a high elevation ground frame $\Phi = \{F; \mathbf{I}_k\}$ where $\mathbf{g} = -32\mathbf{k}$ ft/sec^2. At the instant of interest shown in the figure, the wobble shaft is in the vertical yz-plane. Find the dynamic bearing reactions at A and B at the moment of interest. Ignore the mass of the shaft–axle assembly.



Problem 8.17.

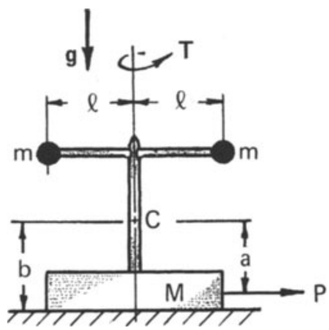
8.18. A pendulum rod of length l and negligible mass is hinged at O to a block of mass m_1 that slides freely on a smooth horizontal shaft parallel to the x-axis. The bob has mass m_2 and oscillates freely without friction in the vertical plane, suspended below m_1 with the placement $\theta(t)$. (a) Derive the coupled equations of motion for the system. (b) Show that when the variables x and θ together with their time derivatives are small quantities whose products are negligible, these equations reduce to two linear equations for x and θ, the latter being the equation of motion for a simple pendulum with period $\tau = 2\pi[m_1 l/g(m_1 + m_2)]^{1/2}$. (c) For small, but otherwise arbitrary initial data find the motions $x(t)$ and $\theta(t)$.

8.19. A pendulum shown in the figure consists of two bobs of equal mass m attached to the ends of a rigid rod of negligible mass. The rod is hinged at O to a torsion spring of stiffness k that supplies a restoring torque proportional to the angular placement θ from the vertical axis. (a) Derive the differential equation for the finite amplitude motion $\theta(t)$ of the system. (b) Determine the angular motion $\theta(t)$ for a small initial placement θ_0 of rest from the vertical position. (c) Discuss the infinitesimal stability of the equilibrium positions of the system.



Problem 8.19.

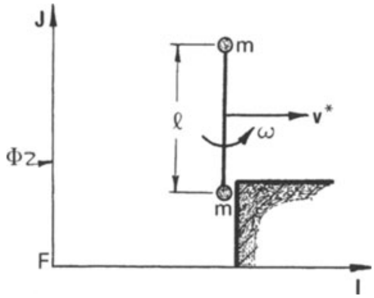
8.20. A block of mass M, subjected to a constant force $\mathbf{P} = P\mathbf{i}$ in the vertical plane at a distance a below the center of mass of the system at C, moves over a rough horizontal surface with coefficient of friction ν. Two small spheres of equal mass m are attached symmetrically to a rigid rod of length 2ℓ which is driven by a constant torque $\mathbf{T} = T\mathbf{j}$. The system shown in the figure is at rest initially when both $\mathbf{P}$ and $\mathbf{T}$ are applied simultaneously. The design geometry is such that the system does not tip. (a) Find the velocity of the block when the system has moved a distance d. (b) Determine the angular speed ω and the angular acceleration $\dot{\omega}$ of the rod after n revolutions. (c) Evaluate the results for the case when $M = 20$ slug, $m = 2$ slug, $\mathbf{P} = 832\mathbf{i}$ lb, $\mathbf{T} = 64\mathbf{k}$ ft · lb, $\nu = 1/3$, $\ell = 4$ ft, $d = 12$ ft, $n = 4$, and $g = 32$ ft/sec^2.



Problem 8.20.

8.21. A small object of mass M is in outer space where all gravitational forces are negligible. The object is initially at rest relative to an inertial frame $\Phi = \{E; \mathbf{i}_k\}$ when suddenly it explodes into two splinters having masses m_1 and m_2. Their subsequent relative velocity of separation is $\mathbf{v}$. Find their velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in Φ, and determine the kinetic energy of the system.

8.22. Two particles of equal mass m are attached to a rigid rod of negligible mass and length ℓ. The system is moving on a smooth horizontal surface with an angular velocity ω and center of mass velocity $\mathbf{v}^*$ when suddenly one end of the rod makes a normal impact with a wall, as illustrated. There is no loss of energy during the collision. (a) Find the instantaneous impulse of the force and the instantaneous torque impulse about the center of mass due to the impact. (b) Show that the impact results in an interchange of translational and rotational kinetic energies of the system. (c) Describe the subsequent motion of the system.

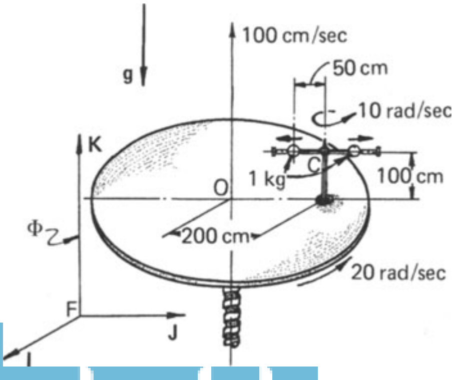


Problem 8.22.

8.23. The center of mass C of a rigid system of two particles of equal mass m separated a distance $2d$ is initially at rest in the vertical plane. The system is given a constant angular velocity $\boldsymbol{\omega} = \omega\mathbf{n}$ in a right-hand sense about an axis $\mathbf{n}$ at C, and the system is released to fall freely under gravity. The axis $\mathbf{n}$ is situated in the vertical plane at a fixed angle ϕ from the line joining the particles. What is the total kinetic energy of the system at time t?

8.24. In a general spatial motion of two particles of masses m_1 and m_2, the velocity of m_2 relative to m_1 is $\mathbf{v}$ and the center of mass has velocity $\mathbf{v}^*$. (a) What is the total kinetic energy of the system? (b) Let a be the perpendicular distance from one particle to the line through the other particle and parallel to $\mathbf{v}$. Show that the moment of momentum relative to the center of mass may be written as $\mathbf{h}_C = av[m_1m_2/(m_1 + m_2)]\mathbf{n}$, where $\mathbf{n}$ is a unit vector perpendicular to the plane containing the particles and the vector $\mathbf{v}$ and $v = |\mathbf{v}|$.

8.25. An antenna coil system consists of a rigid rod that rotates as shown with a constant angular speed of 10 rad/sec relative to a platform which is turning with an angular speed of 20 rad/sec while being raised vertically on a threaded shaft at a speed of 100 cm/sec in the ground frame Φ. At the instant of interest, each of two small coils of equal mass $m = 1$ kg are 50 cm from the center C and moving radially outward with a speed of 200 cm/sec relative to the rod. (a) Find the total momentum of the coil system in Φ. (b) Find the kinetic energy in Φ relative to the center of mass. (c) Determine the kinetic energy of the system in Φ. (d) What is the moment about point O in the platform of the momentum in Φ? (e) Determine the moment about O of the

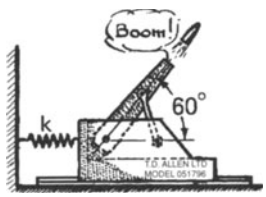


Problem 8.25.

momentum relative to O in Φ. (f) Find the moment about the center of mass C of the momentum in Φ. (g) What is the moment about C of the momentum relative to C in Φ.

8.26. Two particles of masses m and $2m$ are attached to the ends of a rigid rod of length l, negligible mass, and initially at rest along the Y-axis in the vertical plane frame $\Phi = \{O; \mathbf{I}, \mathbf{J}\}$. The mass m, initially at O, is subjected to a propulsive force $\mathbf{P}$ of constant magnitude and directed always perpendicular to the rod. Find the angular speed $\dot{\phi}(t)$ and the angular placement $\phi(t)$ of the system as functions of time. Formulate the problem in two ways: (i) write the work–energy equation with respect to the center of mass and (ii) write the equation for the moment of momentum relative to the center of mass. Notice that in neither case is it necessary to consider the motion of the center of mass.

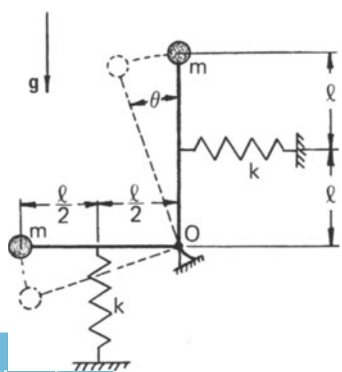
8.27. A 995 lb cannon with a recoil spring of stiffness $k = 193$ lb/ft is mounted on smooth horizontal rails. The gun fires a 5 lb shell with a muzzle velocity of 1500 ft/sec at a 60° angle, relative to the cannon. Determine the ultimate compression of the spring and the impulse reaction of the rails on the system.



Problem 8.27.

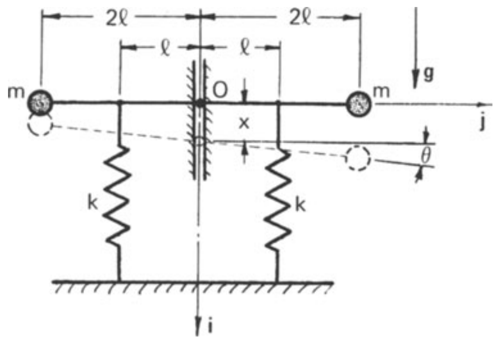
8.28. A shell explodes at the apex of its path into two pieces of equal mass m. One fragment is seen to fall vertically with initial speed $\dot{y}_0$. Find the path of the other splinter in a Cartesian reference frame with origin at the apex. Neglect frictional effects.

8.29. Two particles of equal mass m are attached to the ends of a rigid, right angle frame of negligible mass and supported by a smooth hinge at point O. Identical springs with stiffness k are attached at the midpoint of each rod, the horizontal spring being unstretched in the equilibrium configuration shown in the diagram. The system is given a small angular placement θ_0 and released from rest. (a) Discuss the infinitesimal stability of the equilibrium configuration of the system in terms of the static spring deflection θ_e by (i) use of the moment of momentum principle and (ii) use of the energy equation. (b) Find the motion $\theta(t)$ of the system.



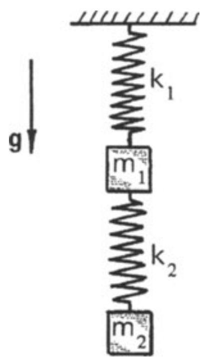
Problem 8.29.

8.30. Two particles of equal mass m are connected to the ends of a massless rigid rod of length 4ℓ and supported symmetrically by identical springs of stiffness k, as shown in the diagram. The center of the rod at O is constrained by smooth vertical rails to move only in the vertical plane. Initially, the rod is held horizontally so that the springs are unstretched, then turned clockwise about O through a small angle θ_0 and released to perform small oscillations in the vertical plane. (a) Apply momentum principles to derive the equations of motion of the system and solve them for the assigned initial conditions. What are the frequencies of the vertical and rotational oscillations? (b) Can the same results be obtained from the energy method? Explain.



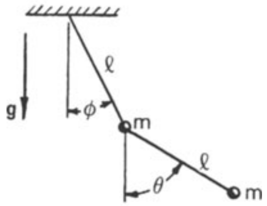
Problem 8.30.

8.31. A coupled system consists of a mass m_1 suspended vertically from a spring of stiffness k_1 and of another mass m_2 suspended from m_1 by a second spring of stiffness k_2. (a) Apply Newton's law to derive the equation of motion for each particle. (b) Write the energy equation for the system. Is it possible to derive from this equation the separate equations of motion for each particle? Explain.



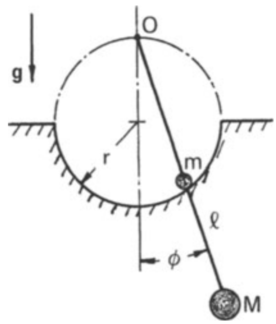
Problem 8.31.

8.32. A double pendulum consists of two bobs of equal mass m attached to the ends of two inextensible strings of equal length ℓ and negligible mass. The pendula are given small displacements shown in the figure and released to perform small plane oscillations. (a) Use Newton's law to derive the equations of motion for each particle. (b) Is it possible to derive these relations from the energy equation for the system? Explain.



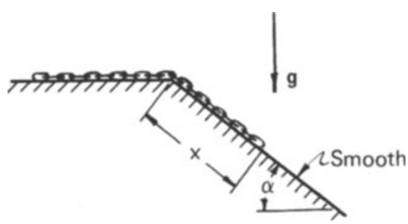
Problem 8.32.

8.33. A pendulum bob of mass M is attached to a smooth rigid rod of negligible mass and length ℓ supported at point O. A second mass m, which can slide freely along the pendulum rod, is constrained to move on a smooth circular surface of radius r as the system swings in the vertical plane, as illustrated. Derive the equation of motion for the system in two ways: (a) by use of the moment of momentum principle and (b) by use of the energy principle. (c) What is the first integral of the equation of motion when the system is released from rest at the placement ϕ_0? (d) What is the small amplitude oscillation frequency of the system?



Problem 8.33.

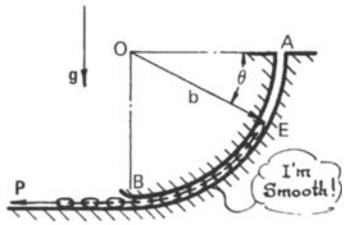
8.34. A uniform, inextensible heavy chain of length l, initially at rest on the horizontal section of the smooth surface (when $x = 0$), is given a small, ignorable disturbance causing it to slide down the inclined plane, shown in the figure. (a) Find the speed v of the chain as a function of its end distance x along the inclined surface. (b) Consider the case when $\alpha = \pi/2$ and the chain has an initial vertical overhang of length $x(0) = a$. Find the speed of the end of the chain as a function of x.



Problem 8.34.

8.35. A uniform, inextensible chain of length $l = \pi b/2$ is pulled by a constant force **P** from a smooth, quarter circle tube of radius b, situated in the vertical plane. The horizontal surface also is smooth. (a) If the chain initially is at rest when its end point E is at A, find its speed $v(\theta)$

as a function of its end placement θ shown in the figure. (b) What is the speed of E when it exits the tube at B? (c) Find $v(\theta)$ for the case when the chain merely slides from its initial state under gravity.



Problem 8.35.

8.36. Consider a body of mass M attached to a very long, uniform and inextensible coiled rope at rest in the horizontal plane. Suppose that the mass M is projected vertically upward from the plane with an initial speed v_0, so that the rope subsequently uncoils and follows vertically behind. This is an example of a variable mass system for which the principle of conservation of energy does not hold, even though the only force acting on this system is the conservative gravitational force. In this case, the first integral of the equation of motion $\mathbf{F} = d\mathbf{p}^*/dt$ does not lead to the work–energy principle. (a) Show that the first integral of this equation for a variable mass $m(t)$ is given by

$$\Delta \frac{1}{2}\mathbf{p}^* \cdot \mathbf{p}^* = \int_{t_0}^{t} \mathbf{F} \cdot \mathbf{p}^* dt. \tag{P8.36}$$

When the total mass is constant, this rule reduces to the familiar work–energy principle for the center of mass; otherwise, it does not. (b) Now, return to the coiled rope problem, let σ denote the mass per unit rope length and determine the maximum height h to which M will ascend. (c) What is the condition on v_0 in order that $h \geq l$ for a rope of length l? Assume that this condition holds for a sufficiently large initial velocity, and find the greatest height $h > l$ attained by M.

8.37. Two putty balls of masses m and $3m$ are moving toward one another with constant velocities $\mathbf{v}_1 = 2v\mathbf{i} + v\mathbf{j}$ and $\mathbf{v}_2 = v\mathbf{k}$, respectively, in $\Phi = \{O; \mathbf{i}_k\}$ when they collide in an oblique, direct impact and stick together. (a) Find the velocity of the single particle formed by the collision. (b) Determine the change in the kinetic energy of the system. Does it increase, decrease, or remain unchanged?

8.38. Two particles of equal mass m are connected by a rigid rod of length l and negligible mass, initially at rest along the X-axis on a smooth horizontal surface in frame $\Phi = \{O; \mathbf{I}, \mathbf{J}\}$. A third particle of mass m moving with velocity $\mathbf{v} = v\mathbf{J}$ in Φ collides with the particle at the right-hand end of the rod in a perfectly elastic, direct central impact. Find the subsequent motion of the center of mass of the original two particle system, and determine the angular speed of the rod.

8.39. A particle of mass m, attached by a light inextensible string to the center of a smooth horizontal table, is moving in a circle of radius r with speed v when it strikes an unconstrained particle of mass M at rest at a point r on the table. (a) Suppose the collision is perfectly inelastic. Find the angular speed after the collision, and show that the tension T in the string is reduced in the ratio $T/T_0 = m/(m + M)$, where T_0 is the initial tension. (b) Suppose the collision is perfectly elastic. Find the angular speed of m and the velocity of M after the impact, and determine the ratio in which the string tension is reduced.

8.40. Two pendulums of equal length l have bobs of masses m_1 and m_2, and both are suspended vertically from the same point O. The mass m_1 is displaced and released from rest at a height h (i.e. at an initial placement θ_0 from m_2). (a) Assume there is no energy loss in the impact, and find the velocities of m_1 and m_2 immediately afterward. (b) Apply the law of restitution to find these velocities. (c) Describe the results for three cases: $m_1 > m_2$, $m_1 < m_2$, and $m_1 = m_2$. (d) Find the common velocity of m_1 and m_2 following a perfectly inelastic collision, and determine the kinetic energy lost in the impact.

9

The Moment of Inertia Tensor

9.1. Introduction

We know that Euler's first law (5.43) relates the total external applied force on a rigid body to the motion of its center of mass, and in the next chapter we shall demonstrate that Euler's second law (5.44) relates the total external applied torque to the body's rotational motion through its moment of momentum vector. The latter involves introduction of the moment of inertia tensor studied here; and, of course, the first law involves the location of the center of mass of the body. We begin, therefore, with the concept of the center of mass of a complex structured body and illustrate its application to a materially nonhomogeneous body having a complex shape and cavities. Then the inertia tensor is introduced, and its components for some special homogeneous bodies are determined. Afterwards, some important physical properties of the moment of inertia tensor, properties actually characteristic of all kinds of symmetric tensors, are derived. Consequently, as an additional benefit, study of the inertia tensor provides tools useful, for example, in the study of the mechanics of deformable solid and fluid materials in which stress, strain, and deformation rate tensors play a major role.

9.2. The Center of Mass of a Complex Structured Rigid Body

The center of mass of a rigid body $\mathscr{B}$ is a unique point that moves with the body and whose position vector $\mathbf{x}^*(\mathscr{B})$ in an arbitrary spatial frame is defined by (5.12), namely,

$$m(\mathscr{B})\mathbf{x}^*(\mathscr{B}) = \int_{\mathscr{B}} \mathbf{x}(P)dm(P). \tag{9.1}$$

Although the center of mass point may or may not be situated at a material point, its motion and momentum are the same as those of a particle of mass $m(\mathscr{B})$, the body's mass. Determination of the center of mass of a homogeneous body having a simple geometrical shape is straightforward. In general, however, bodies are complex structures that often do not have conveniently simple geometrical shapes, they may not be materially homogeneous, and they may have cavities. A body composed of an assembly of materially different homogeneous bodies with holes is a typical example. In this case, the body may be treated as a composition of several simpler bodies each of whose mass and center of mass are readily determined. As a consequence, a complex structured body is called a *composite body*.

To derive the equation for the center of mass of a complex structured body, we consider an arbitrary rigid and possibly materially nonhomogeneous body $\mathscr{B}$ having a complex shape with cavities. Now divide $\mathscr{B}$ into n separate, geometrically or materially simple parts $\mathscr{B}_k$ so that $\mathscr{B} = \cup_{k=1}^{n} \mathscr{B}_k$. Then the total mass of $\mathscr{B}$ may be written as

$$m(\mathscr{B}) = \int_{\cup_{k=1}^{n} \mathscr{B}_k} dm(P) = \sum_{k=1}^{n} \int_{\mathscr{B}_k} dm(P) = \sum_{k=1}^{n} m_k, \tag{9.2}$$

in which $m_k \equiv m(\mathscr{B}_k)$ is the total mass of the k^{th} simple part. Each part may be materially different and nonhomogeneous. This natural result shows that *the total mass of a composite body $\mathscr{B}$ is equal to the sum of the masses of its simple parts $\mathscr{B}_k$*.

If $\mathscr{B}$ has p separate cavities $\mathscr{C}_k$, say, we may imagine that each cavity is filled with material having the same mass distribution as $\mathscr{B}$. In this case, we may consider an auxiliary solid body defined by $\mathscr{B}_S = \mathscr{B} \cup_{k=1}^{p} \mathscr{C}_k$ and apply (9.2) to obtain

$$m(\mathscr{B}_S) = m(\mathscr{B}) + \sum_{k=1}^{p} m(\mathscr{C}_k). \tag{9.3}$$

Therefore, the mass of the actual body is determined by

$$m(\mathscr{B}) = m(\mathscr{B}_S) - \sum_{k=1}^{p} m(\mathscr{C}_k), \tag{9.4}$$

whose interpretation is evident. The same relation follows from (9.2) applied directly to $\mathscr{B} = \mathscr{B}_S \backslash \cup_{k=1}^{p} \mathscr{C}_k$, the auxiliary solid body with all of its separate filled cavities removed. Clearly, (9.4) may be applied to any separate part $\mathscr{B}_j$ having a cavity $\mathscr{C}_j$.

In the same way, the integral in (9.1) for $\mathscr{B} = \cup_{k=1}^{n} \mathscr{B}_k$ may be written as

$$\int_{\mathscr{B}} \mathbf{x}(P) dm(P) = \sum_{k=1}^{n} \int_{\mathscr{B}_k} \mathbf{x}(P) dm(P) = \sum_{k=1}^{n} m(\mathscr{B}_k) \mathbf{x}^*(\mathscr{B}_k), \tag{9.5}$$

in which (9.1) has been applied to each separate part $\mathscr{B}_k$ in the first sum. Thus, by (9.1), *the center of mass* $\mathbf{x}^*(\mathscr{B})$ *of a composite body $\mathscr{B}$ is provided by*

$$m(\mathscr{B}) \mathbf{x}^*(\mathscr{B}) = \sum_{k=1}^{n} m(\mathscr{B}_k) \mathbf{x}^*(\mathscr{B}_k). \tag{9.6}$$

Of course, each part may be materially different and nonhomogeneous. Notice that (9.6) has the same form as (5.5) for the center of mass of a system of particles, each "particle" being a center of mass object.

Similarly, if $\mathscr{B}$ contains p cavities $\mathscr{C}_k$, use of $\mathscr{B} = \mathscr{B}_S \backslash \cup_{k=1}^{p} \mathscr{C}_k$ in (9.1) delivers

$$m(\mathscr{B})\mathbf{x}^*(\mathscr{B}) = m(\mathscr{B}_S)\mathbf{x}^*(\mathscr{B}_S) - \sum_{k=1}^{p} m(\mathscr{C}_k)\mathbf{x}^*(\mathscr{C}_k), \tag{9.7}$$

in which $\mathbf{x}^*(\mathscr{B}_S)$ is the center of mass of the solid body $\mathscr{B}_S$ composed of n solid parts $\mathscr{B}_k^S$, and $\mathbf{x}^*(\mathscr{C}_k)$ is the center of mass of the k^{th} materially similar solid body that fills the hole $\mathscr{C}_k$ in $\mathscr{B}_k$. Clearly, $m(\mathscr{B}_S)$ may be found by use of (9.2) applied to $\mathscr{B}_S$. Then $m(\mathscr{B})$ is given by (9.4), and the first term on the right in (9.7) may be obtained by use of (9.6) applied to $\mathscr{B}_S$, that is, $m(\mathscr{B}_S)\mathbf{x}^*(\mathscr{B}_S) = \sum_{k=1}^{n} m(\mathscr{B}_{Sk})\mathbf{x}^*(\mathscr{B}_{Sk})$, where $\mathscr{B}_{Sk}$ is the k^{th} solid simple part of $\mathscr{B}_S$.

If we view a cavity $\mathscr{C}_k$ as a "body" of negative volume, hence negative mass, and materially similar to the simple body $\mathscr{B}_k$ containing $\mathscr{C}_k$, (9.7) may be rewritten in the same form as (9.6); and (9.4) can be cast in the form of (9.2). *Hence, the rule of composition for the center of mass* $\mathbf{x}^*(\mathscr{B})$ *of a complex structured body* $\mathscr{B}$ *of mass* $m(\mathscr{B})$ is *summarized by the familiar general formula*

$$m(\mathscr{B})\mathbf{x}^*(\mathscr{B}) = \sum_{k=1}^{n} m_k \mathbf{x}_k^*, \qquad \text{with} \quad m(\mathscr{B}) = \sum_{k=1}^{n} m_k. \tag{9.8}$$

Herein $m_k \equiv m(\mathscr{B}_k)$ and $\mathbf{x}_k^* \equiv \mathbf{x}^*(\mathscr{B}_k)$ denote the mass and the center of mass of the k^{th} "body" $\mathscr{B}_k$, respectively.

Example 9.1. For an easy illustration, consider the homogeneous cylinder in Fig. 5.3, page 13. The central cylindrical hole is identified as $\mathscr{C}_1$. The mass of the solid, homogeneous cylinder called $\mathscr{B}_S$ is $m(\mathscr{B}_S) = \rho\pi r_0^2 \ell$, and the mass of a materially similar solid that fills $\mathscr{C}_1$ is $m(\mathscr{C}_1) = \rho\pi r_i^2 \ell$. Thus, from (9.4) or the second equation in (9.8), the total mass of the tube $\mathscr{B} = \mathscr{B}_S \backslash \mathscr{C}_1$ is

$$m(\mathscr{B}) = m(\mathscr{B}_S) - m(\mathscr{C}_1) = \rho\pi\ell\left(r_0^2 - r_i^2\right). \tag{9.9a}$$

In addition, for a homogeneous solid circular cylinder, we have

$$m(\mathscr{B}_S)\mathbf{x}^*(\mathscr{B}_S) = \left(\rho\pi r_0^2\ell\right)\frac{\ell}{2}\mathbf{k}, \qquad m(\mathscr{C}_1)\mathbf{x}_1^*(\mathscr{C}_1) = \left(\rho\pi r_i^2\ell\right)\frac{\ell}{2}\mathbf{k}, \tag{9.9b}$$

in frame φ in Fig. 5.3. Hence, by (9.9a) and (9.7) or the first equation in (9.8),

$$m(\mathscr{B})\mathbf{x}^*(\mathscr{B}) = \rho\frac{\pi\ell^2}{2}\left(r_0^2 - r_i^2\right)\mathbf{k} = m(\mathscr{B})\frac{\ell}{2}\mathbf{k}. \tag{9.9c}$$

Thus, as we know, $\mathbf{x}^*(\mathscr{B}) = \frac{1}{2}\ell\mathbf{k}$ in frame φ.

9.3. The Moment of Inertia Tensor

Lex $\mathbf{x}(P)$ denote the position vector from a base point Q of a rigid body $\mathscr{B}$ to a material parcel of mass $dm(P)$ at P. The tensor $\mathbf{I}_Q(\mathscr{B})$ defined by

$$\mathbf{I}_Q(\mathscr{B}) = \int_{\mathscr{B}} [(\mathbf{x}\cdot\mathbf{x})\,\mathbf{1} - \mathbf{x}\otimes\mathbf{x}]\,dm, \tag{9.10}$$

is called the *moment of inertia tensor relative to* Q, sometimes, briefly, the *inertia tensor*. Herein we recall from (3.31) the identity tensor $\mathbf{1} = \delta_{ij}\mathbf{e}_{ij}$ and from (3.24) the tensor product of vectors $\mathbf{a}$ and $\mathbf{b}$ for which $\mathbf{a}\otimes\mathbf{b} = a_i b_j \mathbf{e}_{ij}$, where $\mathbf{e}_{ij} \equiv \mathbf{e}_i \otimes \mathbf{e}_j$ is the tensor product basis associated with the orthonormal vector basis $\mathbf{e}_k$. Also, we observe the summation rule for repeated indices. From (9.10), $[\mathbf{I}_Q] = [ML^2]$, typical measure units being slug $\cdot$ ft^2 or kg $\cdot$ m^2.

Referred to a frame $\varphi = \{Q; \mathbf{e}_k\}$, the inertia tensor has the representation

$$\mathbf{I}_Q = I_{ij}^Q \mathbf{e}_{ij}, \tag{9.11}$$

in terms of its scalar components $I_{ij}^Q = \mathbf{e}_i \cdot \mathbf{I}_Q \mathbf{e}_j$ referred to $\mathbf{e}_{ij}$, in accordance with (3.15). It is seen from (9.10) that the moment of inertia tensor is symmetric: $\mathbf{I}_Q = \mathbf{I}_Q^T$, that is, $I_{ij}^Q = I_{ji}^Q$. Hence, only six of its nine scalar components are independent. From here onward, to simplify the component notation, the superscript Q usually is written only when we wish to emphasize the reference point being used.

Observe in (9.10) and (9.11) that only the base point need be fixed relative to the body; the reference frame $\varphi = \{Q; \mathbf{e}_k\}$ need not be. If φ is not an imbedded reference frame, however, the moment of inertia tensor referred to φ generally will vary with time as the body turns relative to φ. But if φ is an imbedded frame, then $\mathbf{I}_Q$ is a constant tensor whose components at Q will depend only on the body's fixed orientation in φ. In general then, the components of $\mathbf{I}_Q$ depend on the choice of reference point and on the orientation of the basis directions in the body. We shall return to these aspects later; but first the rectangular Cartesian components of $\mathbf{I}_Q$ in a body reference frame are described and a few general examples are studied.

9.4. Rectangular Cartesian Components of the Inertia Tensor

Let $\mathbf{e}_k = \mathbf{i}_k$ be a rectangular Cartesian reference basis at Q. Then the position vector of P from Q in the body frame $\varphi = \{Q; \mathbf{e}_k\}$ is $\mathbf{x}(P) = x_k\mathbf{i}_k$, and $\mathbf{x}\otimes\mathbf{x} = x_j x_k \mathbf{i}_{jk}$. Use of these relations and $\mathbf{1} = \delta_{jk}\mathbf{i}_{jk}$ in (9.10) yields

$$\mathbf{I}_Q = \left[\int_{\mathscr{B}} (\mathbf{x}\cdot\mathbf{x}\delta_{jk} - x_j x_k)dm\right]\mathbf{i}_{jk}, \tag{9.12}$$

and hence t*he rectangular Cartesian components of the moment of inertia tensor* are given by

$$I_{jk} = \int_{\mathscr{B}} (\mathbf{x} \cdot \mathbf{x} \delta_{jk} - x_j x_k) dm. \tag{9.13}$$

Now let $(x_1, x_2, x_3) = (x, y, z)$ as usual, and note that $\mathbf{x} \cdot \mathbf{x} = x^2 + y^2 + z^2$. Then (9.13) yields the explicit component relations

$$I_{11} = \int_{\mathscr{B}} (y^2 + z^2) dm, \quad I_{22} = \int_{\mathscr{B}} (x^2 + z^2) dm, \quad I_{33} = \int_{\mathscr{B}} (x^2 + y^2) dm, \tag{9.14}$$

$$I_{12} = I_{21} = -\int_{\mathscr{B}} xy dm, \quad I_{13} = I_{31} = -\int_{\mathscr{B}} xz dm, \quad I_{23} = I_{32} = -\int_{\mathscr{B}} yz dm. \tag{9.15}$$

The three components (9.14) are called *normal components of inertia*, and the six symmetric components (9.15) are known as *products of inertia*. It is important to note that in many dynamics books the products of inertia are defined somewhat differently as follows:

$$I_{xy} \equiv -I_{12} = \int_{\mathscr{B}} xy dm, \quad I_{xz} \equiv -I_{13} = \int_{\mathscr{B}} xz dm, \quad I_{yz} \equiv -I_{23} = \int_{\mathscr{B}} yz dm. \tag{9.16}$$

Therefore, the reader must exercise caution when consulting other sources.

It follows from (9.14) that

$$I_{11} + I_{22} = I_{33} + 2 \int_{\mathscr{B}} z^2 dm, \tag{9.17}$$

which occasionally is useful in calculations involving the normal components. Also,

$$\mathrm{tr}\mathbf{I}_Q = I_{11} + I_{22} + I_{33} = 2 \int_{\mathscr{B}} r^2 dm, \tag{9.18}$$

where $r^2 = \mathbf{x} \cdot \mathbf{x}$ is the squared distance from Q to the mass element dm. This rule involves a principal invariant of $\mathbf{I}_Q$ whose value in every reference basis at Q is the same.

For a homogeneous body, the mass density $\rho = dm/dV$ is a constant which may be extracted from the inertia integrals (9.14) and (9.15). The volume integrals that remain define what are known as *volume moments of inertia*.

9.4.1. Moments of Inertia for a Lamina

A thin, flat body $\mathscr{B}$ of negligible thickness h and elemental plane material area $dA(P)$ may be conveniently modeled as a *plane body,* or *lamina* for which $h \to 0$ and $\eta(P) \equiv dm(P)/dA(P)$, the ratio of the element of mass at P to the element of area at P, is the mass density per unit area. Also, $A(\mathscr{B}) = \int_{\mathscr{B}} dA(P)$ defines the total plane material area of $\mathscr{B}$. Let the lamina plane be the xy-plane. Then when $z \to 0$ in (9.14) and (9.15), we obtain the *moment of inertia tensor components for a lamina*:

$$I_{11} = \int_{\mathscr{B}} y^2 dm, \qquad I_{22} = \int_{\mathscr{B}} x^2 dm, \qquad I_{33} = I_{11} + I_{22}, \tag{9.19}$$

$$I_{12} = -\int_{\mathscr{B}} xy dm, \qquad I_{13} = I_{23} = 0. \tag{9.20}$$

For a homogeneous lamina, the constant density η may be removed from these integrals. The area integrals that remain define what are known as *area moments of inertia*.

9.4.2. Moment of Inertia About an Arbitrary Axis and the Radius of Gyration

The normal components of inertia in (9.14) and (9.19) have the general form

$$I_{nn} = \int_{\mathscr{B}} \bar{r}^2 dm, \tag{9.21}$$

in which $\bar{r}$ is the perpendicular distance from the axis n (with unit vector $\mathbf{n}$) to the element of mass dm, as shown in Fig. 9.1. Thus, (9.21) is called the *moment of inertia about the axis n* through the point Q. The result (9.21), however, holds more generally for an arbitrary axis through Q. The general proof is left for the reader in Problem 9.1. Another viewpoint is described in Problem 9.2.

The *radius of gyration about the axis n* is a positive scalar R_n defined by

$$R_n \equiv \sqrt{\frac{I_{nn}}{m(\mathscr{B})}}. \tag{9.22}$$

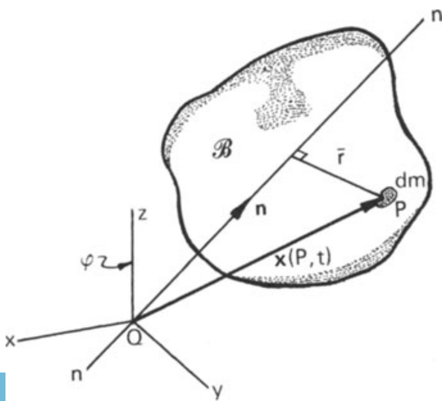


Figure 9.1. Schema for the moment of inertia about an axis n.

Since $m(\mathscr{B})R_n^2 = I_{nn}$, the squared radius of gyration R_n^2 is the average value of the integral (9.21). Further, it can be shown (see Problems 9.3 and 9.4.) that $I_{33} = mR^2$ is the moment of inertia about the central z-axis of a thin circular tube or ring of radius R. Therefore, by (9.22), the radius of gyration of any body $\mathscr{B}$ may be interpreted geometrically as the radius of an equivalent thin circular tube or ring having the same mass $m(\mathscr{B})$ and moment of inertia $I_{nn}(\mathscr{B})$ about the n-axis as those of the given body $\mathscr{B}$.

9.4.3. Moment of Inertia Properties of Symmetric Bodies

When n is the x-, y-, or z-axis, (9.21) coincides with the integrals in (9.14) and (9.19). Thus, $I_{11} = I_{xx}$, $I_{22} = I_{yy}$, $I_{33} = I_{zz}$, the normal Cartesian components, often are called *moments of inertia about the x-, y-, z-axes*, respectively. These integrals always are positive-valued, whereas the products of inertia (9.15) and (9.20) may be positive, negative, or zero. *For a homogeneous body having a plane of symmetry, if one of the coordinate planes contains the body plane of symmetry, the products of inertia involving the coordinate variable perpendicular to this plane will vanish.* Consider, for example, the homogeneous body shown in Fig. 9.2 for which the xz-plane is a body symmetry plane. Then the shape of the body surface to the right of this plane may be written as $y_s = f(x, z)$, and its surface to the left of this plane is described by $y_s = -f(x, z)$. Notice in (9.15) that both I_{12} and I_{23} contain y, the variable perpendicular to the xz-plane, the body symmetry plane. Thus, these products of inertia vanish upon integration with respect to the

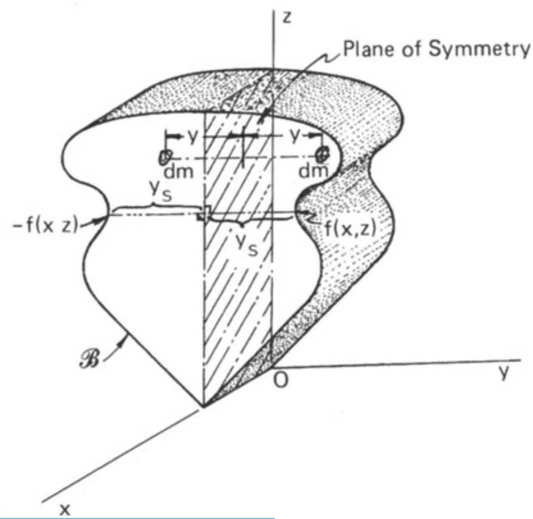


Figure 9.2. A homogeneous body having an xz-plane of symmetry.

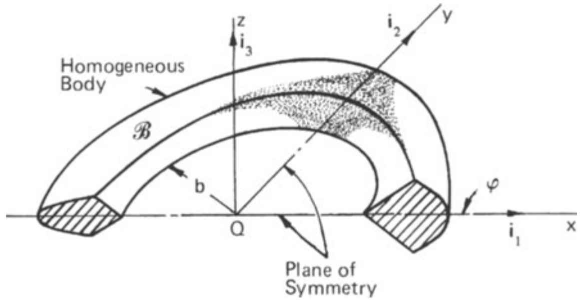


Figure 9.3. A homogeneous semicircular ring of variable cross section having an xy-plane of symmetry.

y variable. To see this, consider I_{12} in (9.15). With $dm = \rho dx dz dy \equiv d\mu dy$, we have

$$I_{12} = -\int_{(x,z)} \left[\int_{-f(x,z)}^{f(x,z)} y dy \right] x d\mu = 0. \tag{9.23}$$

Therefore, referred to coordinate planes that include a body plane of symmetry, at least two of the products of inertia for a homogeneous body will vanish. We shall return to this important property momentarily in some general remarks on axisymmetric bodies.

Example 9.2. What are the matrix and tensor forms of $\mathbf{I}_Q$ for the homogeneous body in Fig. 9.3 referred to the body frame $\varphi = \{Q; \mathbf{i}_k\}$?

Solution. Since the xy-plane is a body plane of symmetry for this homogeneous body, by (9.15), $I_{13} = I_{23} = 0$ in φ. Therefore, referred to the body frame φ in Fig. 9.3, we have the component matrix

$$I_Q = [\mathbf{I}_Q] = \begin{bmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}, \tag{9.24}$$

that is, in its tensor form, $\mathbf{I}_Q = I_{11}\mathbf{i}_{11} + I_{22}\mathbf{i}_{22} + I_{33}\mathbf{i}_{33} + I_{12}(\mathbf{i}_{12} + \mathbf{i}_{21})$. □

Now, consider a body having two orthogonal planes of symmetry. The line formed by the intersection of two orthogonal planes of symmetry of a body is called an *axis of symmetry*, and a body having an axis of symmetry is called *axisymmetric*. The plane geometrical figure formed by a cut through the body normal to an axis of symmetry is called a *cross section*. The cross section describes both the exterior and interior axisymmetric shapes of the body. Any line in the cross section through the axis of symmetry intersects the body at boundary points equidistant from the axis, and hence the point on the axis of symmetry in the cross section is called the *center of symmetry*. Now recall our previous result on a homogeneous body having a coordinate plane of symmetry. *In consequence, if the orthogonal planes of* *symmetry of an axisymmetric homogeneous body are chosen as coordinate planes of a body frame* $\varphi = \{Q; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, *then all products of inertia vanish relative to* Q *in* φ, *and hence the matrix* $[\mathbf{I}_Q]$ *for the moment of inertia referred to* φ *is diagonal*: $I_Q = \text{diag}[I_{11}, I_{22}, I_{33}]$.

A body having a circular (or annular), but not necessarily axially uniform cross section perpendicular to its central axis is called a *body of revolution.* A body of revolution having a cavity may be characterized by an internal surface of revolution different from its exterior surface of revolution, in which case the cross section is a circular annulus whose width varies along the central axis of rotational symmetry. Clearly, a body of revolution is an axisymmetric body for which *every plane* through its axis of symmetry is an identical plane of symmetry; its exterior and interior boundaries in the cross section are circles. *Therefore, if the axis of symmetry of a homogeneous body of revolution is the* z*-axis of a body frame* $\varphi = \{Q; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, *say, then all products of inertia vanish and* $I_{11} = I_{22}$ *relative to every point* Q *on the axis of symmetry*. Hence, in every body frame at Q that includes the $\mathbf{i}_3$ coordinate direction, the moment of inertia tensor for a body of revolution, with or without a cavity of revolution, has the same diagonal form $\mathbf{I}_Q = I_{11}(\mathbf{i}_{11} + \mathbf{i}_{22}) + I_{33}\mathbf{i}_{33}$. A homogeneous solid circular cylinder or tube, an ellipsoid of revolution with a central spherical cavity, and a right circular cone with a central conical cavity are examples of bodies of revolution that share these properties.

Let the frame $\Phi = \{Q; \mathbf{I}_k\}$ be fixed in space, and $\varphi = \{Q; \mathbf{i}_k\}$ fixed in the body at point Q. Suppose that a homogeneous rigid body is turning about a fixed axis of rotational symmetry $\mathbf{I}_3 = \mathbf{i}_3$ through Q. Then at every instant of time the values of the moment of inertia components referred to Φ are indistinguishable from their corresponding values referred to φ, all constant. In fact, the moment of inertia components at every instant of time will have the same values with respect to any reference frame at Q that contains the fixed central axis of rotational symmetry. If the homogeneous body is axisymmetric but not a body of revolution, however, the moments of inertia about orthogonal axes perpendicular to the axis of symmetry and rotation will be independent of time only when these axes are fixed in the body. For illustration, picture a homogeneous right elliptical cylinder, an axisymmetric body, turning about the z-axis, the axis of symmetry through Q, and visualize the (ellipse) frame φ at different instants in its rotation relative to Φ. Clearly, the components I_{11}^Q and I_{22}^Q vary with the orientation of the body when determined in Φ, whereas both are constant when referred to φ. Therefore, we may rightly speculate that dynamical problems generally will be greatly simplified by use of a body reference frame.

9.5. Moments of Inertia for Some Special Homogeneous Bodies

The foregoing examples illustrate some general properties of all homogeneous bodies. In this section, the focus shifts to three specific axisymmetric homogeneous

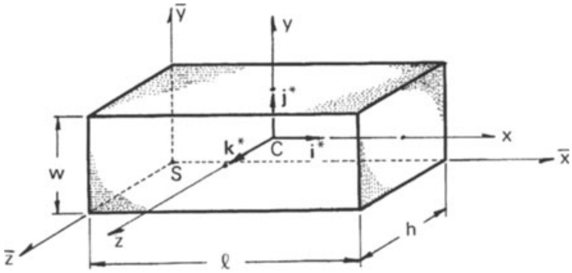


Figure 9.4. Geometry for a homogeneous rectangular block.

bodies—a rectangular body, a circular cylindrical body, and a sphere. The inertia tensors for these and some related slender bodies are derived.

9.5.1. Moment of Inertia Tensor for a Rectangular Body

The moment of inertia tensor $\mathbf{I}_C$ for a homogeneous rectangular block of length ℓ, width w, and height h is determined with respect to a body frame $\varphi = \{C; \mathbf{i}_k^*\}$ at its center of mass C and oriented as shown in Fig. 9.4. The results are then specialized to obtain the inertia tensor for a thin rectangular plate of thickness h and for a thin rod of length ℓ. The body frame at S in Fig. 9.4 is reserved for future use.

The homogeneous block has constant mass density ρ, and hence its center of mass C is at the centroid. The components I_{jk} of the inertia tensor $\mathbf{I}_C$ referred to φ are given by (9.14) and (9.15). Specifically, for the rectangular block,

$$I_{11} = \rho\ell \int_{-h/2}^{h/2} \left[\int_{-w/2}^{w/2} (y^2 + z^2) dy \right] dz = \rho \frac{\ell w h}{12} (w^2 + h^2).$$

That is, with $m \equiv m(\mathscr{B}) = \rho \ell w h$ for the mass of the block,

$$I_{11} = \frac{m(w^2 + h^2)}{12}. \tag{9.25a}$$

Noting the correspondence $(x, y, z) \sim (\ell, w, h)$ and permuting the symbols accordingly in the last two relations in (9.14) while bearing in mind (9.25a), we find

$$I_{22} = \frac{m(\ell^2 + h^2)}{12}, \qquad I_{33} = \frac{m(\ell^2 + w^2)}{12}. \tag{9.25b}$$

All products of inertia are zero. (Why?) Collecting these results in (9.11), we obtain the *moment of inertia tensor for a homogeneous rectangular parallelepiped*:

$$\mathbf{I}_C = \frac{m(w^2 + h^2)}{12} \mathbf{i}_{11}^* + \frac{m(\ell^2 + h^2)}{12} \mathbf{i}_{22}^* + \frac{m(\ell^2 + w^2)}{12} \mathbf{i}_{33}^*, \tag{9.26}$$

referred to the center of mass body frame $\varphi = \{C; \mathbf{i}_k^*\}$. Notice that the matrix of (9.26) is diagonal.

9.5.1.1. *Moment of Inertia of a Thin Rectangular Plate*

A thin plate of negligible thickness h is a lamina, a plane body in the xy-plane, with mass density $\eta = \rho h$ per unit area. Hence, $m = \eta \ell w$. Upon neglecting the h^2 terms in (9.26), we obtain *the moment of inertia tensor for a thin rectangular plate*:

$$\mathbf{I}_C = \frac{mw^2}{12}\mathbf{i}_{11}^* + \frac{m\ell^2}{12}\mathbf{i}_{22}^* + \frac{m(\ell^2 + w^2)}{12}\mathbf{i}_{33}^*, \tag{9.27}$$

referred to the center of mass body frame $\varphi = \{C; \mathbf{i}_k^*\}$. Notice that $I_{33} = I_{11} + I_{22}$ in accordance with the general rule in (9.19) valid for every plane body.

9.5.1.2. *Moment of Inertia of a Thin Rod*

Let $\sigma = \eta w$ denote the mass per unit length of the homogeneous body, so that $m = \sigma \ell$. Now, neglect terms of order w^2 in (9.27) to obtain the *moment of inertia tensor for a thin rod*:

$$\mathbf{I}_C = \frac{m\ell^2}{12}(\mathbf{i}_{22}^* + \mathbf{i}_{33}^*), \tag{9.28}$$

where the rod axis is the $\mathbf{i}_1^*$-axis in $\varphi = \{C; \mathbf{i}_k^*\}$.

9.5.2. Moment of Inertia of a Circular Cylindrical Body

The inertia tensor for a homogeneous, circular cylindrical tube of inside radius r_i, outside radius r_o, and length ℓ is derived relative to a center of mass body frame $\varphi = \{C; \mathbf{i}_k^*\}$ situated at $\ell/2$ from O in Fig. 5.3, page 13, with $\mathbf{i}_3^* = \mathbf{k}$. The result is then applied to find the inertia tensor for a solid cylinder, an annular lamina, a thin circular disk, and a thin rod.

We begin with the cylindrical tube. The moment of inertia about the z-axis is obtained from the last equation in (9.14). Introducing cylindrical coordinates with $x^2 + y^2 = r^2$ and noting that $dm = \rho 2\pi r \ell dr$, we have

$$I_{33} = \int_{\mathscr{B}} r^2 dm = 2\pi\rho\ell \int_{r_i}^{r_o} r^3 dr = \frac{\pi}{2}\rho\ell\left(r_o^4 - r_i^4\right).$$

With $m = \rho A \ell$ and the cross sectional area $A = \pi(r_o^2 - r_i^2)$, we obtain

$$I_{33} = \frac{m}{2}\left(r_o^2 + r_i^2\right). \tag{9.29a}$$

The z-axis is the central axis for this body of revolution; so, $I_{11} = I_{22}$, and hence by (9.17), $2I_{11} = I_{33} + 2\int_{\mathscr{B}} z^2 dm$, in which $dm = \rho A dz$. For a homogeneous material,

$$\int_{\mathscr{B}} z^2 dm = \rho A \int_{-\ell/2}^{\ell/2} z^2 dz = \frac{m\ell^2}{12}, \tag{9.29b}$$

and hence with (9.29a),

$$I_{11} = I_{22} = \frac{m}{4}\left(r_o^2 + r_i^2 + \frac{\ell^2}{3}\right). \tag{9.29c}$$

The products of inertia vanish. (Why?) Collecting (9.29a) and (9.29c), we reach the *moment of inertia tensor for a homogeneous circular cylindrical tube referred to the center of mass body frame* $\varphi = \{C; \mathbf{i}_k^*\}$:

$$\mathbf{I}_C = \frac{m}{4}\left(r_o^2 + r_i^2 + \frac{\ell^2}{3}\right)(\mathbf{i}_{11}^* + \mathbf{i}_{22}^*) + \frac{m}{2}(r_o^2 + r_i^2)\mathbf{i}_{33}^*. \tag{9.30}$$

9.5.2.1. *Moment of Inertia of a Solid Cylinder*

Now consider a solid cylinder for which $r_o = r$ and $r_i = 0$. Then (9.30) reduces to the *moment of inertia tensor for a solid circular cylinder referred to the center of mass body frame* $\varphi = \{C; \mathbf{i}_k^*\}$:

$$\mathbf{I}_C = \frac{m}{4}\left(r^2 + \frac{\ell^2}{3}\right)(\mathbf{i}_{11}^* + \mathbf{i}_{22}^*) + \frac{m}{2}r^2\mathbf{i}_{33}^*, \tag{9.31}$$

where $m = \rho\pi r^2\ell$. Upon neglecting the terms mr^2 in (9.31), we recover the inertia tensor (9.28) for a thin rod, except now the rod axis is $\mathbf{i}_3^*$. Therefore, the actual cross sectional geometry of a slender rod is unimportant.

9.5.2.2. *Moment of Inertia of an Annular Plate*

Relations (9.30) and (9.31) are used next to derive inertia tensors for similar plane bodies having mass density η per unit area. First, consider an annular plate, or flat washer, of negligible thickness ℓ and area $A = \pi(r_o^2 - r_i^2)$. Then with $m = \eta A$, (9.30) yields the *moment of inertia tensor for a homogeneous annular plate relative to the body frame* $\varphi = \{C; \mathbf{i}_k^*\}$:

$$\mathbf{I}_C = \frac{m}{4}(r_o^2 + r_i^2)(\mathbf{i}_{11}^* + \mathbf{i}_{22}^* + 2\mathbf{i}_{33}^*). \tag{9.32}$$

9.5.2.3. Moment of Inertia of a Thin Circular Disk

Finally, with $r_o = R$, $r_i = 0$, and $m = \eta\pi R^2$, (9.32) reduces to the *moment of inertia tensor for a homogeneous, thin circular disk relative to* $\varphi = \{C; \mathbf{i}_k^*\}$:

$$\mathbf{I}_C = \frac{m}{4} R^2 (\mathbf{i}_{11}^* + \mathbf{i}_{22}^* + 2\mathbf{i}_{33}^*). \tag{9.33}$$

Notice that (9.33) also follows directly from (9.31) upon neglecting terms in ℓ^2. In agreement with the last rule in (9.19) for every plane body, it is seen in both (9.32) and (9.33) that $I_{33} = I_{11} + I_{22}$ holds.

9.5.3. Moment of Inertia Tensor for a Sphere

The moment of inertia tensor for a homogeneous sphere of radius R is derived relative to a body frame $\varphi = \{C; \mathbf{i}_k^*\}$ at its center. Clearly, every plane through C is a plane of symmetry, so all products of inertia vanish and $I_{11} = I_{22} = I_{33}$. Hence, by (9.18), $3I_{11} = 2\int_{\mathscr{B}} r^2 dm$. The surface area of a sphere of radius r is $4\pi r^2$, so the elemental mass of a spherical shell of thickness dr is $dm = \rho 4\pi r^2 dr$; and the foregoing relation yields

$$I_{11} = \frac{8}{3}\rho\pi \int_0^R r^4 dr = \frac{2}{5} m R^2,$$

where $m = \frac{4}{3}\rho\pi R^3$ is the total mass of the sphere. *Therefore, in every body reference frame at its center, the moment of inertia tensor for a homogeneous sphere is*

$$\mathbf{I}_C = \frac{2}{5} m R^2 \mathbf{1}. \tag{9.34}$$

Exercise 9.1. Show that, relative to its center C, the moment of inertia tensor for a nonhomogeneous sphere of radius R whose mass density $\rho = \rho(r)$ varies with the radius $r \in [0, R]$ is spherical; i.e., $\mathbf{I}_C = I_C \mathbf{1}$, where $I_C = \frac{8\pi}{3}\int_0^R r^4 \rho(r) dr$. For constant ρ, this returns (9.34). □

This concludes discussion of the moment of inertia tensor for a few homogeneous rigid bodies. Additional results are summarized in the table of properties in Appendix D, and some further examples are provided in the problems.

9.6. The Moment of Inertia of a Complex Structured Body

A complex structured body $\mathscr{B} = \cup_{k=1}^n \mathscr{B}_k$ may be regarded as a composition of several materially or geometrically simple bodies $\mathscr{B}_k$ whose moment of inertia tensors are known or may be readily determined. Hence, for the composite body $\mathscr{B}$, (9.10) may be written as

$$\mathbf{I}_Q(\mathscr{B}) = \sum_{k=1}^{n} \int_{\mathscr{B}_k} [(\mathbf{x}\cdot\mathbf{x})\,\mathbf{1} - \mathbf{x}\otimes\mathbf{x}]\,dm,$$

in terms of the simple parts $\mathscr{B}_k$ of $\mathscr{B}$, each of which may be materially different and nonhomogeneous. Then application of (9.10) to each body $\mathscr{B}_k$ yields the *composition rule for the moment of inertia tensor for a composite body, relative to Q*:

$$\mathbf{I}_Q(\mathscr{B}) = \sum_{k=1}^{n} \mathbf{I}_Q(\mathscr{B}_k). \tag{9.35}$$

Hence, with respect to an assigned body frame $\varphi = \{Q; \mathbf{e}_k\}$, the moment of inertia tensor for a composite body is equal to the sum of the inertia tensors for its simple parts referred to φ.

If $\mathscr{B}$ has p cavities $\mathscr{C}_k$, say, we may imagine that each cavity is filled with material having the same mass distribution as $\mathscr{B}$. The composition rule (9.35) applied to the auxiliary solid body $\mathscr{B}_S = \mathscr{B} \cup_{k=1}^{p} \mathscr{C}_k$ determines the moment of inertia tensor for $\mathscr{B} = \mathscr{B}_S \backslash \cup_{k=1}^{p} \mathscr{C}_k$ in accordance with

$$\mathbf{I}_Q(\mathscr{B}) = \mathbf{I}_Q(\mathscr{B}_S) - \sum_{k=1}^{p} \mathbf{I}_Q(\mathscr{C}_k). \tag{9.36}$$

On the other hand, if a cavity $\mathscr{C}_k$ is viewed as a "body" of negative mass and materially similar to the simple body $\mathscr{B}_k$ containing $\mathscr{C}_k$, (9.36) may be summarized in the form (9.35). Each inertia tensor in the sum (9.35), however, must be referred to the same body frame φ at Q. In consequence, we shall need to know how to transform the inertia tensor components for each of the separate parts, from a reference frame at a point P conveniently chosen for calculation of the components for a separate part, to another reference frame at another point Q appropriate for the tensor components for the composite body. Before we tackle this problem, however, let us consider two examples of homogeneous bodies that require only a single reference frame. The second example is noteworthy because it shows that neither symmetry nor material homogeneity of a body is necessary for the vanishing of products of inertia.

Example 9.3. A homogeneous rectangular parallelepiped of length ℓ and square cross section of side h has a circular hole of radius R drilled lengthwise through its center, as shown in Fig. 9.5. Determine the moment of inertia tensor components of the drilled block referred to the center of mass body frame $\varphi = \{C; \mathbf{i}_k^*\}$.

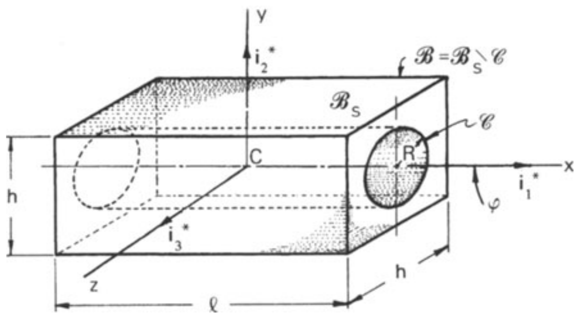


Figure 9.5. A homogeneous block $\mathscr{B}$ having a drilled hole.

Solution. The center of mass of the homogeneous drilled block $\mathscr{B} = \mathscr{B}_S \backslash \mathscr{C}$ is at the center of the hole. Here $\mathscr{B}_S$ denotes the solid rectangular block and $\mathscr{C}$ identifies a homogeneous circular cylinder of the same material which we imagine fills the hole. Thus, with respect to φ, (9.36) yields

$$\mathbf{I}_C(\mathscr{B}) = \mathbf{I}_C(\mathscr{B}_S) - \mathbf{I}_C(\mathscr{C}). \tag{9.37a}$$

Recalling (9.26) for a homogeneous solid parallelepiped with $w = h$ and (9.31) for a homogeneous solid cylinder of radius R, bearing in mind the arrangement of the coordinate axes in Fig. 9.5 and in Fig. 5.3, page 13, for the cylinder, we obtain from (9.37a), referred to the body frame $\varphi = \{C; \mathbf{i}_k^*\}$,

$$\begin{aligned}\mathbf{I}_C(\mathscr{B}) = {} & \frac{m_S h^2}{6}\mathbf{i}_{11}^* + \frac{m_S}{12}(h^2 + \ell^2)(\mathbf{i}_{22}^* + \mathbf{i}_{33}^*) \\ & - \left[\frac{m_C R^2}{2}\mathbf{i}_{11}^* + \frac{m_C}{4}\left(R^2 + \frac{\ell^2}{3}\right)(\mathbf{i}_{22}^* + \mathbf{i}_{33}^*)\right],\end{aligned} \tag{9.37b}$$

wherein the mass m_S of the solid block and m_C of the cavity body are given by

$$m_S \equiv m(\mathscr{B}_S) = \rho \ell h^2, \qquad m_C \equiv m(\mathscr{C}) = \rho \pi \ell R^2. \tag{9.37c}$$

Hence, by (9.4), the mass of the drilled block is

$$m(\mathscr{B}) = m_S - m_C = \rho \ell (h^2 - \pi R^2). \tag{9.37d}$$

Use of (9.37c) and (9.37d) in (9.37b) yields the moment of inertia tensor components for the homogeneous, drilled parallelepiped referred to the center of mass frame φ:

$$I_{11} = m(\mathscr{B})\left[\frac{h^4 - 3\pi R^4}{6(h^2 - \pi R^2)}\right], \qquad I_{22} = I_{33} = \frac{1}{2}I_{11} + \frac{m(\mathscr{B})\ell^2}{12}. \tag{9.37e}$$

□

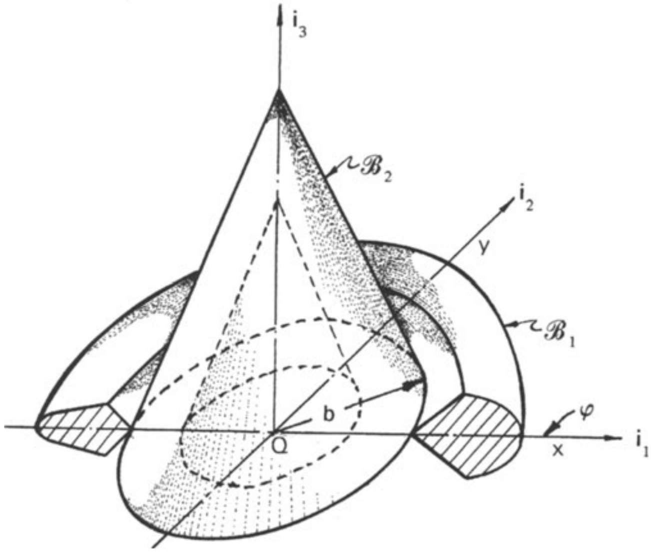


Figure 9.6. A complex structured rigid body composed of homogeneous, but materially different parts $\mathscr{B}_1$ and $\mathscr{B}_2$.

Example 9.4. The complex structured body $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ in Fig. 9.6 consists of a semicircular ring $\mathscr{B}_1$ of variable cross section, shown in Fig. 9.3, and a right conical shell $\mathscr{B}_2$. The bodies are made of different homogeneous materials welded together along their common circular boundary in the xy-plane. Find the matrix of the moment of inertia tensor for the composite body $\mathscr{B}$ referred to $\varphi = \{Q; \mathbf{i}_k\}$.

Solution. Let $\mathbf{I}'_Q(\mathscr{B}_1)$ and $\mathbf{I}''_Q(\mathscr{B}_2)$ denote the moment of inertia tensors for $\mathscr{B}_1$ and $\mathscr{B}_2$ referred to the same body frame $\varphi = \{Q; \mathbf{i}_k\}$ in Fig. 9.6. For the composite body $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$, (9.35) yields $\mathbf{I}_Q(\mathscr{B}) = \mathbf{I}'_Q(\mathscr{B}_1) + \mathbf{I}''_Q(\mathscr{B}_2)$. The matrix of $\mathbf{I}'_Q$ referred to $\varphi = \{Q; \mathbf{i}_k\}$ has the form (9.24), and that of $\mathbf{I}''_Q$ for the conical shell is diagonal with $I''_{11} = I''_{22}$. (Why?) Thus, with $[\mathbf{I}''_Q] = \operatorname{diag}[I''_{11}, I''_{11}, I''_{33}]$ and introduction of the $'$ notation in (9.24) yields the moment of inertia tensor for the nonhomogeneous, composite body $\mathscr{B}$ referred to φ:

$$[\mathbf{I}_Q] = \begin{bmatrix} I'_{11} + I''_{11} & I'_{12} & 0 \\ I'_{12} & I'_{22} + I''_{11} & 0 \\ 0 & 0 & I'_{33} + I''_{33} \end{bmatrix}.$$

The xy-plane clearly is not a plane of geometrical symmetry for $\mathscr{B}$, not to mention the nonhomogeneous nature of the assembly, yet $I_{13} = I_{23} = 0$. Thus, while symmetry of a homogeneous body with respect to an $\alpha\beta$-plane normal to γ implies that the products of inertia $I_{\alpha\gamma} = I_{\beta\gamma} = 0$, body symmetry is not a necessary condition for the vanishing of products of inertia. The general nature of this noteworthy observation will be explored after the transformation laws for the inertia tensor under parallel translation and rotation of the reference frame are in hand. □

9.7. The Parallel Axis Theorem

The parallel axis theorem is a useful transformation rule that provides relations connecting the moment of inertia tensor components in parallel bases at two base points. To deduce this rule, let us consider a rigid body $\mathscr{B}$ of any shape, homogeneous or not, and suppose that $\mathbf{I}_Q(\mathscr{B})$ is known in the body frame $\varphi = \{Q; \mathbf{i}_k\}$ at a base point Q. The problem is to determine $\mathbf{I}_S(\mathscr{B})$ in a parallel body frame $\psi = \{S; \mathbf{i}_k\}$ at another base point S, as shown in Fig. 9.7. The position vectors of a material point P from Q and S are respectively denoted by $\mathbf{x}(P)$ and $\rho(P) = \mathbf{r}(Q) + \mathbf{x}(P)$, where $\mathbf{r}(Q)$ is the position vector of Q from S. Let $\circ$ denote either the $\cdot$ or $\otimes$ operation, recall (9.1), and consider

$$\begin{aligned}\int_{\mathscr{B}} \rho \circ \rho dm &= \mathbf{r} \circ \mathbf{r} \int_{\mathscr{B}} dm + \mathbf{r} \circ \int_{\mathscr{B}} \mathbf{x} dm + \int_{\mathscr{B}} \mathbf{x} dm \circ \mathbf{r} + \int_{\mathscr{B}} \mathbf{x} \circ \mathbf{x} dm \\ &= m(\mathscr{B})[\mathbf{r} \circ \mathbf{r} + \mathbf{r} \circ \mathbf{x}^* + \mathbf{x}^* \circ \mathbf{r}] + \int_{\mathscr{B}} \mathbf{x} \circ \mathbf{x} dm, \end{aligned} \tag{9.38}$$

where $\mathbf{x}^*$ is the position vector of the center of mass C from Q in φ. Thus, use of (9.38) in (9.10) applied at S leads to the general point transformation rule for the moment of inertia tensor:

$$\mathbf{I}_S(\mathscr{B}) = \mathbf{I}_Q(\mathscr{B}) + m(\mathscr{B})[(\mathbf{r} \cdot \mathbf{r})\mathbf{1} - \mathbf{r} \otimes \mathbf{r} + (2\mathbf{r} \cdot \mathbf{x}^*)\mathbf{1} - \mathbf{r} \otimes \mathbf{x}^* - \mathbf{x}^* \otimes \mathbf{r}]. \tag{9.39}$$

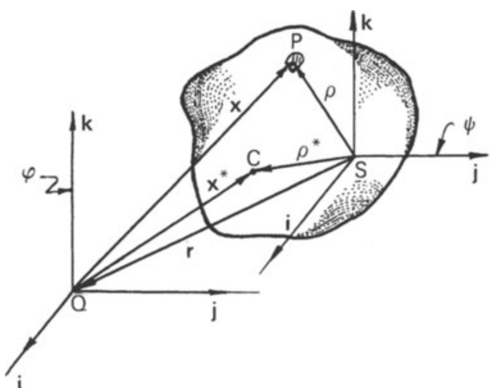


Figure 9.7. Schema for the parallel axis theorem.

This cumbersome expression is readily simplified by our choosing Q at the center of mass C located at $\boldsymbol{\rho}^*$ from S in Fig. 9.7; for then $\mathbf{x}^* = \boldsymbol{\rho}^* - \mathbf{r} = \mathbf{0}$, and (9.39) yields the following reduced point transformation rule:

$$\mathbf{I}_S(\mathscr{B}) = \mathbf{I}_C(\mathscr{B}) + \mathbf{I}_S^*(\mathscr{B}), \tag{9.40}$$

wherein, with $m = m(\mathscr{B})$,

$$\mathbf{I}_S^*(\mathscr{B}) \equiv m[(\boldsymbol{\rho}^* \cdot \boldsymbol{\rho}^*)\mathbf{1} - \rho^* \otimes \boldsymbol{\rho}^*], \tag{9.41}$$

is the *moment of inertia of the center of mass particle relative to* S. Accordingly, rule (9.40) states that *the moment of inertia relative to any point S is equal to the moment of inertia relative to the center of mass point C plus the moment of inertia of the center of mass particle relative to S.*

An important geometrical interpretation of (9.40) derives from its Cartesian component form in a body reference frame $\psi = \{S; \mathbf{i}_k\}$. Write $\boldsymbol{\rho}^* = x^*\mathbf{i} + y^*\mathbf{j} + z^*\mathbf{k}$ for the position vector of C from S in (9.41) to obtain the components of the inertia tensor of the center of mass particle relative to S:

$$[\mathbf{I}_S^*] = \begin{bmatrix} m(y^{*2} + z^{*2}) & -mx^*y^* & -mx^*z^* \\ -mx^*y^* & m(z^{*2} + x^{*2}) & -my^*z^* \\ -mx^*z^* & -my^*z^* & m(x^{*2} + y^{*2}) \end{bmatrix}. \tag{9.42}$$

So far, the frame $\psi = \{S; \mathbf{i}_k\}$ at S may have any orientation relative to $\varphi = \{C; \mathbf{i}_k^*\}$ at C. But we now consider the case when $\mathbf{i}_k = \mathbf{i}_k^*$, so that the coordinate directions at S are parallel to those at C. Then (9.40) yields the six component relations $I_{jk}^S = I_{jk}^C + I_{jk}^{S*}$, and use of (9.42) gives, as example,

$$I_{11}^S = I_{11}^C + md_1^2, \qquad I_{23}^S = I_{23}^C + I_{23}^{S*} = I_{23}^C - my^*z^*, \tag{9.43}$$

in which $d_1^2 = y^{*2} + z^{*2}$ is the square of the perpendicular distance between the parallel axes $\mathbf{i}_1$ at S and $\mathbf{i}_1^* = \mathbf{i}_1$ at C in Fig. 9.8. More generally, the six tensor component relations in a common Cartesian tensor basis $\mathbf{i}_{ab}$ are summarized by

$$I_{aa}^S = I_{aa}^C + md_a^2, \qquad I_{ab}^S = I_{ab}^C + I_{ab}^{S*}, \tag{9.44}$$

in which $a, b = 1, 2, 3$ (no sum) and $b \neq a$. Here d_a^2 is the square of the distance between the parallel $\mathbf{i}_a$-axis at S and the $\mathbf{i}_a^*$-axis at C, $I_{ab}^{S*} \equiv -ma^*b^*$, and $a^*,\ b^*$ are the $a,\ b$ coordinates of the center of mass particle from S. For $a = 2,\ b = 3$, for example, $a^* = y^*,\ b^* = z^*$, and hence $I_{23}^{S*} = -my^*z^*$ in (9.43) and $d_2^2 = x^{*2} + z^{*2}$. In view of (9.44), the point transformation rule (9.40) is characterized by the following useful theorem.

The parallel axis theorem: *The moment of inertia about an axis at S is equal to the moment of inertia about a parallel axis at the center of mass C plus the product of the mass and the square of the perpendicular distance between the parallel axes. Further, the product of inertia with respect to orthogonal axes at S is equal to the product of inertia with respect to the same parallel axes at C plus the corresponding product inertia of the center of mass particle relative to S.*

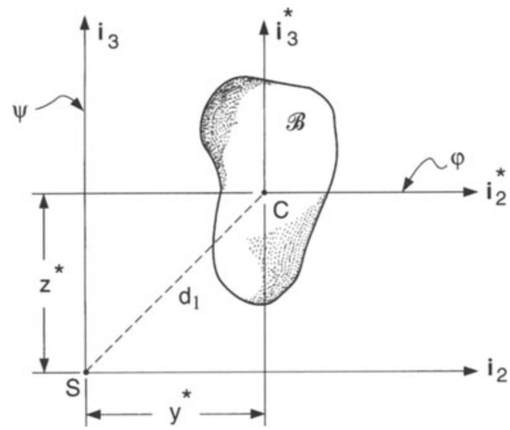


Figure 9.8. Geometrical interpretation of translation terms in the parallel axis rules (9.43).

Now consider a simple parallel shift of the axes along a coordinate direction, say, the z-axis. Then both base points S and C at $d = z^*$ from S are on the z-axis, and $x^* = y^* = 0$. Therefore, all products of inertia of the center of mass particle vanish in (9.42), and (9.44) simplifies to $I_{11}^S = I_{11}^C + md^2$, $I_{22}^S = I_{22}^C + md^2$, $I_{33}^S = I_{33}^C$, and $I_{ab}^S = I_{ab}^C$, $a \neq b = 1, 2, 3$, in accordance with the parallel axis theorem. If the body is homogeneous and the z-axis also is an axis of symmetry, then $I_{ab}^S = I_{ab}^C = 0$ as well. (What can be said if the mass density of the axisymmetric body varies with z?)

Example 9.5. The moment of inertia tensor $\mathbf{I}_C$ for a homogeneous rectangular block is given in (9.26). Find the inertia tensor components with respect to parallel axes at the corner point S in Fig. 9.4, page 364.

Solution. To determine $\mathbf{I}_S$, we apply the point transformation law (9.40). First, note that the center of mass C has coordinates $(x^*, y^*, z^*) = (\ell/2, w/2, h/2)$ in the parallel frame at S; therefore, (9.42) yields relative to S the moment of inertia component matrix for the center of mass particle:

$$[\mathbf{I}_S^*] = \begin{bmatrix} \frac{m}{4}(w^2 + h^2) & -\frac{m}{4}\ell w & -\frac{m}{4}\ell h \\ -\frac{m}{4}\ell w & \frac{m}{4}(h^2 + \ell^2) & -\frac{m}{4}wh \\ -\frac{m}{4}\ell h & -\frac{m}{4}wh & \frac{m}{4}(\ell^2 + w^2) \end{bmatrix}. \tag{9.45a}$$

The moment of inertia components of the block in a parallel frame at C are given in (9.26), and, with (9.45a), the point transformation law (9.40) delivers the inertia

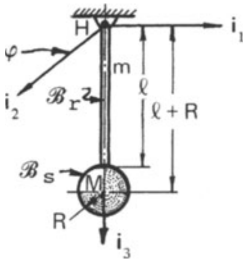


Figure 9.9. Application of the parallel axis theorem to a composite pendulum.

tensor components of the block in the parallel frame at S:

$$[\mathbf{I}_S] = \begin{bmatrix} \frac{m}{3}(w^2 + h^2) & -\frac{m}{4}\ell w & -\frac{m}{4}\ell h \\ -\frac{m}{4}\ell w & \frac{m}{3}(h^2 + \ell^2) & -\frac{m}{4}wh \\ -\frac{m}{4}\ell h & -\frac{m}{4}wh & \frac{m}{3}(\ell^2 + w^2) \end{bmatrix}. \tag{9.45b}$$

Notice that no products of inertia occur in (9.26) for $\mathbf{I}_C$, whereas all products of inertia appear in (9.45b) for $\mathbf{I}_S$. Of course, the same result may be obtained from (9.44). This is left as an exercise for the reader. □

Example 9.6. A complex structured pendulum assembly in Fig. 9.9 consists of a homogeneous spherical body $\mathscr{B}_s$ of radius R and mass M fastened to a homogeneous, but materially different thin rod $\mathscr{B}_r$ of length ℓ, mass m, and supported by a small hinge pin at H. Find the moment of inertia tensor for the pendulum assembly referred to the body frame $\varphi = \{H; \mathbf{i}_k\}$.

Solution. The moment of inertia tensor relative to H for the composite pendulum assembly $\mathscr{B} = \mathscr{B}_s \cup \mathscr{B}_r$ is given by (9.35):

$$\mathbf{I}_H(\mathscr{B}) = \mathbf{I}_H(\mathscr{B}_s) + \mathbf{I}_H(\mathscr{B}_r). \tag{9.46a}$$

Therefore, we shall need to determine the inertia tensor for each simple part $\mathscr{B}_k$ in the body frame $\varphi = \{H; \mathbf{i}_k\}$ in Fig. 9.9. Since the separate homogeneous parts $\mathscr{B}_k$ are materially different, the composite body $\mathscr{B}$ is neither homogeneous nor materially uniform. Nevertheless, because each part is a homogeneous body of revolution that shares the same $\mathbf{i}_3$-axis of symmetry, all products of inertia for the separate bodies, and hence for the assembly, vanish with respect to all base points of parallel frames along the common $\mathbf{i}_3$-axis, specifically, relative to φ at H.

First, consider the homogeneous sphere of radius R and mass $M = \frac{4}{3}\pi\rho R^3$ for which $\mathbf{I}_C(\mathscr{B}_s)$ is given by (9.34) in every reference frame at its center C;

and note that $\ell + R$ is the distance between the parallel axes $\{\mathbf{i}_1^*, \mathbf{i}_2^*\}$ at C for the sphere and $\{\mathbf{i}_1, \mathbf{i}_2\}$ at H. Then, by the parallel axis theorem in (9.44), $I_{11}^H = I_{22}^H = \frac{2}{5}MR^2 + M(\ell + R)^2$, $I_{33}^H = \frac{2}{5}MR^2$, and hence the inertia tensor for the sphere relative to H in φ is

$$\mathbf{I}_H(\mathscr{B}_s) = \left[\frac{2}{5}MR^2 + M(\ell + R)^2\right](\mathbf{i}_{11} + \mathbf{i}_{22}) + \frac{2}{5}MR^2\mathbf{i}_{33}. \tag{9.46b}$$

The same result follows easily from the parallel axis theorem in (9.40), from which $\mathbf{I}_H(\mathscr{B}_s) = \mathbf{I}_C(\mathscr{B}_s) + \mathbf{I}_H^*(\mathscr{B}_s)$, wherein $\mathbf{I}_H^*(\mathscr{B}_s) = M(\ell + R)^2(\mathbf{i}_{11} + \mathbf{i}_{22})$, the moment of inertia of the center of mass of the sphere relative to H. Thus, with (9.34), (9.46b) follows.

Now recall (9.28) for a thin rod of length ℓ, note in Fig. 9.9 that the rod axis is $\mathbf{i}_3$, and rewrite (9.28) to obtain, with $m \equiv m(\mathscr{B}_r) = \sigma\ell$,

$$\mathbf{I}_C(\mathscr{B}_r) = \frac{m\ell^2}{12}(\mathbf{i}_{11} + \mathbf{i}_{22}), \tag{9.46c}$$

where now C is the center of mass of the rod which is at $\ell/2$ from H, i.e. the distance between the parallel axes $\{\mathbf{i}_1^*, \mathbf{i}_2^*\}$ at C for the rod and $\{\mathbf{i}_1, \mathbf{i}_2\}$ at H. Then, by the parallel axis theorem in (9.44), $I_{11}^H = I_{22}^H = m\ell^2/12 + m\ell^2/4$, $I_{33}^H = 0$; so, the inertia tensor for the homogeneous thin rod relative to H in φ is

$$\mathbf{I}_H(\mathscr{B}_r) = \frac{m\ell^2}{3}(\mathbf{i}_{11} + \mathbf{i}_{22}). \tag{9.46d}$$

As before, the same result follows from the tensor form of the parallel axis theorem in (9.40), from which $\mathbf{I}_H(\mathscr{B}_r) = \mathbf{I}_C(\mathscr{B}_r) + \mathbf{I}_H^*(\mathscr{B}_r)$, wherein $\mathbf{I}_H^*(\mathscr{B}_r) = (m\ell^2/4)(\mathbf{i}_{11} + \mathbf{i}_{22})$, the moment of inertia of the center of mass of the rod relative to H.

We are now prepared to calculate $\mathbf{I}_H(\mathscr{B})$ for the pendulum assembly. Substitution of (9.46b) and (9.46d) into (9.46a) delivers the inertia tensor for the pendulum assembly relative to the hinge H in the body frame φ:

$$\mathbf{I}_H(\mathscr{B}) = \left[\frac{2}{5}MR^2 + M(\ell + R)^2 + \frac{m}{3}\ell^2\right](\mathbf{i}_{11} + \mathbf{i}_{22}) + \frac{2}{5}MR^2\mathbf{i}_{33}. \tag{9.46e}$$

$\square$

Let us return briefly to the first expression in (9.44) and notice that each of its terms is positive; hence, $I_{aa}^S > I_{aa}^C$ for every point S and for each choice of axis a. *Therefore, the moment of inertia about an axis has its smallest value at the center of mass.* A similar statement does not hold for the products of inertia in the second relation in (9.44), because their signs are indefinite; but their smallest absolute values at every point plainly are zero. Since we have infinitely many choices for center of mass axes, however, these observations prompt an interesting question: For what directions at the center of mass, or any other point, are the moments of inertia greatest and least? We shall return to this question later. First, we shall need to consider the transformation of tensor components induced by a rotation of the frame of reference.

9.8. Moment of Inertia Tensor Transformation Law

The inertia tensor $\mathbf{I}_Q$ referred to Cartesian frames $\varphi = \{Q; \mathbf{e}_k\}$ and $\varphi' = \{Q; \mathbf{e}'_k\}$ at Q has the same representation (9.11), and hence

$$\mathbf{I}_Q = I_{jk}\mathbf{e}_{jk} = I'_{lm}\mathbf{e}'_{lm}. \tag{9.47}$$

The change of vector basis defined by $\mathbf{e}'_j = A_{jk}\mathbf{e}_k$, or by its inverse $\mathbf{e}_k = A_{jk}\mathbf{e}'_j$, in which $A_{jk} \equiv \cos\langle \mathbf{e}'_j, \mathbf{e}_k\rangle$, induces a change of the corresponding tensor product bases in accordance with (3.101). Consequently, under a change of frame by a rotation of the bases at Q, in terms of the matrix notation in (3.108), we obtain *the Cartesian tensor transformation law for the moment of inertia components at Q:*

$$I'_Q = A I_Q A^T \quad \text{or} \quad I_Q = A^T I'_Q A, \tag{9.48}$$

where $A = [A_{jk}] = [\cos\langle \mathbf{e}'_j, \mathbf{e}_k\rangle]$. The reader may confirm this important rule by tracing its derivation starting with the change of basis in (9.47).

Example 9.7. Find the moment of inertia tensor for a homogeneous rectangular block of mass $m = 6$ slug, referred to body frames $\varphi = \{C; \mathbf{i}_k\}$ and $\varphi' = \{C; \mathbf{i}'_k\}$ defined in Fig. 9.10 at the center of mass C.

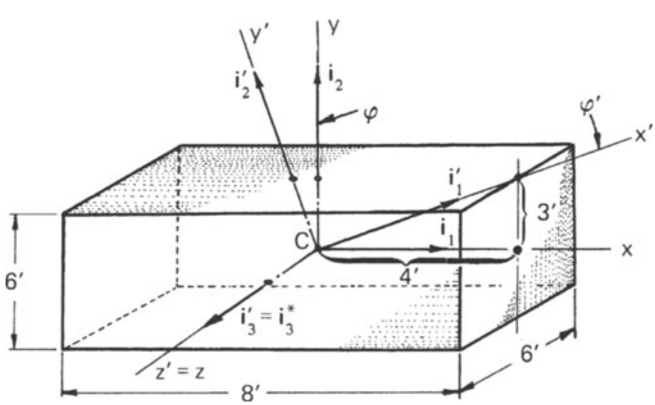


Figure 9.10. Application of the transformation law for the moment of inertia tensor referred to a rotated frame φ'.

Solution. The component matrix I_C of the moment of inertia tensor for the homogeneous block may be read from (9.26). With $m = 6$ slug, $\ell = 8$ ft, $w = h = 6$ ft, we find in $\varphi = \{C; \mathbf{i}_k\}$ the tensor component matrix

$$I_C = \begin{bmatrix} 36 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix}. \tag{9.49a}$$

The matrix I'_C in φ' is obtained from (9.48). It is seen from Fig. 9.10 that the transformation matrix $A = [\cos\langle \mathbf{i}'_p, \mathbf{i}_q \rangle]$ is given by

$$A = \begin{bmatrix} 4/5 & 3/5 & 0 \\ -3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{9.49b}$$

and its use in the first rule in (9.48) provides

$$I'_C = \begin{bmatrix} 4/5 & 3/5 & 0 \\ -3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 36 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} \begin{bmatrix} 4/5 & -3/5 & 0 \\ 3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9.49c}$$

Hence, the matrix I'_C of the inertia tensor referred to $\varphi' = \{C; \mathbf{i}'_k\}$ in Fig. 9.10 is

$$I'_C = \begin{bmatrix} 1026/25 & 168/25 & 0 \\ 168/25 & 1124/25 & 0 \\ 0 & 0 & 50 \end{bmatrix} \text{slug} \cdot \text{ft}^2. \tag{9.49d}$$

Notice that all products of inertia vanish in φ but not in φ'.

It is useful to note the invariance of $\operatorname{tr}\mathbf{I}_Q$ as a check on the calculation. By (9.49d), $\operatorname{tr}\mathbf{I}'_C = 1026/25 + 1124/25 + 50 = 136$. This agrees with $\operatorname{tr}\mathbf{I}_C = 136$ obtained from (9.49a). □

Exercise 9.2. Because of the symmetry of a homogeneous sphere with respect to every plane through its center C, its inertia tensor (9.34) has the same components in every reference frame at C. (i) Consider any tensor $\mathbf{T}$ whose matrix T in a Cartesian basis $\mathbf{e}_k$ is a scalar multiple of the identity matrix: $T = \tau I$, say. Apply the tensor transformation law (3.108) to show that $\mathbf{T}$ has the same components in every basis $\mathbf{e}'_k$; indeed, $\mathbf{T} = \tau\mathbf{1}$, and hence $\mathbf{T}$ is called a *spherical tensor.* (ii) A homogeneous cube, of course, does not have global symmetry with respect to every plane through its center C; so it is surprising that its inertia tensor is spherical. Show that the *moment of inertia tensor for a homogeneous cube of side a with mass* $m = \rho a^3$ *is*

$$\mathbf{I}_C = \frac{1}{6} m a^2 \mathbf{1}. \tag{9.50}$$

Other striking examples are the hemispherical shell shown in Fig. D.11 and the hemisphere in Fig. D.13 of Appendix D. See Problems 9.10 and 9.11. □

9.9. Extremal Properties of the Moment of Inertia Tensor

We have seen that all of the products of inertia vanish for every homogeneous body having at least two orthogonal planes of symmetry with respect to a body reference frame. A homogeneous cube, however, is a particularly striking example in that (9.50) shows that the products of inertia for a cube vanish in every reference frame at its center, underscoring our earlier observation that geometrical symmetry of a body is not necessary for the vanishing of its products of inertia in a body reference frame. But the most remarkable part of the story is yet untold. We are going to show that however complex the body geometry and regardless of its material distribution, there always exists at each point of a body an imbedded reference frame, called a *principal frame*, with respect to which the products of inertia vanish. Moreover, two of these are the directions for which the normal components of the inertia tensor assume their greatest and least values at each point. To demonstrate this, however, it is convenient to first study the method of Lagrange multipliers, a neat analytical technique that enables one to determine the stationary values of a function of several variables related by some specified constraint conditions.

The principal problem is introduced next. Then the Lagrange method of undetermined multipliers is described, and the method is illustrated in an easily visualized mechanical control problem. Afterwards, the extremal properties of the normal components of the inertia tensor are determined by Lagrange's method, and these properties are then characterized geometrically by Cauchy's inertia ellipsoid. The procedure for finding the body axes relative to which all products of inertia vanish at a specified body point is illustrated.

9.9.1. Introduction to the Principal Values Problem

Let $\mathbf{n}$ be an arbitrary unit vector at Q in an assigned body frame $\varphi = \{Q; \mathbf{e}_k\}$. The moment of inertia about the axis $\mathbf{n}$ at Q is the normal component of the inertia tensor $\mathbf{I}_Q$ for the direction $\mathbf{n}$ defined by

$$I_{nn}^{Q} = \mathbf{n} \cdot \mathbf{I}_{\mathbf{Q}}\mathbf{n}. \tag{9.51}$$

Clearly, the value of I_{nn}^{Q} depends on the direction $\mathbf{n}$. *The main problem, therefore, is to find the directions* $\mathbf{n} = \nu_k \mathbf{e}_k$ *in the body frame* φ *at* Q *for which the normal components of the moment of inertia tensor are largest and least.*

The normal component I_{nn}^{Q} is a function of the three direction cosines ν_k of the unit vector $\mathbf{n}$, thus subject to the *constraint equation*

$$\mathbf{n} \cdot \mathbf{n} = \nu_k \nu_k = 1. \tag{9.52}$$

Therefore, the three variables ν_k are not independent. The constraint equation (9.52) can be used to express any one of the ν_k in terms of the others, the result substituted into (9.51), and the stationary values I_{nn}^{Q} then determined in the usual way. This procedure, though straightforward in principle, often proves tedious; and sometimes it does not give a correct solution (see Problem 9.46). A more convenient, systematic scheme, applicable to a function of p variables related by $q < p$ constraint equations, is provided by Lagrange's method of undetermined multipliers.

9.9.2. The Method of Lagrange Multipliers

Consider a scalar-valued function $D(\mathbf{u})$ of the Cartesian components u_k of the p-dimensional vector field variable $\mathbf{u} = u_k \mathbf{i}_k$. Let $\partial S(\mathbf{u})/\partial \mathbf{u} \equiv (\partial S/\partial u_k)\mathbf{i}_k$ define the gradient of a general scalar function $S(\mathbf{u})$ with respect to $\mathbf{u}$. With no constraints on $\mathbf{u}$, a necessary condition that $D(\mathbf{u})$ have a stationary value is that

$$dD(\mathbf{u}) = \frac{\partial D(\mathbf{u})}{\partial u_k} du_k = \frac{\partial D(\mathbf{u})}{\partial \mathbf{u}} \cdot d\mathbf{u} = 0, \tag{9.53}$$

hold for all values of the differentials du_k, that is, for all vectors $d\mathbf{u}$. Since the variables u_k are assumed independent, their differentials can be assigned arbitrarily. We are thus led by (9.53) to p equations $\partial D/\partial u_k = 0$ in the p scalar components u_k of $\mathbf{u}$, that is, $\partial D(\mathbf{u})/\partial \mathbf{u} = \mathbf{0}$. These are the usual necessary conditions for existence of extrema of $D(\mathbf{u})$.

Now suppose that the components u_k must satisfy a *constraint equation* $F(\mathbf{u}) = 0$. Then only $p - 1$ of the p components u_k are independent. In addition to (9.53), $\mathbf{u}$ also must satisfy

$$dF(\mathbf{u}) = \frac{\partial F(\mathbf{u})}{\partial \mathbf{u}} \cdot d\mathbf{u} = 0. \tag{9.54}$$

Since du_k cannot be varied arbitrarily in (9.53) and (9.54), the extreme values of $D(\mathbf{u})$ are no longer determined by the p equations $\partial D(\mathbf{u})/\partial u_k = 0$, nor equivalently by $\partial D/\partial \mathbf{u} = \mathbf{0}$. Also, in general, $\partial F(\mathbf{u})/\partial u_k \neq 0$. Observe, however, that (9.53) and (9.54) show that the vectors $\partial D/\partial \mathbf{u}$ and $\partial F/\partial \mathbf{u}$ are perpendicular to the same vector $d\mathbf{u}$ for which any $p - 1$ components du_k can be varied arbitrarily, the p^{th} component being fixed by the constraint equation $F(\mathbf{u}) = 0$. These conditions imply that $\partial D/\partial \mathbf{u}$ and $\partial F/\partial \mathbf{u}$ must be parallel vectors so that $\partial D/\partial \mathbf{u} = \lambda \partial F/\partial \mathbf{u}$ at the stationary point, where λ is an unspecified, essentially arbitrary scalar independent of $\mathbf{u}$, called a *Lagrange multiplier,* to be determined as needed.

To prove this, we introduce an auxiliary function

$$G(\mathbf{u}) \equiv D(\mathbf{u}) - \lambda F(\mathbf{u}), \tag{9.55}$$

where λ is an arbitrary scalar to be determined as needed. Then the extreme values of $D(\mathbf{u})$ subject to the constraint $F(\mathbf{u}) = 0$ are determined from the extrema of $G(\mathbf{u})$ upon setting $\partial G(\mathbf{u})/\partial \mathbf{u} = \mathbf{0}$. Indeed, since

$$dG(\mathbf{u}) = \left(\frac{\partial D}{\partial \mathbf{u}} - \lambda \frac{\partial F}{\partial \mathbf{u}}\right) \cdot d\mathbf{u} = \left(\frac{\partial D}{\partial u_k} - \lambda \frac{\partial F}{\partial u_k}\right) du_k = 0, \tag{9.56}$$

must hold for an arbitrary value of λ, we may choose λ so that the coefficient of any one of the p differentials du_k in (9.56) vanishes, assuming of course that *for this choice* $\partial F/\partial u_k \neq 0$. Then the components of $d\mathbf{u}$ that remain in (9.56) are independent and can be varied arbitrarily. In consequence, it follows that all coefficients of the differentials in (9.56) must vanish. Therefore, the necessary condition for an extremum of $G(\mathbf{u})$ is provided by

$$\frac{\partial G}{\partial \mathbf{u}} = \frac{\partial D}{\partial \mathbf{u}} - \lambda \frac{\partial F}{\partial \mathbf{u}} = \mathbf{0}. \tag{9.57}$$

Thus, (9.57) determines the stationary values of $D(\mathbf{u})$ subject to the constraint

$$F(\mathbf{u}) = 0. \tag{9.58}$$

Indeed, the system of $p + 1$ equations (9.57) and (9.58) determine the p components u_k and the scalar multiplier λ for which $G(\mathbf{u})$ has an extremum. Now, at an extremal point $\mathbf{u} = \mathbf{u}^*$, say, the constraint $F(\mathbf{u}^*) = 0$ must be satisfied, and hence (9.55) shows that $G(\mathbf{u}^*) = D(\mathbf{u}^*)$; that is, the stationary values of D are the same as those of G. This procedure is known as *the method of Lagrange multipliers*.

The method may be extended to $q < p$ constraints by introduction of q undetermined Lagrange multipliers λ_r. In this case, we introduce the auxiliary function $G(\mathbf{u}) \equiv D(\mathbf{u}) - \sum_{r=1}^{q} \lambda_r F_r(\mathbf{u})$, in which the q constraints to be satisfied at the stationary points are $F_r(\mathbf{u}) = 0$. Then the necessary conditions for an extremum of $D(\mathbf{u}) = 0$ subject to these constraints are given by $\partial G(\mathbf{u})/\partial \mathbf{u} = \partial D(\mathbf{u})/\partial \mathbf{u} - \sum_{r=1}^{q} \lambda_r \partial F_r(\mathbf{u})/\partial \mathbf{u} = \mathbf{0}$. By setting $\partial G(\mathbf{u})/\partial \lambda_r = -F_r(\mathbf{u}) = 0$, we may recover the q constraint equations.

An application of Lagrange's method to a mechanical control problem whose solution is easily visualized follows.

Example 9.8. A bell crank mechanism having a telescopic control arm OP is shown in Fig. 9.11. The control pin P is constrained to move in a straight slot defined by the equation $y = 1 - x$. To design the crank, the designer must know the shortest distance d from the origin to the line of motion of P, an easy geometry

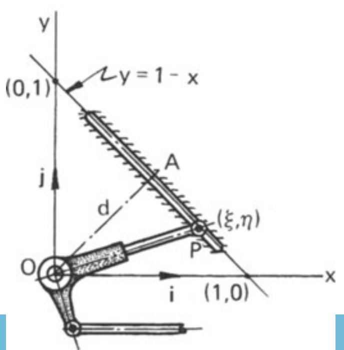


Figure 9.11. Application of the method of Lagrange multipliers to the design analysis of a bell crank mechanism.

problem. Find by geometry and then by the method of Lagrange multipliers the point on this line which is closest to the origin, and thus determine d.

Solution. The geometrical solution is evident in Fig. 9.11. The shortest line OA is the perpendicular bisector of the hypotenuse of the isosceles right triangle whose length is $\sqrt{2}$. Hence, $d = \sqrt{2}/2$ is the shortest distance from O to the line of motion of P, the nearest point to O being the midpoint A at $\mathbf{x} = \frac{1}{2}(\mathbf{i} + \mathbf{j})$.

Now let us see how the method of Lagrange multipliers is used to find the place $\mathbf{x} = \xi\mathbf{i} + \eta\mathbf{j}$ on the line $y = 1 - x$ which is nearest the origin in Fig. 9.11. The problem is to minimize the function $d(P) = (\mathbf{x} \cdot \mathbf{x})^{1/2}$, or more conveniently, the related squared distance function

$$D(\mathbf{x}) \equiv d^2(P) = \mathbf{x} \cdot \mathbf{x} = \xi^2 + \eta^2, \tag{9.59a}$$

subject to the constraint relation

$$F(\mathbf{x}) \equiv \xi + \eta - 1 = 0, \tag{9.59b}$$

specifying that the point (ξ, η) is constrained to the line $y + x = 1$. Notice that neither $\partial F(\mathbf{x})/\partial \xi$ nor $\partial F(\mathbf{x})/\partial \eta$ vanishes, as required below (9.56). Now use (9.59a) and (9.59b) to form the auxiliary function

$$G(\mathbf{x}) \equiv D(\mathbf{x}) - \lambda F(\mathbf{x}) = \xi^2 + \eta^2 - \lambda(\xi + \eta - 1), \tag{9.59c}$$

in accordance with (9.55). Then, by (9.57), the extremal points are determined by

$$\frac{\partial G(\mathbf{x})}{\partial \xi} = 2\xi - \lambda = 0, \qquad \frac{\partial G(\mathbf{x})}{\partial \eta} = 2\eta - \lambda = 0. \tag{9.59d}$$

Consequently, $\lambda = 2\xi = 2\eta$, and use of this result in the constraint equation (9.59b) yields the nearest point coordinates $\xi = \eta = \frac{1}{2}$, from which (9.59a) delivers the minimum value $D(\mathbf{x}) = d^2 = \frac{1}{2}$. Therefore, the nearest point on the line from O is at $\mathbf{x} = \frac{1}{2}(\mathbf{i} + \mathbf{j})$, at a distance $d = \sqrt{2}/2$ from O. □

Lagrange's systematic method of undetermined multipliers in this example is just about as easy as the elementary geometrical solution. Now consider another example where the conclusion is not so apparent.

Example 9.9. An atomic particle is confined to a rectangular box of sides a, b, c in which its ground state energy is $\mathscr{E} = k(1/a^2 + 1/b^2 + 1/c^2)$, where k is a constant. Find the dimensions of a box of constant volume for which the energy is least.

Solution. The rectangular box has volume $V(a, b, c) = abc = \gamma$, a constant. The problem is to find the smallest value of $D(a, b, c) \equiv \mathscr{E}(a, b, c)$ among all positive values of a, b, c for which the volume constraint $F(a, b, c) \equiv V(a, b, c) - \gamma = 0$ holds. To apply Lagrange's method, we form the auxiliary function

$$G(a, b, c) \equiv \mathscr{E}(a, b, c) - \lambda(V(a, b, c) - \gamma) = k\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - \lambda(abc - \gamma),$$

in accordance with (9.55). Notice that $\partial F/\partial a = bc \neq 0$, for example, and hence the assumption below (9.56) is satisfied. The extremal values of $G(a, b, c)$ are then determined by

$$\frac{\partial G}{\partial a} = -\frac{2k}{a^3} - \lambda bc = 0, \qquad \frac{\partial G}{\partial b} = -\frac{2k}{b^3} - \lambda ac = 0, \qquad \frac{\partial G}{\partial c} = -\frac{2k}{c^3} - \lambda ab = 0,$$

from which, with the aid of the constraint condition,

$$\frac{2k}{a^2} = \frac{2k}{b^2} = \frac{2k}{c^2} = -\lambda abc = -\lambda\gamma.$$

Hence, $a = b = c$; that is, the box for which the ground state energy is least is a cube of side a. Consequently, $\mathscr{E}_{\min} = 3k/a^2$ is the smallest value of the ground state energy consistent with the constant volume constraint. □

See Problems 9.23 through 9.30 and 9.45 for additional examples. We now return to the major problem posed earlier below (9.51).

9.9.3. Principal Values and Directions for the Inertia Tensor

Consider a rigid body of any sort, homogeneous or not. The principal problem is to find all directions $\mathbf{n}$ for which the normal components

$$D(\mathbf{n}) \equiv \mathbf{I}_Q\mathbf{n} \cdot \mathbf{n}, \tag{9.60}$$

of the inertia tensor at a point Q in a body reference frame are greatest and least, subject to the unit vector constraint condition

$$F(\mathbf{n}) \equiv \mathbf{n} \cdot \mathbf{n} - 1 = 0. \tag{9.61}$$

The problem is best solved by the method of Lagrange multipliers. We thus introduce the auxiliary function (9.55),

$$G(\mathbf{n}) \equiv \mathbf{I}_Q\mathbf{n} \cdot \mathbf{n} - \lambda(\mathbf{n} \cdot \mathbf{n} - 1), \tag{9.62}$$

in which λ is an undetermined scalar, independent of $\mathbf{n}$. The extreme values of (9.62) are then obtained by differentiation with respect to the three scalar components ν_k of $\mathbf{n} = \nu_k \mathbf{i}_k$ in accordance with (9.57). Bearing in mind the symmetry of $\mathbf{I}_Q$, we find in direct notation

$$\frac{\partial G(\mathbf{n})}{\partial \mathbf{n}} = 2\mathbf{I}_Q\mathbf{n} - 2\lambda\mathbf{n} = \mathbf{0}.$$

Hence, the extremal directions and values of the inertia tensor are determined by the vector equation

$$(\mathbf{I}_Q - \lambda \mathbf{1})\mathbf{n} = \mathbf{0}, \tag{9.63}$$

called the *principal vector equation*, together with the *constraint equation* (9.61). In index notation, bearing in mind the summation rule, this system of four algebraic equations for λ and ν_k is written as

$$\left(I_{kj}^Q - \lambda \delta_{kj}\right)\nu_j = 0, \qquad \nu_k \nu_k = 1, \tag{9.64}$$

or, explicitly, in expanded notation with the superscript Q suppressed,

$$\begin{aligned} (I_{11} - \lambda)\nu_1 + I_{12}\nu_2 + I_{13}\nu_3 &= 0, \\ I_{21}\nu_1 + (I_{22} - \lambda)\nu_2 + I_{23}\nu_3 &= 0, \\ I_{31}\nu_1 + I_{32}\nu_2 + (I_{33} - \lambda)\nu_3 &= 0, \\ \nu_1^2 + \nu_2^2 + \nu_3^2 &= 1. \end{aligned} \tag{9.65}$$

Nontrivial solutions of the homogeneous system (9.63), i.e. the first three equations of (9.65), exist if and only if

$$\det(\mathbf{I}_Q - \lambda \mathbf{1}) = \det \begin{bmatrix} I_{11} - \lambda & I_{12} & I_{13} \\ I_{12} & I_{22} - \lambda & I_{23} \\ I_{13} & I_{23} & I_{33} - \lambda \end{bmatrix} = 0. \tag{9.66}$$

This yields a cubic equation for λ, called the *characteristic equation*:

$$f(\lambda) \equiv -\lambda^3 + J_1\lambda^2 - J_2\lambda + J_3 = 0, \tag{9.67}$$

in which J_1, J_2, J_3 are defined by

$$J_1 \equiv \operatorname{tr}\mathbf{I}_Q, \qquad J_2 \equiv \frac{1}{2}\left(J_1^2 - \operatorname{tr}\mathbf{I}_Q^2\right), \qquad J_3 \equiv \det \mathbf{I}_Q. \tag{9.68}$$

These are the *principal invariants of the moment of inertia tensor* $\mathbf{I}_Q$. See (3.113).

The real cubic equation (9.67) has at least one real, positive root $\lambda = \lambda_1$, say, so there exists at least one real extremal direction $\mathbf{n} = \mathbf{n}_1$ determined by the system (9.65). In fact, it is proved later that because $\mathbf{I}_Q$ is a real-valued symmetric tensor, (9.67) always has three real roots λ_k, all positive. Therefore, there exist three corresponding mutually perpendicular directions $\mathbf{n}_k$ determined by (9.65) that define a special basis at Q, called the *principal basis* or *principal directions*. Two of these directions are the directions at Q with respect to which the normal components of the inertia tensor assume their maximum and minimum values, all determined by the roots λ_k of (9.67), called the *principal values* of $\mathbf{I}_Q$. To see this, let $\mathbf{n}$ be a principal direction for the characteristic root λ. Then from (9.63) and (9.61), $\mathbf{n} \cdot \mathbf{I}_Q\mathbf{n} = \lambda$ for each extremal direction $\mathbf{n}$. That is, the three roots λ_k of the characteristic equation (9.67) are the extreme values of the function (9.60) for which the constraint (9.61) is satisfied. Moreover, these are the moments of inertia about the axes defined by the corresponding principal directions $\mathbf{n}_k$. So, the principal values also are known as the *principal moments of inertia*.

For future notational convenience, henceforward, the principal basis is denoted by $\hat{\mathbf{e}}_k$, and $\hat{I}_{jk}$ denote the corresponding principal components of the inertia tensor at Q. Then, for the principal value $\lambda = \lambda_k$ and its corresponding principal direction $\mathbf{n} = \hat{\mathbf{e}}_k$, for a fixed value of $k = 1, 2, 3$, (9.63) becomes

$$\mathbf{I}_Q \hat{\mathbf{e}}_k = \lambda_k \hat{\mathbf{e}}_k \text{ (no sum on } k), \tag{9.69}$$

and the principal components $\hat{I}_{jk}$ of $\mathbf{I}_Q$ are given by

$$\hat{I}_{jk} = \hat{\mathbf{e}}_j \cdot \mathbf{I}_Q \hat{\mathbf{e}}_k = \lambda_k \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k = \lambda_k \delta_{jk} \text{ (no sum on } k). \tag{9.70}$$

Consequently, in the principal reference basis $\hat{\mathbf{e}}_j$ at Q, we have

$$\begin{gathered} \hat{I}_{11} = \lambda_1, \qquad \hat{I}_{22} = \lambda_2, \qquad \hat{I}_{33} = \lambda_3, \\ \hat{I}_{12} = \hat{I}_{21} = \hat{I}_{13} = \hat{I}_{31} = \hat{I}_{23} = \hat{I}_{32} = 0. \end{gathered} \tag{9.71}$$

Therefore, all products of inertia vanish, and thus attain their smallest absolute values, in the principal reference frame. We shall have no need to determine their maximum absolute values. (See Problem 9.45.) It follows from the first three relations in (9.71) that the three principal values λ_k are the moments of inertia about the principal axes, and these values may be ordered so that

$$\hat{I}_{11} \geq \hat{I}_{22} \geq \hat{I}_{33}, \tag{9.72}$$

and hence, $\hat{I}_{11}$ and $\hat{I}_{33}$ are the largest and smallest values of I_{nn}^Q among all possible normal components of $\mathbf{I}_Q$ at Q. We thus have the following remarkable result.

The principal axes theorem: *At each point Q of an arbitrary rigid body, homogeneous or not, there exists an orthonormal basis $\hat{\mathbf{e}}_k$ with respect to which the products of inertia are zero, the moments of inertia about these axes assume their greatest and least values, and the inertia tensor at Q has the unique representation*

$$\mathbf{I}_Q(\mathscr{B}) = \hat{I}_{11}^Q \hat{\mathbf{e}}_{11} + \hat{I}_{22}^Q \hat{\mathbf{e}}_{22} + \hat{I}_{33}^Q \hat{\mathbf{e}}_{33}, \tag{9.73}$$

referred to the principal tensor basis $\hat{\mathbf{e}}_{jk} = \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k$.

Recall that among all moments of inertia about parallel axes, regardless of the geometry and material distribution of the body, the smallest value occurs about an axis at the center of mass. Therefore, by the principal axes theorem, *the absolute minimum moment of inertia of a body about an axis is given by the smallest principal moment of inertia at its center of mass.* The principal axes of the inertia tensor for a homogeneous body having geometrical symmetry often are readily determined by inspection, as shown in earlier examples. For a nonhomogeneous or composite body, however, it is usually necessary to apply the principal axes analysis. The importance of the analysis rests on the reduction of the inertia tensor to its simplest diagonal form (9.73). The procedure is illustrated next in a numerical example.

Example 9.10. Find the principal values and directions for an inertia tensor $\mathbf{I}_Q$ whose component matrix at Q in frame $\varphi = \{Q; \mathbf{i}_k\}$ is

$$I_Q = \begin{bmatrix} 5/2 & -3/2 & 0 \\ -3/2 & 5/2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{kg} \cdot \text{m}^2. \tag{9.74a}$$

Solution. The principal values of $\mathbf{I}_Q$ at Q are determined by the characteristic equation (9.66) for the matrix (9.74a):

$$f(\lambda) \equiv \det \begin{bmatrix} \frac{5}{2} - \lambda & -\frac{3}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} - \lambda & 0 \\ 0 & 0 & 3 - \lambda \end{bmatrix} = (3 - \lambda)\left[\left(\frac{5}{2} - \lambda\right)^2 - \frac{9}{4}\right];$$

that is,

$$f(\lambda) = (3 - \lambda)(\lambda - 4)(\lambda - 1) = 0. \tag{9.74b}$$

Hence, the three principal values ordered so that $\lambda_1 \geq \lambda_2 \geq \lambda_3$ are

$$\lambda_1 = 4, \qquad \lambda_2 = 3, \qquad \lambda_3 = 1. \tag{9.74c}$$

It follows that the greatest and least normal components of the inertia tensor at Q are $\hat{I}_{11} = \lambda_1 = 4\ \text{kg} \cdot \text{m}^2$ and $\hat{I}_{33} = \lambda_3 = 1\ \text{kg} \cdot \text{m}^2$, respectively.

The principal directions at Q are determined by the system (9.65). With (9.74a), these take the form

$$\begin{aligned} \left(\frac{5}{2} - \lambda\right) \nu_1 - \frac{3}{2}\nu_2 &= 0, \\ -\frac{3}{2}\nu_1 + \left(\frac{5}{2} - \lambda\right) \nu_2 &= 0, \\ (3 - \lambda)\nu_3 &= 0, \\ \nu_1^2 + \nu_2^2 + \nu_3^2 &= 1. \end{aligned} \tag{9.74d}$$

This system of equations in ν_k is to be solved for each principal value in (9.74c). For $\lambda = \lambda_1 = 4$, (9.74d) yields $\nu_1 = -\nu_2 = \pm\sqrt{2}/2$, $\nu_3 = 0$, and hence the first principal direction $\hat{\mathbf{e}}_1 = \nu_k \mathbf{i}_k$ referred to φ is

$$\hat{\mathbf{e}}_1 = \pm\frac{\sqrt{2}}{2}(\mathbf{i}_1 - \mathbf{i}_2) \sim \lambda_1 = 4. \tag{9.74e}$$

Use of $\lambda = \lambda_2 = 3$ in (9.74d) delivers $\nu_1 = \nu_2 = 0,\ \nu_3 = \pm 1$. Thus, the second principal axis is

$$\hat{\mathbf{e}}_2 = \pm \mathbf{i}_3 \sim \lambda_2 = 3. \tag{9.74f}$$

Finally, the third principal direction orthogonal to $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ is given directly by $\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2$:

$$\hat{\mathbf{e}}_3 = \mp \frac{\sqrt{2}}{2}(\mathbf{i}_1 + \mathbf{i}_2) \sim \lambda_3 = 1. \tag{9.74g}$$

The triple of vectors $\hat{\mathbf{e}}_k$ define the principal directions of a frame $\hat{\varphi} = \{Q; \hat{\mathbf{e}}_k\}$ with respect to which the products of inertia vanish and the inertia tensor has diagonal form at Q. Notice that six unit vectors have been found. The difference in sign means only that either $\hat{\mathbf{e}}_k$ or its opposite may be chosen as a principal vector. It is customary, however, to select the principal basis to form a right-hand set, in which case the signs for $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, say, may be chosen arbitrarily and $\hat{\mathbf{e}}_3$ is then determined in accordance with the right-hand rule. With this concluded, the results referred to the original Cartesian body frame at Q are

$$\begin{aligned} \lambda_1 &= \hat{I}_{11} = 4\ \mathrm{kg}\cdot\mathrm{m}^2, & \hat{\mathbf{e}}_1 &= \frac{\sqrt{2}}{2}(\mathbf{i}_1 - \mathbf{i}_2), \\ \lambda_2 &= \hat{I}_{22} = 3\ \mathrm{kg}\cdot\mathrm{m}^2, & \hat{\mathbf{e}}_2 &= \mathbf{i}_3, \\ \lambda_3 &= \hat{I}_{33} = 1\ \mathrm{kg}\cdot\mathrm{m}^2, & \hat{\mathbf{e}}_3 &= -\frac{\sqrt{2}}{2}(\mathbf{i}_1 + \mathbf{i}_2). \end{aligned} \tag{9.74h}$$

The principal vectors $\hat{\mathbf{e}}_k$ are described relative to the original body frame $\varphi = \{Q; \mathbf{i}_k\}$. The orthogonal transformation matrix A: $\mathbf{i}_k \to \hat{\mathbf{e}}_k$, i.e. $A_{jk} = \cos\langle \hat{\mathbf{e}}_j, \mathbf{i}_k \rangle$, from the frame φ to the principal body frame $\hat{\varphi} = \{Q; \hat{\mathbf{e}}_k\}$ may be read from (9.74h):

$$A = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \end{bmatrix}. \tag{9.74i}$$

This matrix diagonalizes the original matrix I_Q in (9.74a), as the reader may confirm from the tensor transformation law (9.48) written as $\hat{I}_Q = A I_Q A^T$. Thus, in the principal basis,

$$\mathbf{I}_Q = 4\hat{\mathbf{e}}_{11} + 3\hat{\mathbf{e}}_{22} + \hat{\mathbf{e}}_{33}\ \mathrm{kg}\cdot\mathrm{m}^2. \tag{9.74j}$$

□

The canonical form (9.73) of the inertia tensor in the principal reference system at Q is independent of the shape of the body and its distribution of material. In particular, however, the principal axes of inertia for a homogeneous rigid body with two orthogonal planes of symmetry are easily identified at a point Q on the axis of symmetry, the first principal axis; call it $\hat{\mathbf{e}}_1$. The other two principal axes

$\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ lie in the orthogonal planes of symmetry, perpendicular to $\hat{\mathbf{e}}_1$ and to each other at Q, ordered so that $\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2$. This identifies the principal basis at Q relative to which the products of inertia of a homogeneous axisymmetric body vanish and $\mathbf{I}_Q$ has the canonical form (9.73). When $\hat{\mathbf{e}}_3$ is the axis of a homogeneous body of revolution, any orthogonal pair of planes through the axis are identical orthogonal principal planes of symmetry, and hence every direction in the cross section is a principal body axis for which $\hat{I}_{11}^Q = \hat{I}_{22}^Q$.

More generally, however, the foregoing analysis for an arbitrary body has avoided the important special cases in which two or three of the principal values λ_k are equal. This topic is explored geometrically in the next section and analytically later. It is found that when two principal values are equal, say $\lambda_1 = \lambda_2 \neq \lambda_3$, every direction in the plane perpendicular to $\hat{\mathbf{e}}_3$ is a principal direction for $\mathbf{I}_Q$. This occurs, for example, in the special case described above, when $\hat{\mathbf{e}}_3$ is the axis of a homogeneous body of revolution, as illustrated in equations (9.30) through (9.33) for homogeneous circular cylinders and tubes. Moreover, if all three principal values of inertia of an arbitrary body are equal, then $\mathbf{I}_Q = \lambda\mathbf{1}$, and every direction at Q is a principal direction. The inertia tensors for a homogeneous sphere (9.34) and for a cube (9.50) have these spherical tensor properties. A thin hemispherical shell and a homogeneous hemisphere are especially striking additional examples for which the inertia tensor is spherical. (See Figs. D.11 and D.13 in Appendix D.)

An illuminating geometrical interpretation of the variation of the normal components of the inertia tensor with direction follows. The maximum and minimum normal components of the inertia tensor are related to the geometry of a quadric surface, and the principal directions corresponding to equal principal values are characterized.

9.9.4. Cauchy's Inertia Ellipsoid

Consider an arbitrary rigid body whose inertia tensor $\mathbf{I}_Q$ is known in frame $\varphi = \{Q; \mathbf{e}_k\}$, and introduce an arbitrary axis n through Q with direction $\mathbf{n}$ so that $\mathbf{x} \equiv R\mathbf{n}$ is the position vector on this radial line of a point P at $R = |\mathbf{x}|$ from Q, as shown in Fig. 9.12. Recall (9.51) and note that

$$\mathbf{x} \cdot \mathbf{I}_Q \mathbf{x} = R^2 I_{nn}^Q, \tag{9.75}$$

is a quadratic form. Of course, the normal component I_{nn}^Q will vary with the direction $\mathbf{n}$, and R will change with the position of P. On each line n through Q, however, let us choose P so that its squared distance R^2 from Q is inversely proportional to I_{nn}^Q, the moment of inertia about that line; that is, let C be an arbitrary, positive constant and measure R along each axis n so that

$$R = \frac{C}{\sqrt{I_{nn}^Q}}. \tag{9.76}$$

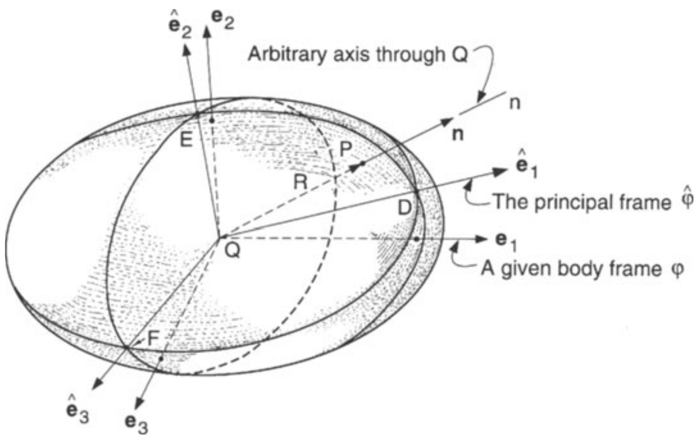


Figure 9.12. Cauchy's inertia ellipsoid at Q.

Then $R^2 I_{nn}^Q = C^2$, and (9.75), with $\mathbf{x} = x_k \mathbf{e}_k$ and $\mathbf{I_Q} = I_{jk}\mathbf{e}_{jk}$ in the body frame φ, describes the equation of a quadric surface, $\mathbf{x} \cdot \mathbf{I_Q x} = I_{jk} x_j x_k = C^2$ centered at Q. In expanded notation this becomes

$$I_{11}x_1^2 + I_{22}x_2^2 + I_{33}x_3^2 + 2I_{12}x_1x_2 + 2I_{13}x_1x_3 + 2I_{23}x_2x_3 = C^2. \quad (9.77)$$

In the principal reference frame $\hat{\varphi} = \{Q; \hat{\mathbf{e}}_k\}$, the products of inertia vanish and the position vector of P in $\hat{\varphi}$ is $\mathbf{x} = \hat{x}_k \hat{\mathbf{e}}_k$. Thus, (9.77) is transformed in the principal frame $\hat{\varphi}$ at Q to its simplest canonical form

$$\hat{I}_{11}\hat{x}_1^2 + \hat{I}_{22}\hat{x}_2^2 + \hat{I}_{33}\hat{x}_3^2 = C^2. \quad (9.78)$$

Equation (9.78), all of whose coefficients are positive, describes the closed quadratic surface shown in Fig. 9.12—an ellipsoid known as *Cauchy's inertia ellipsoid*, or *the momental ellipsoid*.

Equation (9.77) describes Cauchy's ellipsoid in the rotated position of the initially assigned body frame $\varphi = \{Q; \mathbf{e}_k\}$ in Fig. 9.12, with respect to which the inertia tensor $\mathbf{I}_Q = I_{jk}\mathbf{e}_{jk}$ is known. Equation (9.78) describes the same invariant ellipsoid in the principal body frame $\hat{\varphi} = \{Q; \hat{\mathbf{e}}_k\}$. For each choice of basis at the same point Q, equation (9.77) representing the ellipsoid in terms of the new component coefficients I_{jk} for a given rigid body $\mathscr{B}$ will be different. Nevertheless, each of these quadratic equations can be transformed to the unique canonical form (9.78); they all describe the same invariant ellipsoid centered at Q. Indeed, the momental ellipsoid has the invariant form $\mathbf{x} \cdot \mathbf{I}_Q\mathbf{x} = C^2$. However, if Q is changed to another point S, say, the inertia ellipsoid at Q will change to another invariant ellipsoid $\mathbf{X} \cdot \mathbf{I}_S\mathbf{X} = B^2$, say, centered at S; and at the new point S there exists another basis in which the ellipsoid is described by the canonical form (9.78) with principal values determined for $\mathbf{I}_S$.

We are now able to visualize the extremal properties of the inertia tensor in terms of the geometrical properties of its ellipsoid. Fix C in (9.76) to have any convenient constant value, say $C = 1$. Then $R = 1/\sqrt{I^Q_{nn}}$, and *the squared distance along a radial line from Q to a point P on the momental ellipsoid is numerically equal to the reciprocal of the moment of inertia of the body about that line*. In this case, with $\hat{R}_k \equiv 1/\sqrt{\hat{I}_{kk}},\ k = 1, 2, 3$, the inertia ellipsoid (9.78) in the principal reference frame at Q is described by

$$\left[\frac{\hat{x}_1}{\hat{R}_1}\right]^2 + \left[\frac{\hat{x}_2}{\hat{R}_2}\right]^2 + \left[\frac{\hat{x}_3}{\hat{R}_3}\right]^2 = 1, \tag{9.79}$$

where $\hat{R}_1 \geq \hat{R}_2 \geq \hat{R}_3$ are the ordered lengths of the three principal semidiameters of the inertia ellipsoid centered at Q, shown as $\overline{QD}$, $\overline{QE}$, and $\overline{QF}$ in Fig. 9.12. Among all possible lines from Q to any point P on this surface, none can be greater than $\hat{R}_1$ nor smaller than $\hat{R}_3$, the largest and least of the principal semidiameters. Accordingly, we have $\hat{I}^Q_{33} \geq \hat{I}^Q_{22} \geq \hat{I}^Q_{11}$; therefore, among all possible moments of inertia about axes through Q, none can be larger than $\hat{I}^Q_{33}$ nor smaller than $\hat{I}^Q_{11}$, the greatest and least of the principal moments of inertia. Moreover, if two of the principal components of the inertia tensor are equal, then two of the semidiameters of the inertia ellipsoid also are equal. Suppose, for example, that $\hat{R}_2 = \hat{R}_3 = \rho$. Then the surface is an ellipsoid of revolution about $\hat{\mathbf{e}}_1$ in $\hat{\varphi}$ and thus has a circular cross section for which no direction in its plane is distinguished. Consequently, the moment of inertia about every axis in this plane is numerically equal to $1/\rho^2$, and hence every axis in the plane at Q perpendicular to $\hat{\mathbf{e}}_1$ is a principal axis for the inertia tensor. If all three semidiameters of the ellipsoid, and therefore all three principal values of the inertia tensor, are equal, the ellipsoid is a sphere for which every axis is a principal axis of inertia at Q.

Exercise 9.3. Set $C = \sqrt{m(\mathscr{B})}$ in (9.76) and recall (9.22). Then $R = 1/R_n$, and hence in Fig. 9.12 *the distance from Q to the point P where the n axis intersects the inertia ellipsoid is numerically equal to the reciprocal of the radius of gyration of the body about that axis.* Review the properties of the momental ellipsoid in these terms. □

Not every ellipsoid centered at Q can be an inertia ellipsoid. *The class of inertia ellipsoids is restricted by the condition that the sum of any two normal Cartesian components of the inertia tensor is not less than the third.* It follows from (9.17), for example, that in any Cartesian frame $\varphi = \{Q; \mathbf{e}_j\}$ at Q, $I_{11} + I_{22} \geq I_{33}$ must hold, *the equality holding in φ if and only if the body is a plane body* for which $z = 0$ in φ, in accordance with (9.19); otherwise, the strict inequality holds. Thus, *if the body is not a plane body*, (9.14) yields the three constraints

$$I_{11} + I_{22} > I_{33}, \qquad I_{22} + I_{33} > I_{11}, \qquad I_{33} + I_{11} > I_{22}. \tag{9.80}$$

Plainly, the same relations hold for the principal components. *Consequently, if the body is not a plane body, the greatest principal moment of inertia must be smaller than the sum of the other two.*

Each invariant in (9.68), and specifically,

$$J_1 \equiv \operatorname{tr}\mathbf{I}_Q = I_{11} + I_{22} + I_{33}, \tag{9.81}$$

has the same value in every reference frame at Q. So, (9.80) and (9.81) are useful as a quick check on numerical computations.

In Example 9.10, page 385, for instance, we find from (9.74a) that $J_1 = 8$, and an easy check on the principal values in (9.74c) confirms the same sum. Further, (9.74a) shows that

$$I_{11} + I_{22} = 5 > I_{33} = 3, \qquad I_{22} + I_{33} = \frac{11}{2} > I_{11} = \frac{5}{2}, \qquad I_{11} = I_{22}, \tag{9.82a}$$

in the assigned frame φ. Similarly, for the principal values (9.74c), we obtain

$$\lambda_1 + \lambda_2 = 7 > \lambda_3 = 1, \qquad \lambda_2 + \lambda_3 = 4 = \lambda_1, \qquad \lambda_3 + \lambda_1 = 5 > \lambda_2 = 3, \tag{9.82b}$$

at the same point Q in the principal frame $\hat{\varphi} = \{Q; \hat{\mathbf{e}}_k\}$. The equality in the second principal axes relation in (9.82b) means that the body is a plane (thin) body in the principal 23-plane of $\hat{\varphi}$, a fact that is not evident from any relations in frame φ. In all cases, the constraints (9.80) are satisfied.

Similar geometrical interpretations of the normal components of symmetric tensors for stress and strain in terms of their ellipsoids arise in the study of the mechanics of deformable solids. This is a reflection of the analytical properties shared by all symmetric tensors, the principal aspects of which are sketched above. But to complete the picture some details beg further discussion and analytical clarification presented below. The reader who may wish to move on to the next chapter, however, will experience no serious loss of continuity.

9.10. Loose Ends and Generalities for Symmetric Tensors

The principal axes analysis developed for the inertia tensor is applicable to any symmetric tensor, and symmetric tensor quantities occur often in all areas of engineering, specifically in the study of mechanics of solids and fluids. In these and other areas, the tensor entities generally have real-valued components, and the tensor is said to be real-valued. When $\mathbf{T}$ is a real-valued symmetric tensor, its principal values and directions have important special properties—the principal values must be real, the principal vectors are mutually orthogonal regardless of any multiplicity of these principal values, and the component matrix referred to the principal basis is diagonal. Our objective is to explore these properties in general terms applicable to all symmetric tensor quantities and thus dispose of a few loose ends mentioned only briefly and previously described geometrically.

9.10.1. Summary of the Principal Values Problem

The principal values and directions for an arbitrary tensor **T** are determined by the principal vector equation

$$(\mathbf{T} - \lambda\mathbf{1})\mathbf{n} = \mathbf{0}, \tag{9.83}$$

subject to the unit vector constraint

$$\mathbf{n} \cdot \mathbf{n} = 1. \tag{9.84}$$

The homogeneous system of algebraic equations (9.83) for the components of **n** has a nontrivial solution if and only if

$$f(\lambda) \equiv \det(\mathbf{T} - \lambda\mathbf{1}) = 0. \tag{9.85}$$

This provides the characteristic equation for λ:

$$f(\lambda) = -\lambda^3 + J_1\lambda^2 - J_2\lambda + J_3 = 0, \tag{9.86}$$

where the principal invariants J_1, J_2, J_3 of the tensor **T** are defined by

$$J_1 = \operatorname{tr}\mathbf{T}, \qquad J_2 = \frac{1}{2}\left(J_1^2 - \operatorname{tr}\mathbf{T}^2\right), \qquad J_3 = \det(\mathbf{T}). \tag{9.87}$$

The real cubic equation (9.86) has at least one real root λ_1, say. Depending on the nature of **T**, the other two roots λ_2, λ_3 are either real or they are complex conjugates. In any case, (9.86) may be expressed in terms of its factors λ_k to obtain

$$f(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0. \tag{9.88}$$

Comparison of the coefficients in (9.86) and (9.88) shows that

$$J_1 = \lambda_1 + \lambda_2 + \lambda_3, \qquad J_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \qquad J_3 = \lambda_1\lambda_2\lambda_3. \tag{9.89}$$

9.10.2. Reality of the Principal Values of a Symmetric Tensor

Recall that for real numbers a and b the conjugate of a complex number $z = a + ib$ is denoted by $\bar{z} = a - ib$. Then $\bar{z} = z$ is a real number if and only if $b = 0$. The magnitude of z (or $\bar{z}$) is defined by $|z|^2 = z \cdot \bar{z} = a^2 + b^2$. Similarly, let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be real vectors, i.e. vectors having real components. Then $\boldsymbol{\eta} = \boldsymbol{\alpha} + i\boldsymbol{\beta}$ and $\bar{\boldsymbol{\eta}} = \boldsymbol{\alpha} - i\boldsymbol{\beta}$ are complex conjugate vectors for which $\boldsymbol{\eta} = \bar{\boldsymbol{\eta}}$ is a real vector when and only when $\boldsymbol{\beta} = \mathbf{0}$. The magnitude of $\boldsymbol{\eta}$ (or $\bar{\boldsymbol{\eta}}$) is defined by $|\boldsymbol{\eta}|^2 = \boldsymbol{\eta} \cdot \bar{\boldsymbol{\eta}} = \boldsymbol{\alpha} \cdot \boldsymbol{\alpha} + \boldsymbol{\beta} \cdot \boldsymbol{\beta}$. Thus, if $\boldsymbol{\eta}$ (hence $\bar{\boldsymbol{\eta}}$) is a unit vector, then $\boldsymbol{\eta} \cdot \bar{\boldsymbol{\eta}} = 1$.

We consider only real symmetric tensors, i.e. those having real-valued components in every real basis. Equation (9.86), however, requires only that at least one characteristic value must be real, while the others might be complex conjugates. Therefore, it appears that three real principal values might not exist for a real symmetric tensor, in which case there would be no way to transform the tensor to a diagonal form whose components would be real. We now prove that *the principal values of a real symmetric tensor are real.*

Suppose that two of the principal values of a real symmetric tensor $\mathbf{T}$ are complex conjugates λ and $\bar{\lambda}$, and let $\mathbf{n} = \boldsymbol{\alpha} + i\boldsymbol{\beta}$ and $\bar{\mathbf{n}} = \boldsymbol{\alpha} - i\boldsymbol{\beta}$ denote the corresponding complex conjugate principal vectors for λ and $\bar{\lambda}$. Then, for any real tensor $\mathbf{T}$, by (9.83),

$$\mathbf{T}\mathbf{n} = \lambda\mathbf{n}, \qquad \mathbf{T}\bar{\mathbf{n}} = \bar{\lambda}\bar{\mathbf{n}}, \tag{9.90a}$$

where the unit principal vectors satisfy

$$\mathbf{n} \cdot \bar{\mathbf{n}} = 1. \tag{9.90b}$$

Now form the inner product of the first equation in (9.90a) by $\bar{\mathbf{n}}$, the second by $\mathbf{n}$, introduce (9.90b), and recall the transpose rule (3.42) to obtain

$$\mathbf{n} \cdot (\mathbf{T}^T - \mathbf{T})\bar{\mathbf{n}} = \lambda - \bar{\lambda}.$$

Thus, if $\mathbf{T} = \mathbf{T}^{\mathbf{T}}$, then $\lambda = \bar{\lambda}$, and hence λ is real. That is, the principal values of a real symmetric tensor are real. An alternative proof is provided as an exercise.

Exercise 9.4. Let $\lambda = a + ib$ and $\mathbf{n} = \boldsymbol{\alpha} + i\boldsymbol{\beta}$ be a principal pair for a real symmetric tensor $\mathbf{T}$. Use only the first equation in (9.90a), identify its real and imaginary parts, and prove from these relations that $b = 0$, and hence λ is real. □

9.10.3. Orthogonality of Principal Directions

Let λ_α and λ_β be any two *distinct* principal values of a symmetric tensor $\mathbf{T}$ with corresponding principal vectors $\mathbf{n}_\alpha$ and $\mathbf{n}_\beta$. Then, by (9.83),

$$\mathbf{T}\mathbf{n}_\alpha = \lambda_\alpha \mathbf{n}_\alpha, \qquad \mathbf{T}\mathbf{n}_\beta = \lambda_\beta \mathbf{n}_\beta \text{ (no sum on } \alpha \text{ and } \beta),$$

and by the previous argument, we reach

$$\mathbf{n}_\alpha \cdot (\mathbf{T}^T - \mathbf{T})\mathbf{n}_\beta = (\lambda_\alpha - \lambda_\beta)\mathbf{n}_\alpha \cdot \mathbf{n}_\beta \text{ (no sum on } \alpha \text{ and } \beta),$$

where $\lambda_\alpha \neq \lambda_\beta$. Hence, if $\mathbf{T}$ is symmetric, $\mathbf{n}_\alpha \cdot \mathbf{n}_\beta = 0$. *Consequently, the principal vectors corresponding to distinct principal values of a real symmetric tensor are mutually perpendicular.* The proof breaks down if any two principal values are equal.

9.10.4. Multiplicity of Principal Values

If a real tensor $\mathbf{T}$ has repeated principal values, then all must be real, whether $\mathbf{T}$ is symmetric or not; but it is not evident that the principal vectors must be orthogonal, nor in fact if they need be distinct. Suppose, for example, that all principal values of $\mathbf{T}$ are unity; then (9.83) provides only one system of equations $\mathbf{T}\mathbf{n} = \mathbf{n}$

and $\mathbf{n} \cdot \mathbf{n} = 1$ for the principal vectors corresponding to $\lambda = 1$. This raises the question of whether the number of principal vectors also reduces to a single vector. The answer is no. *If* $\mathbf{T}$ *is symmetric and has a principal value* λ_1 *of multiplicity* $m = 2$ *or* 3, *so that the characteristic equation* (9.88) *has a factor* $(\lambda - \lambda_1)^m$, *then there exist at least m orthogonal principal vectors corresponding to the same* λ_1. The proof follows.

We know that there exists at least one principal pair $\lambda = \lambda_3$ and $\mathbf{n} = \hat{\mathbf{e}}_3$, say, such that $\mathbf{T}\hat{\mathbf{e}}_3 = \lambda\hat{\mathbf{e}}_3$. Let $\mathbf{e}_k$ be an orthonormal basis for which $\mathbf{e}_3 = \hat{\mathbf{e}}_3$. Then $\mathbf{e}_1 \cdot \mathbf{T}\mathbf{e}_3 = T_{13} = T_{31} = 0,\ \mathbf{e}_2 \cdot \mathbf{T}\mathbf{e}_3 = T_{23} = T_{32} = 0,\ \mathbf{e}_3 \cdot \mathbf{T}\mathbf{e}_3 = T_{33} = \lambda_3$, and so the symmetric tensor $\mathbf{T}$ has the component matrix

$$T = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}; \tag{9.91a}$$

that is, referred to $\varphi = \{O; \mathbf{e}_1, \mathbf{e}_2, \hat{\mathbf{e}}_3\}$, $\mathbf{T}$ has the representation

$$\mathbf{T} = \mathbf{S} + \lambda_3 \hat{\mathbf{e}}_{33}, \qquad \mathbf{S} \equiv T_{\alpha\beta}\mathbf{e}_{\alpha\beta},\ (\alpha, \beta = 1, 2), \tag{9.91b}$$

in which $\mathbf{S}$ is a two-dimensional symmetric tensor.

Regardless of the possible multiplicity of the principal values for $\mathbf{T}$, we now wish to determine if it is possible to find a nonzero vector $\mathbf{u} = u_\alpha \mathbf{e}_\alpha$, $(\alpha = 1, 2)$ in the plane P of $\mathbf{e}_1$ and $\mathbf{e}_2$ for which $\mathbf{T}\mathbf{u} = \lambda\mathbf{u}$ holds. From (9.91b), we note that $\mathbf{T}\mathbf{u} = \mathbf{S}\mathbf{u}$; therefore, we seek a vector $\mathbf{u} \neq \mathbf{0}$ in P such that

$$(\mathbf{S} - \lambda\mathbf{1})\mathbf{u} = \mathbf{0};\ \text{ that is, }\ (T_{\alpha\beta} - \lambda\delta_{\alpha\beta})u_\beta = 0,\ (\alpha, \beta = 1, 2). \tag{9.91c}$$

The homogeneous system (9.91c) will have a nontrivial solution $\mathbf{u}$ provided that $\det(\mathbf{S} - \lambda\mathbf{1}) = 0$. Because $\mathbf{S}$ is real and symmetric, this real quadratic equation in λ has two real roots. In consequence, there exists at least one vector $\mathbf{u}$ in P for which (9.91c) holds. Therefore, $\lambda = \lambda_2$ and $\mathbf{u} \equiv \hat{\mathbf{e}}_2$, say, is a principal pair for $\mathbf{S}$, hence also a second principal pair for $\mathbf{T}$. Since the basis directions $\mathbf{e}_k$ in the plane P were arbitrary, we may now assign them so that $\mathbf{e}_2 = \hat{\mathbf{e}}_2$. Then referred to $\varphi = \{O; \mathbf{e}_k\}$, we have $\mathbf{S}\mathbf{e}_2 = \lambda_2\mathbf{e}_2$, and therefore $\mathbf{e}_2 \cdot \mathbf{S}\mathbf{e}_2 = T_{22} = \lambda_2,\ \mathbf{e}_1 \cdot \mathbf{S}\mathbf{e}_2 = T_{12} = T_{21} = 0$, wherein we recall the second equation in (9.91b). Thus, the matrix (9.91a) referred to $\varphi = \{O; \mathbf{e}_k\} = \{O; \mathbf{e}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ now has the diagonal form

$$T = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{9.91d}$$

Finally, in view of (9.91d), we have $\mathbf{T}\mathbf{e}_1 = T_{k1}\mathbf{e}_k = T_{11}\mathbf{e}_1$; and hence $\mathbf{e}_1$ also is a principal direction for $\mathbf{T}$ and $T_{11} = \lambda_1$ is the corresponding principal value. Notice that we have nowhere assumed that the principal values of $\mathbf{T}$ must be distinct. *Consequently, regardless of the possible multiplicity of principal values for a symmetric tensor* $\mathbf{T}$, *there always exist at least three mutually orthogonal directions* $\mathbf{e}_k$ *that may be chosen as a principal basis* $\hat{\mathbf{e}}_k$ *with respect to which* $\mathbf{T}$

has the diagonal form

$$\mathbf{T} = \lambda_1 \hat{\mathbf{e}}_{11} + \lambda_2 \hat{\mathbf{e}}_{22} + \lambda_3 \hat{\mathbf{e}}_{33}. \tag{9.92}$$

This is called the *spectral representation* for $\mathbf{T}$.

Suppose, however, that $\mathbf{T}$ has a principal value of multiplicity $m = 2$, say, $\lambda_1 = \lambda_2 = \lambda$. Then the foregoing theorem assures existence of at least two orthogonal directions $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ corresponding to the same λ so that $\mathbf{T}\hat{\mathbf{e}}_\alpha = \lambda \hat{\mathbf{e}}_\alpha$ $(\alpha = 1, 2)$. Now, an arbitrary unit vector $\mathbf{n}$ in the plane of $\hat{\mathbf{e}}_\alpha$ may be written as

$$\mathbf{n} = (\mathbf{n} \cdot \hat{\mathbf{e}}_\alpha)\hat{\mathbf{e}}_\alpha;$$

so, we have

$$\mathbf{Tn} = (\mathbf{n} \cdot \hat{\mathbf{e}}_\alpha)\mathbf{T}\hat{\mathbf{e}}_\alpha = (\mathbf{n} \cdot \hat{\mathbf{e}}_\alpha)\lambda\hat{\mathbf{e}}_\alpha = \lambda \mathbf{n}.$$

In consequence, every vector $\mathbf{n}$ *in the plane perpendicular to* $\hat{\mathbf{e}}_3$*, the direction corresponding to the distinct principal value for* $\mathbf{T}$*, is a principal direction for* $\mathbf{T}$ *corresponding to the repeated principal value* $\lambda_1 = \lambda_2 = \lambda$*. Similarly, if* $\mathbf{T}$ *has three equal principal values* $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$*, then* $\mathbf{T} = \lambda \mathbf{1}$*, and every spatial direction* $\mathbf{n}$ *is a principal direction for* $\mathbf{T}$.

The multiplicity properties of the symmetric tensor $\mathbf{T}$ are precisely those described geometrically by the Cauchy momental ellipsoid. A Cauchy ellipsoid with two equal principal radii is an ellipsoid of revolution, every direction in the cross section being a principal direction. When all three principal radii are equal, the ellipsoid is a sphere for which every direction is a principal direction. Some further topics on symmetric tensors are described in Problems 9.34, 9.37, 9.41, and 9.45. See also Problem 9.40 in which $\mathbf{T} \neq \mathbf{T}^T$. We conclude with two examples.

Example 9.11. A symmetric tensor $\mathbf{T}$ has scalar components

$$T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{9.93a}$$

referred to $\varphi = \{O; \mathbf{e}_k\}$. Determine the principal values and directions for $\mathbf{T}$.

Solution. The principal values for $\mathbf{T}$ in (9.93a) are determined by (9.85):

$$\det(\mathbf{T} - \lambda \mathbf{1}) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)[(1-\lambda)^2 - 4] = 0,$$

which has three real roots

$$\lambda_1 = 3, \qquad \lambda_2 = -1, \qquad \lambda_3 = 1. \tag{9.93b}$$

Hence, $\lambda_2 = -1$ is the algebraically smallest normal component of $\mathbf{T}$ and $\lambda_1 = 3$ is the greatest. We note that $\mathrm{tr}T = 3$ from (9.93a) and confirm that the sum of the principal values (9.93b) is the same.

With $\mathbf{n} = \nu_k \mathbf{e}_k$ in φ and use of (9.93a), the principal vector equation (9.83) and the constraint (9.84) may be expanded as

$$\begin{aligned} (1-\lambda)\nu_1 + 2\nu_2 &= 0, \\ 2\nu_1 + (1-\lambda)\nu_2 &= 0, \\ (1-\lambda)\nu_3 &= 0, \\ \nu_1^2 + \nu_2^2 + \nu_3^2 &= 1. \end{aligned} \tag{9.93c}$$

With $\lambda = \lambda_3 = 1$, (9.93c) yields $\nu_1 = \nu_2 = 0$, $\nu_3 = \pm 1$; thus $\mathbf{n} \equiv \hat{\mathbf{e}}_3 = \pm\, \mathbf{e}_3$. Similarly, for $\lambda = \lambda_2 = -1$, $\nu_1 = -\nu_2 = \pm\sqrt{2}/2$, $\nu_3 = 0$, and hence $\mathbf{n} \equiv \hat{\mathbf{e}}_2 = \pm\,(\sqrt{2}/2)(\mathbf{e}_1 - \mathbf{e}_2)$. The third principal vector orthogonal to $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ is given by $\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 = \mp(\sqrt{2}/2)(\mathbf{e}_1 + \mathbf{e}_2)$. The signs are fixed as we please, but such that the triple $\hat{\mathbf{e}}_k$ forms a right-hand basis. We thus find the following principal values and directions for $\mathbf{T}$:

$$\begin{aligned} &\lambda_1 = 3 \sim \hat{\mathbf{e}}_1 = (\sqrt{2}/2)(\mathbf{e}_1 + \mathbf{e}_2), \\ &\lambda_2 = -1 \sim \hat{\mathbf{e}}_2 = (\sqrt{2}/2)(-\mathbf{e}_1 + \mathbf{e}_2), \\ &\lambda_3 = 1 \sim \hat{\mathbf{e}}_3 = \mathbf{e}_3. \end{aligned} \tag{9.93d}$$

Hence, by (9.92), in the principal basis $\hat{\mathbf{e}}_k$,

$$\mathbf{T} = 3\hat{\mathbf{e}}_{11} - \hat{\mathbf{e}}_{22} + \hat{\mathbf{e}}_{33}. \tag{9.93e}$$

Notice that $\mathbf{T}$ cannot be a moment of inertia tensor for a rigid body, because $\hat{T}_{22} < 0$. This is not evident from (9.93a) for which all of the diagonal components are positive and the inequalities (9.80) are satisfied, which is not true for (9.93e).

Finally, it is useful to note from (9.93d) the orthogonal transformation matrix $A : \mathbf{e}_k \to \hat{\mathbf{e}}_k$ for which $A_{ij} = \cos\langle \hat{\mathbf{e}}_i, \mathbf{e}_j \rangle$:

$$A = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9.93f}$$

Hence, the transformation A that diagonalizes T in (9.93a) in accordance with the tensor transformation law $\hat{T} = ATA^T$ describes a 45° counterclockwise rotation of $\mathbf{e}_k \to \hat{\mathbf{e}}_k$ about their common axis $\mathbf{e}_3 = \hat{\mathbf{e}}_3$, to yield the matrix of the tensor $\mathbf{T}$ in (9.93e). □

Example 9.12. If the T_{33} component of T in (9.93a) is replaced by $T_{33} = -1$, the characteristic equation for the new tensor has a repeated root $\lambda_3 = \lambda_2 = -1$ and $\lambda_1 = 3$. Hence, every vector in the plane normal to $\hat{\mathbf{e}}_1$ is a principal direction corresponding to $\lambda_2 = -1$. In particular, the same two vectors $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ given in (9.93d) are principal vectors for the new tensor.

References

1. Bowen, R. M., *Introduction to Continuum Mechanics for Engineers*, Plenum, New York, 1989. Appendix A presents a parallel development of the elements of tensor algebra in notation similar to that used here. The principal values and vectors for a tensor and the Cayley–Hamilton theorem also are discussed there.
2. Buck, R. C., *Advanced Calculus*, 2nd Edition, McGraw-Hill, New York, 1965. The method of Lagrange multipliers is described in Chapter 6.
3. Greenwood, D. T., *Principles of Dynamics*, Prentice-Hall, Englewood Cliffs, New Jersey, 1965. This intermediate level text is a good source for general collateral study. Some subtle aspects of the momental ellipsoid and its relation to the body are discussed in Chapter 7.
4. Kane, T. R., *Dynamics*, Holt, Reinhart and Winston, New York, 1968. Moments of inertia are nicely described in Chapter 3 as the components of both the second moment vector (See Problem 9.2.) and also as dyadic (tensor) components. Some further examples may be found here and in Kane's earlier work *Analytical Elements of Mechanics*, Vol. 1, *Dynamics*, Academic, New York, 1961.
5. Rosenberg, R. M., *Analytical Dynamics of Discrete Systems*, Plenum, New York, 1977. The moment of inertia tensor, its transformation properties, and description in terms of Cauchy's ellipsoid are presented in both index and expanded notation. Recommended for advanced readers.
6. Shames, I. H., *Engineering Mechanics*, Vol. 2, *Dynamics*, 2nd Edition, Prentice-Hall, Englewood Cliffs, New Jersey, 1966. An alternative formulation of the properties of the inertia tensor in expanded notation is presented in Chapter 16. See also Chapter 9 of the 3rd Edition, 1980.
7. Yeh, H., and Abrams, J. I., *Principles of Mechanics of Solids and Fluids*, Vol. 1, *Particle and Rigid Body Mechanics*, McGraw-Hill, New York, 1960. Chapter 11 deals with the inertia tensor mainly in expanded notation, though index notation also is used sparingly. Cauchy's momental ellipsoid is introduced to characterize the principal moments of inertia.

Problems

9.1. Let $\mathbf{n}$ be a unit vector along an arbitrary imbedded axis n through a base point Q in Fig. 9.1, page 360, and let $\mathbf{x}$ be the position vector from Q to the mass element dm. Begin with definition (9.10) and derive (9.21), wherein $I_{nn}^{Q} \equiv \mathbf{n} \cdot \mathbf{I}_Q \mathbf{n}$.

9.2. The *second moment vector* $\mathbf{I}_n^Q(\mathscr{B})$ relative to Q for a fixed direction $\mathbf{n}$ in Fig. 9.1 is defined by

$$\mathbf{I}_n^Q(\mathscr{B}) = \int_{\mathscr{B}} \mathbf{x} \times (\mathbf{n} \times \mathbf{x}) dm. \tag{P9.2a}$$

(i) Show by vector algebra that $\mathbf{n} \cdot \mathbf{I}_n^Q = I_{nn}^Q$, the integral in (9.21). (ii) More generally, expand the triple product to show in direct notation that

$$\mathbf{I}_n^Q(\mathscr{B}) = \mathbf{I}_Q(\mathscr{B})\mathbf{n}, \tag{P9.2b}$$

and thus prove that the component of the second moment vector in the direction $\mathbf{m}$ is given by $\mathbf{m} \cdot \mathbf{I}_n^Q = \mathbf{m} \cdot \mathbf{I}_Q \mathbf{n} = I_{mn}^Q$. Now show that $\mathbf{m} \cdot \mathbf{I}_n^Q = \mathbf{n} \cdot \mathbf{I}_m^Q$, and hence $I_{mn}^Q = I_{nm}^Q$. Notice that if $\mathbf{m} = \mathbf{n}$, this yields (9.21) for the moment of inertia about the axis $\mathbf{n}$; and if $\mathbf{m} \cdot \mathbf{n} = 0$, I_{mn}^Q is the product of inertia for the orthogonal directions $\mathbf{m}$ and $\mathbf{n}$.

9.3. Although the inertia tensor for a thin body may be derived as a limit case of a similar thick body, it is also straightforward to obtain results for thin bodies directly. Apply (9.14) to derive the inertia properties referred to $\varphi = \{C; \mathbf{i}_k^*\}$ at the center of mass for (a) the homogeneous thin tube in Fig. D.5 of Appendix D, (b) the homogeneous thin rod shown in Fig. D.7, and (c) the homogeneous thin spherical shell in Fig. D.10.

9.4. Find the mass, the center of mass, and the components of the moment of inertia tensor for the thin homogeneous rod forming the circular sector in Fig. D.8. Apply the results to determine these properties for a homogeneous semicircular wire and a thin circular ring.

9.5. Determine the mass, center of mass, and inertia tensor $\mathbf{I}_O$ for a homogeneous, circular cylindrical sector having a central angle 2θ, inner radius R_i, outer radius R_o, and length L. Refer all quantities to a body frame $\varphi = \{O; \mathbf{i}_k\}$ at the central point O with $\mathbf{i}_3$ being the cylinder axis and $\mathbf{i}_1$ bisecting both the central angle and the length. Derive as limit cases the properties for (a) a thin-walled circular sector, and (b) a thin circular rod described in Fig. D.8.

9.6. Find the mass, center of mass, and moment of inertia tensor for the homogeneous thin conical shell in Fig. D.4, referred to $\varphi = \{O; \mathbf{i}_k\}$.

9.7. Determine the mass, center of mass, and moment of inertia tensor for the homogeneous semicylinder of length ℓ and radius R shown in Fig. D.9, referred to the body frame $\varphi = \{O; \mathbf{i}_k\}$.

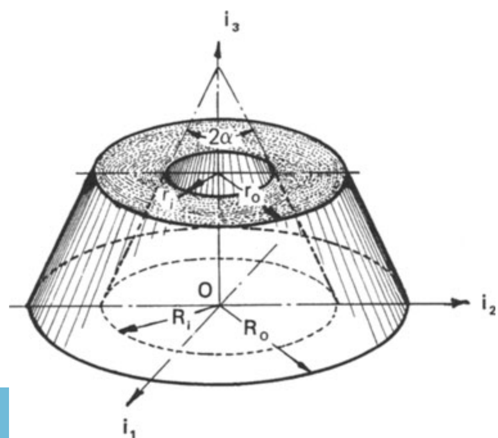
9.8. Derive the inertia tensor properties for the homogeneous right rectangular pyramid described in Fig. D.2.

9.9. Apply (9.29c) to derive the moment of inertia tensor for a homogeneous thin-walled circular tube of mean radius r, referred to $\varphi = \{C; \mathbf{i}_k^*\}$. Use the result to deduce the inertia tensor for a plane circular wire of radius R.

9.10. (a) Find the mass, center of mass, and moment of inertia tensor at O for the homogeneous thin hemispherical shell in Fig. D.11. (b) Derive from these results the same properties for the entire thin spherical shell in Fig. D.10.

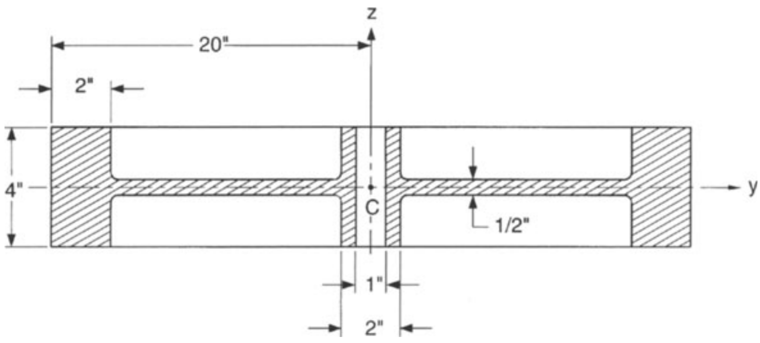
9.11. (a) Find the mass, center of mass, and moment of inertia tensor at O for the homogeneous hemisphere in Fig. D.13. (b) Derive from these results the same properties for a sphere of radius R. (c) Use the solution for a solid sphere to deduce the inertia tensor for the thick-walled, homogeneous spherical shell in Fig. D.10.

9.12. A portion of a thick-walled, homogeneous projectile casing whose inner and outer parallel surfaces are frustums of similar coaxial cones is shown in the figure. Apply the properties of a homogeneous right circular cone in Fig. D.3 to determine the mass of the casing and its moment of inertia about the z-axis, referred to the body frame $\varphi = \{O; \mathbf{i}_k\}$.



Problem 9.12.

9.13. The figure shows an arbitrary diametral cross section of a homogeneous flywheel made of a grade of steel of density $\rho = 15$ slug/ft^3. Determine its moment of inertia about the z-axis. What is the radius of gyration about the z-axis?



Problem 9.13.

9.14. The moment of inertia tensor for a sector of a homogeneous circular rod is given in Fig. D.8. Derive its inertia tensor in a parallel frame $\varphi = \{C; \mathbf{i}_k^*\}$ at its center of mass. What is the inertia tensor for a semicircular wire referred to φ?

9.15. Derive by integration the inertia tensor for the thin rod in Fig. D.7, referred to frame $\phi = \{O; \mathbf{i}_k\}$ at its end point O. Confirm the result by use of the parallel axis theorem applied to the tensor $\mathbf{I}_C$ given there.

9.16. A nonhomogeneous thin rigid rod of length l has a mass density $\rho(x)$ that varies with the distance x from one end O such that $d\rho(x)/dx = \rho_1$, a constant, and $\rho(0) = \rho_0$. (a) Find the mass of the rod and determine its moments of inertia relative to O. (b) Find the moments of inertia relative to the center of mass of the rod. (c) Derive from the results in (a) and (b) the corresponding properties for a homogeneous thin rod. (d) Consider a rod for which $\rho(l) = 2\rho_0$, and thus determine all of the properties found more generally in (a) and (b).

9.17. Apply the properties in Fig. D.9 for a homogeneous semicylinder to derive the moment of inertia tensor for (a) a solid cylinder referred to frame $\varphi = \{O; \mathbf{i}_k\}$, and (b) a semicylinder in a parallel frame $\phi^* = \{C; \mathbf{i}_k^*\}$ at its center of mass.

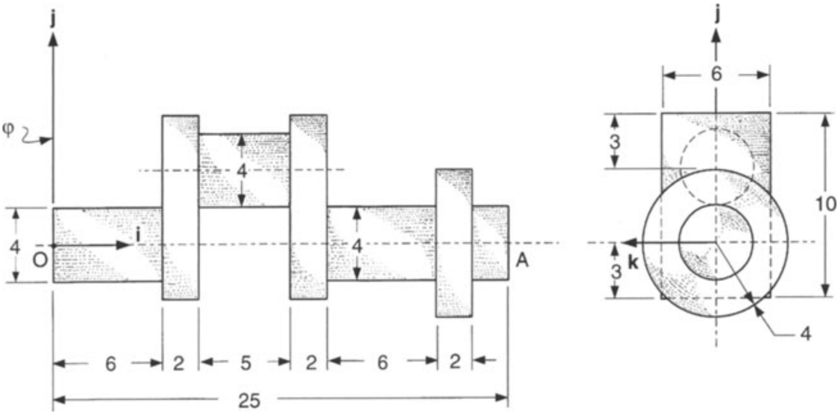
9.18. Derive the moment of inertia tensor for the homogeneous right circular cone in Fig. D.3, referred to frame $\varphi = \{O; \mathbf{i}_k\}$ in its base, and to a parallel frame at its center of mass.

9.19. Use the properties of the solid cone in Fig. D.3 to find its moment of inertia tensor $\mathbf{I}_Q$ referred to a parallel frame $\psi = \{Q; \mathbf{i}_k\}$ at its vertex Q. Let P be a point on the base circle at $\mathbf{r} = r\mathbf{i}_1$ from O, and determine at Q the moment of inertia tensor component about the edge line QP, referred to ψ.

9.20. A model of a crankshaft assembly for a one cylinder engine is shown in the figure. Use the table of properties in Appendix D to find the radius of gyration of the assembly about the axis OA, referred to $\varphi = \{O; \mathbf{i}_k\}$.

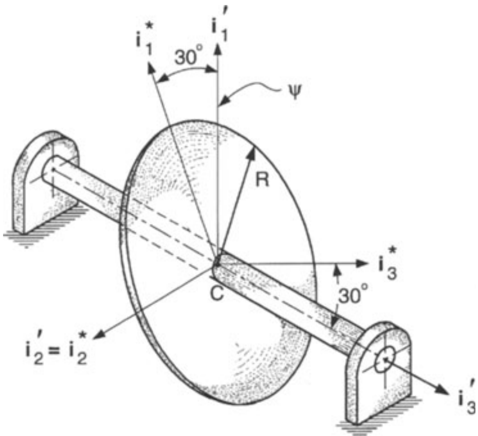
9.21. The plane of a homogeneous thin disk makes a 30° angle with the vertical plane, as shown in the figure. Determine the inertia tensor $\mathbf{I}_C$ for the disk in the body frame $\psi = \{C; \mathbf{i}'_k\}$.

9.22. A homogeneous, thin rectangular plate of mass $m = 2$ kg is welded to a horizontal shaft, as shown in the diagram. Find its inertia tensor $\mathbf{I}_C$ in the plate frame $\psi = \{C; \mathbf{e}_k\}$.

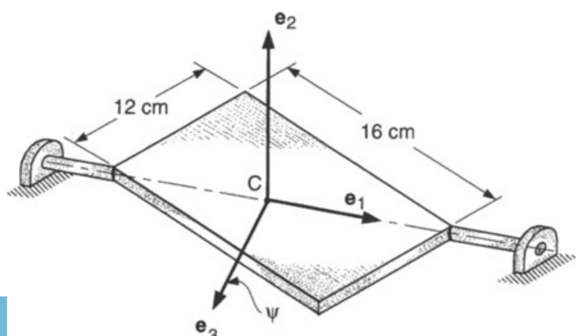


Note: All Dimensions are in Centimeters.

Problem 9.20.



Problem 9.21.



Problem 9.22.

9.23. Find by the method of Lagrange multipliers the point P in the plane $x_1 + x_2 + x_3 = 3$ nearest to the origin. Sketch the portion of this surface for $x_k \geq 0$ and provide a geometrical description of the point P in this region.

9.24. A particle moves on a curve defined by the intersection of the plane $2x + 4y = 5$ and the paraboloid $x^2 + z^2 = 2y$. What is the greatest elevation $z = h$ that the particle may reach in the motion? Note that there are two constraints here.

9.25. A particle P initially at the place $(0, 0, 12)$ is constrained by forces to move in the plane $2x + 3y + z = 12$. Find the equation of the straight path for which the motion of P passes the point where the potential energy function $V(\mathbf{x}) = 4x^2 + y^2 + z^2$ has a minimum.

9.26. What are the volume V and the moment of inertia tensor $\mathbf{I}_C$ for the largest homogeneous rectangular block having sides parallel to the coordinate planes and which can be inscribed in the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$? The constants a, b, c are the principal semidiameters of the ellipsoid.

9.27. A particle P moves along the line through the points $(1, 0, 0)$ and $(0, 1, 0)$. Find the point on this line at which P is nearest to the line $x = y = z$. What is the shortest distance between these lines?

9.28. In continuum mechanics, a Bell material is a constrained elastic solid material for which the first principal invariant $J_1 = \operatorname{tr}\mathbf{V}$ of the symmetric Cauchy–Green deformation tensor $\mathbf{V}$ must satisfy the rule $J_1 = 3$ in every deformation from the undeformed state where $\mathbf{V} = \mathbf{1}$. The three principal values λ_k of $\mathbf{V}$, all positive, are called principal stretches. The principal invariants of $\mathbf{V}$ are defined by (9.89). Determine the extremal values of the second and third principal invariants of $\mathbf{V}$. Are these extrema their largest or smallest values?

9.29. Find the work done in moving a particle from a place at $\mathbf{x} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ to the place at which the potential energy function $V(x, y, z) = x - 2y + 2z$ has a maximum value among all points (x, y, z) located on a sphere of radius 3.

9.30. An electron moves in the plane $ax + by + cz + d = 0$. Find the point in this plane at which the electron can be closest to the origin in frame $\varphi = \{O; \mathbf{i}_k\}$, and determine its shortest distance from O.

9.31. Consider the symmetric tensor $\mathbf{T} = \mathbf{1} - 4(\mathbf{e}_{12} + \mathbf{e}_{21})$ referred to the Cartesian frame $\varphi = \{O; \mathbf{e}_k\}$. (i) Find the principal values and directions for $\mathbf{T}$. (ii) Identify the basis transformation matrix A: $\mathbf{e}_k \rightarrow \hat{\mathbf{e}}_k$, the principal basis for $\mathbf{T}$, and describe its geometrical character. (iii) Apply the tensor transformation law to demonstrate that A diagonalizes the matrix T in $\mathbf{e}_k$ to form $\hat{T}$ in $\hat{\mathbf{e}}_k$. Could $\mathbf{T}$ be the inertia tensor for some body in φ? Appropriate units are assumed.

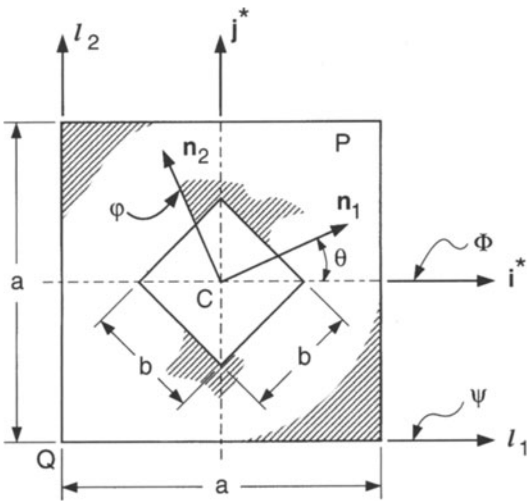
9.32. Consider two symmetric tensors: $\mathbf{T} = 2\mathbf{e}_{11} + 5\mathbf{e}_{22} - \mathbf{e}_{33} + 4(\mathbf{e}_{23} + \mathbf{e}_{32})$ in $\mathbf{e}_k$ and $\mathbf{U} = -\bar{\mathbf{e}}_{11} + 6\bar{\mathbf{e}}_{22} + \bar{\mathbf{e}}_{33} + 2(\bar{\mathbf{e}}_{23} + \bar{\mathbf{e}}_{32})$ in $\bar{\mathbf{e}}_k$. Determine the principal values and directions for $\mathbf{T}$ in $\mathbf{e}_k$. Is the tensor $\mathbf{U}$ the same as $\mathbf{T}$ but merely referred to another basis $\bar{\mathbf{e}}_k$?

9.33. A homogeneous, thin rectangular plate has sides $\hat{x} = a$ and $\hat{y} = 2a$. (a) Find by integration the moment of inertia tensor $\mathbf{I}_O$ referred to a Cartesian frame $\varphi = \{O; \hat{\mathbf{e}}_k\}$ at the corner point O and parallel to the plate edges $\hat{x}$ and $\hat{y}$. (b) Confirm the result by application of (9.27) and the parallel axis theorem. (c) Determine in φ the principal values and directions for the inertia tensor at O.

9.34. A certain symmetric tensor $\mathbf{T}$ is given by $\mathbf{T} = 15\mathbf{e}_{11} + 25\mathbf{e}_{22} + 30\mathbf{e}_{33} - 10(\mathbf{e}_{13} + \mathbf{e}_{31})$ in $\varphi = \{Q; \mathbf{e}_k\}$. (a) Write the equation for its ellipsoid in φ, and find its principal ellipsoid. (b) Could $\mathbf{T}$ be the inertia tensor at Q for some rigid body? Could it be the inertia tensor for a plane body?

9.35. Let T be the matrix in a Cartesian frame φ of a tensor $\mathbf{T}$, and consider another tensor $\mathbf{U}$ whose matrix in φ is $U = \alpha T$, where α is a scalar. (a) Prove that the same proportional relation holds in every Cartesian reference system. What can be said about the corresponding principal values and directions for **U** and **T**? (b) Let $\alpha = 1/5$ and consider the tensor **T** defined in Problem 9.34. Solve that problem for the tensor **U**.

9.36. A homogeneous, thin square plate of side a has a square hole of side b punched through its center, as illustrated. (a) First, consider the plate without the hole. Find by integration the inertia tensor for the solid plate referred to the frame $\Phi = \{C; \mathbf{i}_k^*\}$, and then read from this the result referred to the frame $\varphi = \{C; \mathbf{n}_k\}$ making an angle θ with Φ in the plane of the plate, as shown. What is the radius of gyration of the plate about the axis $\mathbf{n}_1$ and about any other axis in the plane? (b) Now consider the plate with the hole. Determine the inertia tensor for the punched plate (i) referred to Φ at C and (ii) referred to a parallel frame $\psi = \{Q; \mathbf{l}_k\}$ at the corner Q.



Problem 9.36.

9.37. The matrix in $\psi = \{Q; \mathbf{l}_k\}$ of the inertia tensor $\mathbf{I}_Q$ for the punched plate described in the previous problem has the general form

$$I_Q = \begin{bmatrix} A & -B & 0 \\ -B & A & 0 \\ 0 & 0 & 2A \end{bmatrix}, \tag{P9.37}$$

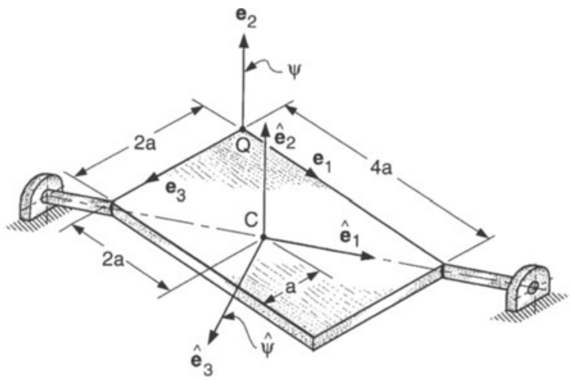
where $A > B > 0$. (a) Find the principal values and directions for $\mathbf{I}_Q$. (b) Interpret the geometry of the principal directions for $\mathbf{I}_Q$, and relate it to the geometry of the punched plate. Are these directions evident from the plate geometry? (c) Use the results of Problem 9.36 to find by the parallel axis theorem the principal values of the inertia tensor at Q.

9.38. The inertia tensor $\mathbf{I}_O$ in the frame $\psi = \{O; \mathbf{i}_k\}$ has the component matrix

$$I_O = \begin{bmatrix} \frac{1}{2}(\alpha+\beta) & \frac{1}{2}(\alpha-\beta) & 0 \\ \frac{1}{2}(\alpha-\beta) & \frac{1}{2}(\alpha+\beta) & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \tag{P9.38}$$

where α, β, γ are positive constants. Find the principal values and directions for the inertia tensor. Describe the angular orientation of the principal frame relative to ψ.

9.39. (a) Find by integration the inertia tensor $\mathbf{I}_C$ for the homogeneous, rectangular plate of mass m in a plate frame $\psi = \{C; \mathbf{e}_k\}$, and thus determine $\mathbf{I}_C$ in the plate frame $\hat{\psi} = \{C; \hat{\mathbf{e}}_k\}$ shown in the figure. (b) Find the inertia tensor $\mathbf{I}_Q$ at the corner Q in the plate frame $\psi = \{Q; \mathbf{e}_k\}$. (c) Determine in ψ the principal values and directions for $\mathbf{I}_Q$ at Q. Here the $\hat{\mathbf{e}}_k$ are not principal vectors.



Problem 9.39.

9.40. Because a tensor and its transpose have the same principal invariants, they have the same characteristic equation and hence the same principal values. Their principal vectors, however, need not be the same. (a) Prove that $\mathbf{T}$ and $\mathbf{T}^T$ have the same principal invariants. (b) Now consider the tensor $\mathbf{T} = \mathbf{e}_{11} + \mathbf{e}_{22} + 3\mathbf{e}_{33} + \mathbf{e}_{12} + 2\mathbf{e}_{21}$ in the Cartesian frame $\varphi = \{O; \mathbf{e}_k\}$. Find the principal values and directions for $\mathbf{T}$ and for $\mathbf{T}^T$, and determine their principal invariants. (c) Determine the angles between the principal vectors for $\mathbf{T}$, and do the same for $\mathbf{T}^T$. Sketch the principal vectors for both tensors in φ, and describe the geometry. (d) Are the principal vectors of $\mathbf{T}$ mutually orthogonal? Are those of $\mathbf{T}^T$ mutually orthogonal?

9.41. Consider the symmetric tensor $\mathbf{T} = \frac{5}{2}(\mathbf{e}_{11} + \mathbf{e}_{22}) - \frac{3}{2}(\mathbf{e}_{12} + \mathbf{e}_{21}) + 3\mathbf{e}_{33}$ referred to $\mathbf{e}_k$. Show that the principal values of $\mathbf{T}$ are non-negative. Therefore, $\mathbf{T}$ has a unique, positive symmetric square root defined by $\mathbf{T}^{1/2} = \sqrt{\lambda_1}\hat{\mathbf{e}}_{11} + \sqrt{\lambda_2}\hat{\mathbf{e}}_{22} + \sqrt{\lambda_3}\hat{\mathbf{e}}_{33}$ in the principal basis $\hat{\mathbf{e}}_k$ of $\mathbf{T}$. Find $\mathbf{T}^{1/2}$ in $\mathbf{e}_k$ and check your solution by the matrix multiplication $T = T^{1/2}T^{1/2}$ in $\mathbf{e}_k$.

9.42. Determine in the principal basis $\hat{\mathbf{e}}_k$ the positive square root of the symmetric tensor $\mathbf{U}$ whose component matrix referred to $\varphi = \{O; \mathbf{e}_k\}$ is

$$U = \begin{bmatrix} 5 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}. \tag{P9.42}$$

Let $\hat{U}^{1/2}$ denote the principal matrix of $\mathbf{U}^{1/2}$ in $\hat{\mathbf{e}}_k$. Identify the basis transformation matrix required to transform $\hat{U}^{1/2}$ into $U^{1/2}$ in φ. See Problem 9.41.

9.43. Since $\mathbf{I}_Q$ is a positive, symmetric tensor, in accordance with the theorem stated in Problem 9.41, it has a unique positive, symmetric square root $\mathbf{I}_Q^{1/2}$. Hence, we may define the unique *gyration tensor*

$$\mathbf{G}_Q \equiv \frac{1}{\sqrt{m(\mathscr{B})}}\mathbf{I}_Q^{1/2}, \tag{P9.43a}$$

so that $m\mathbf{G}_Q^2 = \mathbf{I}_Q$. The tensors $\mathbf{G}_Q$ and $\mathbf{I}_Q$ have the same principal directions, and the principal values of $\mathbf{G}_Q$ are the familiar radii of gyration (9.22) about the principal axes $\hat{\mathbf{e}}_k$, namely,

$$\hat{R}_n = \sqrt{\frac{\hat{I}_{nn}}{m}} = \hat{G}_{nn}. \tag{P9.43b}$$

The matrix (P9.37) shows that $R_1^Q = R_2^Q = \sqrt{A/m},\ R_3^Q = \sqrt{2A/m}$ are the radii of gyration about the $\mathbf{I}_k$ axes at Q. (a) What are the principal components of $\mathbf{G}_Q$ for the tensor with matrix (P9.37)? (b) Identify the principal directions from Problem 9.37, and determine the components of $\mathbf{G}_Q$ referred to $\psi = \{Q; \mathbf{I}_k\}$ in terms of the principal radii of gyration (P9.43b). Of course, the normal components G_{nn}^Q of the gyration tensor $\mathbf{G}_Q$ in ψ generally are *not* the same as the radii of gyration R_n^Q given above.

9.44. A tensor $\mathbf{W}$ has the Cartesian component matrix

$$W = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{P9.44}$$

referred to $\varphi = \{O; \mathbf{e}_k\}$. Find the positive square root of $\mathbf{W}$ referred to φ. See Problem 9.41.

9.45. Although the greatest values of the products of inertia of a rigid body are unimportant, the determination of the extremal values of the nondiagonal components of a symmetric tensor is important in the study of the properties of stress and strain tensors in continuum mechanics, for example. Let $\mathbf{T}$ be a symmetric tensor and $\mathbf{m}$ and $\mathbf{n}$ orthogonal unit vectors. The *orthogonal shear* or *product component* of $\mathbf{T}$ for the pair $(\mathbf{m}, \mathbf{n})$ is defined by $T_{mn} \equiv \mathbf{m} \cdot \mathbf{Tn}$. (a) Apply the method of Lagrange multipliers to derive a system of two vector equations that determine among all possible pairs $(\mathbf{m}, \mathbf{n})$ those orthogonal directions for which T_{mn} has its greatest absolute value $|T_{mn}|_{\max}$. (b) How are the Lagrange multipliers related to the components of $\mathbf{T}$ in the extremal directions and to $|T_{mn}|_{\max}$? (c) Derive from the vector equations in (a) two equations for the sum and difference of the extremal directions $\mathbf{m}$ and $\mathbf{n}$. Interpret these equations in terms of the principal values τ_a and principal directions $\hat{\mathbf{e}}_a$ for $\mathbf{T}$, and thus show that the product components of $\mathbf{T}$ have their maximum absolute value with respect to a basis with directions $\mathbf{m}$ and $\mathbf{n}$ that bisect the principal directions for $\mathbf{T}$. That is, show that*

$$|T_{mn}|_{\max} = \max\left\{\left|\frac{\tau_1 - \tau_2}{2}\right|, \left|\frac{\tau_2 - \tau_3}{2}\right|, \left|\frac{\tau_3 - \tau_1}{2}\right|\right\}, \tag{P9.45a}$$

where $\mathbf{m}$ and $\mathbf{n}$ are the orthogonal directions

$$\mathbf{m} = \frac{\sqrt{2}}{2}(\hat{\mathbf{e}}_a + \hat{\mathbf{e}}_b), \qquad \mathbf{n} = \frac{\sqrt{2}}{2}(\hat{\mathbf{e}}_a - \hat{\mathbf{e}}_b) \tag{P9.45b}$$

(or their opposites) for which $|T_{mn}|_{\max} = \frac{1}{2}|\tau_a - \tau_b|$ in (9P.45a). Of course, the least absolute value for the orthogonal shear or product components of $\mathbf{T}$ occurs for the principal basis where they all vanish.

* A very short elegant proof of the maximum orthogonal shear component of a symmetric tensor is given by Ph. Boulanger and M.A. Hayes, Shear, shear stress and shearing, *Journal of the Mechanics and Physics of Solids* **40**, 1449–1457 (1992). An earlier alternate proof that uses the geometrical properties of pairs of conjugate semi-diameters of ellipses is provided by M.A. Hayes, A note on maximum orthogonal shear stress and shear strain, *Journal of Elasticity* **21**, 117–120 (1989).

9.46. †Consider a particle P with potential energy $V(x, y) = 4x + y + y^2$. The motion of P is constrained so that the point (x, y) lies on the circle $x^2 + 2x + y^2 + y = 1$. Determine the extremal values of the potential energy. (a) First, apply the constraint equation to write $V(x, y) = \bar{V}(x)$ and thus show that the usual substitution procedure fails to deliver a real solution for any extrema of $V(x, y)$. (b) Apply the method of Lagrange multipliers and show that the potential energy has both maximum and minimum values at distinct points on the circle. Find these points and determine the energy extrema.

† I thank Professor Michael A. Hayes for suggesting this example and for recalling the aforementioned references on maximum orthogonal shear.

10

Dynamics of a Rigid Body

10.1. Introduction

Newton's *Principia* (1687) stands among the world's greatest scientific and intellectual achievements, and justly so, but it does not address all of the general principles of mechanics. In Newton's book there is no theory of general dynamical systems nor of rigid bodies, and nothing on the mechanics of deformable solid and fluid continua. Newton's theory for mass points is just insufficiently general to deliver a unifying method for their study. More than half a century of research and struggle with solutions of special problems would pass before the first of the general principles of mechanics applicable to all bodies was discovered by Euler in 1750, and thereafter.

The first time that the two principles of momentum and moment of momentum, though not explicitly stated, were applied in a system of differential equations occurred in the study by Euler (1744) of the finite plane motion of a chain of rigid links and of a loaded string, both being constrained systems of discrete material points. In a subsequent paper on celestial mechanics in 1747, following a series of successes with specific mechanical problems like those just mentioned, Euler now sees clearly that for all discrete systems the equations of motion are of the form $m_k \ddot{\mathbf{X}}_k = \mathbf{F}_k$, equations that appear nowhere in works prior to 1747, not even in Newton's *Principia**. Three years later, the general importance of the momentum principle as a set of differential equations for application to all mechanical problems, whether discrete or continuous, is finally recognized. Euler is now able to discard all of the special mechanical axioms used in earlier works by himself and others, and to formulate by his "new principle of mechanics" the governing equations of motion for each mechanical system he studies. We now have, in 1750,

* Some historians (e.g. Bixby, see Chapter 5, References, page 85) suggest that Newton likely had this equation in mind but avoided its formulation and use in problem solutions in terms of his emerging new calculus, in favor of fashionable geometrical methods that were well understood by scholars and prospective *Principia* readers of the day. Perhaps, but it is nonetheless curious that he never subsequently published his principle in any mathematical form.

Euler's first general principle $\mathbf{F} = \int_{\mathscr{P}} \ddot{\mathbf{X}} dm$ valid for every part $\mathscr{P}$ of any body. (This is the same as (5.42).) In this monumental memoir, Euler succeeds in deriving the equations for the general motion of a rigid body about its center of mass, and he proves that a rigid body may spin freely about an axis provided that the axis is a principal axis through the center of mass, as we shall see later. These grand and novel accomplishments are based entirely on Euler's discovery of the first general principle of motion, not on the less general Newtonian theory, as remarked in Truesdell's *Essays*.

> The discovery of this principle seems so easy, from the Newtonian ideas, that it has never been attributed to anyone but Newton; such is the universal ignorance of the true history of mechanics. It is an incontestable fact that more than sixty years of research using complicated methods even for rather simple problems took place before this "new principle" was seen (by Euler).
>
> C. A. Truesdell, *Essays in the History of Mechanics.*

Ten years later in the work published in 1760, Euler formally defines and names the center of mass for a general deformable body as a property of the body independent of forces, and he shows that the momentum principle thus reduces to an equation of motion for the center of mass; he defines moments of inertia, discovered earlier, and calculates them for homogeneous bodies; he clarifies the concept of principal axes of inertia, studies separately the motion of a rigid body under zero force and under gravity alone, and finally investigates the general motion of a rigid body by its decomposition into motion of its center of mass and motion relative to the center of mass. Here Euler emphasizes the relevance of a moving reference frame fixed in the body, and by a certain transformation, at last, he discovers his famous equations of motion for a rigid body referred to principal axes of inertia at the center of mass.

Finally, in 1771, Euler perceives that the key to the solution of all mechanical problems for bodies is to regard the moment of momentum principle as an independent basic law of mechanics that he now records in the form $\mathbf{M}_O = \int_{\mathscr{P}} \mathbf{x}_O \times \ddot{\mathbf{X}} dm$, in which $\mathbf{M}_O$ is the total torque about a fixed point O, including couples, and $\mathscr{P}$ is any part of any body. (This is the same as (5.44).) In his final memoir of 1775, 25 years after his initial *Discovery of a new principle of mechanics,* Euler sets down the two fundamental and independent principles governing the mechanics applicable to every part of every body. Thus, essentially from 1750 onwards, the solution of mechanical problems was reduced by Euler's principles to a problem in analysis, a problem in the theory of differential equations. With this heritage as foundation and with earlier results in hand, we are now prepared to develop Euler's theory on the general motion of rigid bodies and to illustrate its physical applications in several examples. These are the major objectives ahead.

First, let us backup a bit and recall that a rigid body has at most six degrees of freedom—three translational degrees of freedom of a base point and three independent rotational degrees of freedom about suitably chosen axes. So, at most six scalar equations, equivalent to two vector equations of motion that derive from *Euler's*

Laws, together with appropriately assigned initial data, are needed to determine the general motion of a rigid body. Determination of this motion, however, does not always split neatly into a translational part and a rotational part. Complications arise from circumstances that couple these motions. In fact, Euler's general system of equations for rigid bodies themselves constitute a strongly coupled system of ordinary nonlinear differential equations. In spite of the mathematical difficulties imposed by Euler's equations, we shall find that the analytical solution of a great variety of physical problems is possible.

Our studies begin with a formal introduction to Euler's laws for continua, from which the general equations of motion for a rigid body are then deduced. The role of the center of mass and the importance of the moment of inertia tensor in characterizing the motion of a rigid body are described. Then Euler's equations of motion for rigid bodies are applied in the solution of several problems, including the motion of a flywheel, the motion and stability of a spinning rod, the motion of a gyrocompass, and the impulsive motion of a billiard ball, for example. A general energy equation for the motion of a rigid body also is derived from a work–energy principle; and some additional applications are illustrated.

10.2. Euler's Laws of Motion

Euler's laws relate the forces and torques that act on a body to its translational and rotational motions. We recall from Chapter 1 that force is a primitive concept characterized as a vector measure of the push–pull action between pairs of bodies in the universe. Similarly, *torque* is a primitive concept of mechanics. Any kind of twisting or turning effect about a point or line arises in response to torque. We wind a clock, and a spring driven mechanism responds to turn its hands; we twist a door knob to gain entry; and we toss a football by snapping the wrist to impart a stabilizing spin to it. A rapidly twisted toy top spins on its tip, its axis tilted at an angle opposing the effect of gravity and turning about a vertical line, eventually wobbling up and down, and in the end dramatically collapsing under gravity. A rubber bar having one end bonded to a rigid support and the other end attached to a circular disk, when twisted about its axis and released will induce the disk to oscillate about that axis. An electric motor, without contact between its parts, is turned by electromagnetic action. Moreover, the effect of a twist clearly depends on its axial direction—twisting a body through an angle about an axis generally does not produce the same effect when twisting the body equally about a different axis or direction. Restoring a screw cap on a bottle certainly is not the same as our taking it off—the same degree of twist is applied in reverse. All of these actions describe various physical effects of torques. But they do not tell us what torque is. Torque is an undefined term—a primitive concept. These experiences, however, reveal that torque is a *vector quantity*. A torque about a point is exerted by one body on another with a certain magnitude about a certain direction; it is identified in mechanics as a vector measure of the twist-turn action between pairs of bodies in the universe. Thus, to the primitive terms of kinematics, namely, particle, position, and time, the primitive terms mass, force, and torque about a point are introduced to study the dynamics of all continuous media.

Now let us recall that for a body $\mathscr{B}$ the total mass $m(\mathscr{B})$, momentum $\mathbf{p}(\mathscr{B}, t)$, and moment of momentum $\mathbf{h}_O(\mathscr{B}, t)$ about a point O in an *arbitrary* reference frame $\psi = \{Q; \mathbf{e}_k\}$ are respectively defined in (5.10), (5.11), and (5.33), in which $\mathbf{v}(P, t)$ is the velocity of the particle P in ψ and $\mathbf{x}_O(P, t)$ is the position vector of P from O. Since every part of a body is itself a body, these definitions apply to every part of $\mathscr{B}$.

Notice that the velocity of every particle of the body appears in (5.11) and (5.33). We shall say that the *motion of the body* is known when and only when it is possible to determine the motion $\mathbf{x}(P, t)$ in ψ of all of its material points. In particular, a body is said to have a *uniform motion* in a reference frame ψ, if and only if every particle of the body has the same constant velocity relative to ψ. The body is said to be *stationary* or *at rest* in ψ, if and only if all of its particles are at rest in ψ. As a consequence, it is evident that a necessary, though not sufficient, condition that a (possibly deformable) body may have a uniform motion in ψ is that the motion of its center mass (5.12) be uniform in ψ, i.e.

$$\mathbf{x}(P, t) = \mathbf{X}(P) + \mathbf{v}_0 t \quad \Rightarrow \quad \mathbf{x}^*(\mathscr{B}, t) = \mathbf{X}^*(\mathscr{B}) + \mathbf{v}^*(\mathscr{B})t, \tag{10.1}$$

for all $P \in \mathscr{B}$. Here $\mathbf{X}(P) \equiv \mathbf{x}(P, 0)$ and $\mathbf{X}^*(\mathscr{B}) \equiv \mathbf{x}^*(\mathscr{B}, 0)$ are the respective initial positions of P and of the center of mass of $\mathscr{B}$, and $\mathbf{v}^*(\mathscr{B}) = \mathbf{v}(P) \equiv \mathbf{v}_0$ for all $P \in \mathscr{B}$ is its constant velocity. Plainly, if the motion is uniform, the acceleration $\mathbf{a}(P, t) = \mathbf{a}^*(\mathscr{B}, t) = \mathbf{0}$ for all $P \in \mathscr{B}$ and for all time t. For the converse, however, we need to say more.

The *axiom of continuity of matter* that comprises a continuum, namely a body, specifies that no material element of finite dimension can be deformed into one of zero or infinite dimension. Now, plainly, if the acceleration of every material point P of a body $\mathscr{B}$ vanishes, then every point of $\mathscr{B}$ must have a uniform motion. So, if the constant velocity of material points differed, the relative position vector for an arbitrary pair of particles would grow or shrink indefinitely over time, contrary to the axiom of continuity. Therefore, the acceleration of every material point of a body vanishes for all time t only when $\mathbf{v}(P) = \mathbf{v}_0$, a constant, for all particles $P \in \mathscr{B}$. Consequently, *the acceleration of every material point of a body $\mathscr{B}$ vanishes when and only when the motion of $\mathscr{B}$ is uniform.*

In all of these definitions the reference frame ψ may be any convenient frame. In the laws that follow, however, we shall assume the existence of a particular frame, called the *preferential* or *inertial frame* Φ, with respect to which the laws are asserted to hold. With this in mind, we now set down the basic principle of determinism modeled after Newton's first law.

The principle of determinism: *In every material universe, the motion of a body (i.e. all of its particles) in a preferential reference frame Φ is determined by the action of forces and torques about an arbitrary point O fixed in Φ such that the total force and the total torque vanish together when the body either is at rest* *or has a uniform motion in* Φ. *That is, a body at rest or in uniform motion in the preferential frame continues in that state until compelled by forces and torques, acting separately or together, to change it.*

The preferred frame is identified as the Newtonian reference frame discussed in Chapter 5. We recall that every nonrotating, uniformly translating frame is an equivalent inertial reference frame. In the real world, the preferred frame may be identified as any frame which has at most a uniform translational velocity relative to the astronomical frame fixed in the distant stars. In any case, however, the point O must be fixed in Φ. Continuing, we further postulate existence of a material universe, called the *world*, wherein the following laws hold.

Euler's Laws of Motion

1. **The first law of motion:** *The total force* $\mathbf{F}(\mathscr{B}, t)$ *exerted on a body* $\mathscr{B}$ *in the inertial frame* Φ *is equal to the time rate of change of the total linear momentum of* $\mathscr{B}$ *relative to* Φ:

$$\mathbf{F}(\mathscr{B}, t) = \frac{d\mathbf{p}(\mathscr{B}, t)}{dt} = \frac{d}{dt}\int_{\mathscr{B}} \mathbf{v}(P, t)\, dm(P). \tag{10.2}$$

2. **The second law of motion:** *With respect to a point* O *fixed in* Φ, *the total torque* $\mathbf{M}_O(\mathscr{B}, t)$ *that acts on a body* $\mathscr{B}$ *in the inertial frame* Φ *is equal to the time rate of change of the total moment about* O *of the momentum of* $\mathscr{B}$ *relative to* Φ:

$$\mathbf{M}_O(\mathscr{B}, t) = \frac{d\mathbf{h}_O(\mathscr{B}, t)}{dt} = \frac{d}{dt}\int_{\mathscr{B}} \mathbf{x}_O(P, t) \times \mathbf{v}(P, t)\, dm(P). \tag{10.3}$$

Because every part $\mathscr{P}$ of a body is itself a body, Euler's laws hold for every part $\mathscr{P}$ of every body $\mathscr{B}$. We recall that (10.2) and (10.3) are, respectively, the same as (5.42) and (5.44) introduced informally in Chapter 5. The symbolic quantities used in (10.2) and (10.3) are shown schematically in Fig. 10.1. It is important to recall that $\mathbf{X}(P, t)$ is the position vector of P from F in the inertial frame $\Phi = \{F; \mathbf{I}_k\}$, and hence $\mathbf{v}(P, t) = \dot{\mathbf{X}}(P, t)$, whereas $\mathbf{x}_O(P, t)$ is the position vector of P from the *fixed* point O in Φ. Sometimes, as indicated in Fig. 10.1, we write $\mathbf{x}(P, t) \equiv \mathbf{x}_O(P, t)$. Clearly, because O is fixed, $\dot{\mathbf{x}}(P, t) = \dot{\mathbf{X}}(P, t)$. The situation when the moment point O may be moving in Φ is explored later. The substance of Euler's laws will be discussed first. We begin with some remarks on forces and torques.

10.2.1. Forces and Torques

The total force and the total torque about a specified point are frame invariant physical entities—they are the same for all observers. Of course, the direction in which a force or torque may act will appear differently to different observers, but

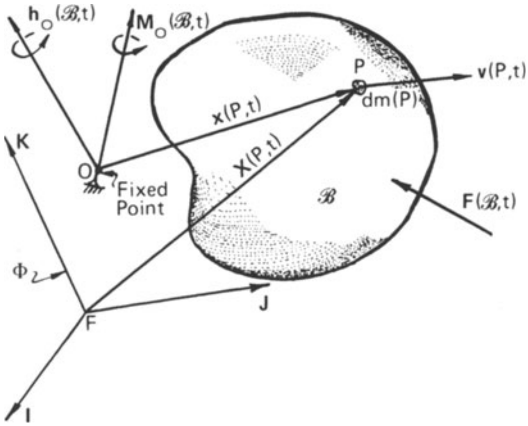


Figure 10.1. Schema for quantities appearing in Euler's laws.

their physical nature does not change. Gravitational force is invariably directed toward the Earth, and at a specified place relative to the Earth, the weight of a body is the same for all observers, moving or not. The motion of a reference frame, however, induces inertial forces that may alter the apparent weight of a body, though in fact the gravitational force at the same place is not affected in any manner by the observer's motion. Because the Earth is a frame whose motion induces such forces, we have agreed previously to adopt the apparent weight as our measure of the gravitational force due to the Earth. We also recall that the force exerted by a linear spring is always proportional to its extension along its axis, regardless of whether the spring is situated on a rotating table or simply hung from a fixed laboratory support. Similarly, the torque exerted by a linear torsion spring is proportional to the angle of twist about its axis. The motion of a reference frame in which the spring may be fastened does not alter this invariant physical property. Of course, the torque varies with the choice of reference point about which the moment is determined. To refer a force or a torque vector to any desired reference frame is a straightforward geometrical problem. This simply means that the force or the torque is represented in terms of the vector basis that defines the frame, but a change of frame does not change in any way the physical nature of the force or the torque that acts on a body.

The torques that act on a body may be classified in the same terms introduced in Chapter 5 for forces. *Contact torque* arises from the mutual action of bodies that touch one another, while *body torque* is produced by the interaction between pairs of separated bodies. Electric and magnetic torques depending on fields **E** and **B**, respectively, are examples of body torques. The torque exerted on a part $\mathscr{P}$ of a body $\mathscr{B}$ by another part of the same body is called an *internal torque*. The torque exerted on a part $\mathscr{P}$ of $\mathscr{B}$ by another part $\mathscr{P}^*$ of a separate body $\mathscr{B}^*$ that is not a part of $\mathscr{B}$ is called an *external torque*. The various torques that may act on a body are assumed to be totally additive. Thus, the total torque acting on a body $\mathscr{B}$ is the sum of the total contact torque and the total body torque exerted on $\mathscr{B}$.

10.2.2. The Principle of Determinism

The principle of determinism specifies that any disturbance of a body from a stationary or uniform state of motion relative to an inertial frame can occur only in response to force and torque, acting separately or together. An arbitrary uniform motion or trivial stationary state requires neither of these. Since a stationary state is a special uniform motion for which $\mathbf{v}(P, t) = \mathbf{0}$ for every particle of the body, there is no intrinsic difference between a uniform motion and a state of ease; so, we often refer to these states collectively as a uniform motion. Let $\mathbf{F}(\mathscr{B}, t)$ denote the total force and $\mathbf{M}_O(\mathscr{B}, t)$ the total torque about an arbitrary point O fixed in an inertial frame Φ. Then, in analytical terms, the principle of determinism states that in every material universe the motion $\mathbf{X}(P, t)$ of *every* material point P of a body $\mathscr{B}$ relative to an inertial frame Φ is determined by two *vector equations of motion* (written in the standard form adopted in Chapter 5),

$$\mathbf{F}(\mathscr{B}, t) = \mathscr{F}(\mathbf{X}(P, t)), \qquad \mathbf{M}_O(\mathscr{B}, t) = \mathscr{T}_O(\mathbf{X}(P, t)), \tag{10.4}$$

in which the general functionals $\mathscr{F}$ and $\mathscr{T}_O$ vanish *together* for all time when the motion of the body is uniform with constant velocity $\mathbf{v}_0$ in Φ, namely,

$$\mathscr{F}(\mathbf{X}_0(P) + \mathbf{v}_0 t) = \mathbf{0}, \qquad \mathscr{T}_O(\mathbf{X}_0(P) + \mathbf{v}_0 t) = \mathbf{0}. \tag{10.5}$$

That is, by (10.4), for every particle P of an assigned body $\mathscr{B}$ and for all t in Φ,

$$\mathbf{X}(P, t) = \mathbf{X}_0(P) + \mathbf{v}_0 t \quad \Rightarrow \quad \mathbf{F}(\mathscr{B}, t) = \mathbf{0} \quad \text{and} \quad \mathbf{M}_O(\mathscr{B}, t) = \mathbf{0}, \tag{10.6}$$

where $\mathbf{X}_0(P) \equiv \mathbf{X}(P, 0)$. Notice from (5.24) that when (10.6) holds for all t, the total torque about *every* point Q in Φ must vanish for all t: $\mathbf{M}_Q(\mathscr{B}, t) = \mathbf{0}$, and hence the force system is equipollent to zero.

The principle of determinism thus restricts the class of admissible functionals $\mathscr{F}$ and $\mathscr{T}_O$ in (10.4) to those for which the two relations in (10.5) hold in every material universe. Equations (10.4) and (10.5), on the other hand, are applicable to any body regardless of its constitution. Everybody knows, however, that solids, fluids, and gasses behave differently under the same forces and torques. In general, therefore, the motion of the material points of a body cannot be found until the special constitutive nature of the body is assigned. Is the body a solid or a fluid? And what kind of solid or fluid? The principle of determinism says nothing about this, rather, it tacitly assumes a potential constitutive assignment for the body but places no a priori restrictions on it. We may ultimately specify, for example, that the body is rigid so that its geometry is known for all time. In this case, the constraint of rigidity of all material points of the body suffices to determine the motion of a rigid body (from Euler's laws). Thus, when the applied forces and torques are specified, the two fundamental equations (10.4) subject to (10.5), in principle, determine in any specified material universe the motion of every particle of a rigid body relative to an inertial reference frame. We see later how this may be done for the material universe in which Euler's laws hold.

The principle of determinism includes for all bodies a *law of equilibrium* that is the same for every material universe. Recall that a motion of a body is uniform when and only when the acceleration of every particle of the body is zero for all time. Therefore, (10.6) may be written as

$$\mathbf{a}(P,t) = \mathbf{0} \quad \text{for all} \quad P \in \mathscr{B} \quad \Rightarrow \quad \mathbf{F}(\mathscr{B},t) = \mathbf{0} \quad \text{and} \quad \mathbf{M}_O(\mathscr{B},t) = \mathbf{0}, \tag{10.7}$$

in an inertial frame Φ, for all t. By definition, a uniform motion in Φ is called an *equilibrium configuration* of $\mathscr{B}$ in Φ. Therefore, in accordance with (10.6) and (10.7), the principle of determinism implies that *in every material universe, a necessary condition for equilibrium of a body in an inertial frame is that the total force and the total torque about an arbitrary point O fixed in* Φ *shall vanish together, and this is possible provided that the acceleration in* Φ *of every particle of the body is zero*:

$$\begin{aligned} \text{Equilibrium} \quad &\Leftrightarrow \quad \mathbf{a}(P,t) = \mathbf{0} \text{ for all } P \in \mathscr{B} \text{ and for all } t \\ &\Rightarrow \quad \mathbf{F}(\mathscr{B},t) = \mathbf{0} \quad \text{and} \quad \mathbf{M}_O(\mathscr{B},t) = \mathbf{0} \text{ for all } t. \end{aligned} \tag{10.8}$$

Notice that equilibrium is merely a sufficient condition for which both the total force and the total torque must vanish. Unlike the corresponding principle embodied in Newton's first law, however, the vanishing of both the total force and the total torque does not imply equilibrium of a body. The reason for this will become clear later on. It is then proved, conversely, that for a rigid body *initially* at ease or in uniform motion in Φ at an instant t_0, the simultaneous null equations in (10.8) also suffice for equilibrium of the body for all t.

10.2.3. Euler's Laws

The form of the equations of motion (10.4) will depend on the nature of the particular material universe considered, and Euler's laws are definite about this. These laws postulate for a particular material universe, called the world, the specific functional relations (10.2) and (10.3) relating force and torque to the motion of a body by rules that best describe the nature of things in the world, our mathematical abstraction of the real world. It is easy to verify that, the mass being constant, (10.2) and (10.3) satisfy (10.6) in accordance with the principle of determinism. Thus, in accord with this principle, there may exist infinitely many material universes, all having the same principle of equilibrium but each characterized by special constitutive laws of inertia of its own, perhaps different from (10.2) and (10.3).

So far the concepts of torque and force have been treated on the same abstract level as separate unrelated entities. There is nothing, however, that prohibits a relationship between force and torque. Henceforward, torque is defined as the moment of force about a point,† whatever may be the nature of the force. Thus, if $d\mathbf{F}(P, t)$ denotes the elemental force distribution acting at time t at a particle P of a body, the *total force* and the *total moment about a point* Q of the force acting on $\mathscr{B}$ at time t are respectively defined by

$$\mathbf{F}(\mathscr{B}, t) = \int_{\mathscr{B}} d\mathbf{F}(P, t), \tag{10.9}$$

$$\mathbf{M}_Q(\mathscr{B}, t) = \int_{\mathscr{B}} \mathbf{x}_Q(P, t) \times d\mathbf{F}(P, t), \tag{10.10}$$

wherein $\mathbf{x}_Q(P, t)$ is the vector of P from Q. The point Q may be any point at rest or in motion relative to Φ, whereas Euler's second law requires in (10.3) that the point O must be fixed in Φ. Further, since the distribution of force may consist of both contact and body force, $\mathbf{M}_Q$ will be the sum of their total moments about Q. These moments are the total contact torque and the total body torque acting on the body about Q. Concentrated forces and their moments about Q are included as pointwise distributions.

If all torques are moments of forces about any fixed point in an inertial frame $\Phi = \{F; \mathbf{e}_k\}$ and, as usual, mass is conserved, the second law (10.3) may be derived from the first law applied to an incremental distribution of force $d\mathbf{F}(P, t) = \dot{\mathbf{v}}(P, t)dm(P)$, the increment being a continuous function of P and t acting on the material parcel of mass $dm(P)$ at P. First, recall the notation in Fig. 10.1. Then with respect to any fixed point O in Φ, by (10.10),

$$\begin{aligned}\mathbf{M}_O(\mathscr{B}, t) &= \int_{\mathscr{B}} \mathbf{x}_O(P, t) \times \dot{\mathbf{v}}(P, t)\, dm \\ &= \frac{d}{dt}\int_{\mathscr{B}} \mathbf{x}_O(P, t) \times \mathbf{v}(P, t)\, dm = \frac{d\mathbf{h}_O(\mathscr{B}, t)}{dt},\end{aligned}$$

wherein $\mathbf{v} = \dot{\mathbf{X}} = \dot{\mathbf{x}} \equiv \dot{\mathbf{x}}_O$, because O is fixed in Φ. This is Euler's second law (10.3). Note, however, that in order to move the time derivative outside the integral, it is necessary to use a fixed referential configuration of the body, which may be deformable and changing with time. Without getting into these details, it suffices to know that this is always possible, and it is certainly true for any rigid body $\mathscr{B}$, our principal concern here. Conversely, if all torques are moments of forces about an arbitrary point O fixed in Φ and mass is conserved, as before, then the first of Euler's laws may be derived from the second. Indeed, by (10.3) and (10.10), for a fixed referential configuration of the body, specifically for a rigid body, we may

† *Generalized couples* that include contact couples and body couples whose total torque $\boldsymbol{\mu}(\mathscr{B}, t)$ is independent of any reference point Q are introduced in certain continuum theories of deformable bodies. In this case, the additional torque $\boldsymbol{\mu}(\mathscr{B}, t)$ must be added to the right-hand side of (10.10). But these generalized couples are not identified as moments of force. The familiar idea of a force couple comprised of a pair of opposite, equal forces whose total is zero and whose torque about a point is independent of the moment point (see the discussion at the end of Section 5.3 in Chapter 5) are included in the total (10.10).

form the integral

$$\int_{\mathscr{B}} \mathbf{x}_O(P,t) \times \left(\frac{d\mathbf{F}(P,t)}{dm(P)} - \dot{\mathbf{v}}(P,t) \right) dm(P) = \mathbf{0}.$$

We then suppose that the term in parentheses, which is independent of the reference point O, varies continuously with P and t over all parcels $dm(P)$ of the body. Since O is an arbitrary point fixed in Φ, it follows that $d\mathbf{F}/dm - \dot{\mathbf{v}} = \mathbf{0}$, and hence $\mathbf{F}(P,t) = \int_{\mathscr{B}} \dot{\mathbf{v}}(P,t) dm(P)$. This is Euler's first law (10.2). In spite of their mutual dependence for the assigned conditions, however, it is essential that both principles be applied in the analysis of the general motion of bodies. In fact, earlier studies of statics emphasize the essential nature of both laws.

We learned in Chapter 5 that the momentum of a body is equal to the momentum of its center of mass: $\mathbf{p}(\mathscr{B},t) = \mathbf{p}^*(\mathscr{B},t)$. As a consequence, and since the mass of the body is conserved, Euler's first law (10.2) may be cast in the well-known classical form (5.43), namely,

$$\mathbf{F}(\mathscr{B},t) = \dot{\mathbf{p}}^*(\mathscr{B},t) = m(\mathscr{B})\mathbf{a}^*(\mathscr{B},t). \tag{10.11}$$

Hence, only the motion of the center of mass point is determined by (10.11), and $\mathbf{F}(\mathscr{B},t) = \mathbf{0}$, if and only if the motion of the center of mass is uniform. Clearly, the other material points of the body, and hence the body itself, need not have uniform motion. Euler's second law (10.3), therefore, is essential to the determination of the general motion of all points of the body. Indeed, in his memoir of 1750, Euler introduced and applied his new first principle of mechanics, and by taking the moment about the center of mass of the incremental momentum of a rigid body parcel he thereby derived the general equations of motion for a rigid body relative to its center of mass. On that occasion, however, there was no explicit identification of his second law; this followed in the work published in 1771. Discussion of Euler's laws will continue following some observations on the law of action and reaction.

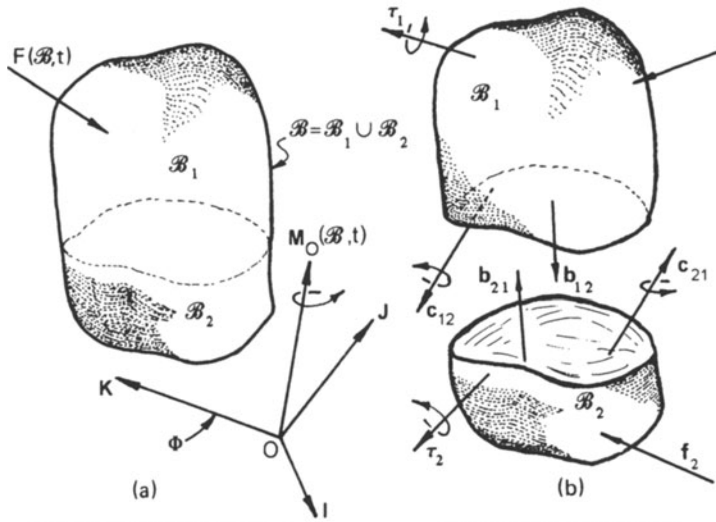
10.3. The Law of Mutual Action

So far, a principle of mutual action corresponding to Newton's third law, already applied repeatedly to interactions between pairs of bodies, has not been formally set down. We are going to show that the law of mutual action for bodies may be derived from Euler's laws of motion.‡

Consider the force distribution $d\mathbf{F}(P,t)$ at time t acting over the free body $\mathscr{B}$ shown in Fig. 10.2a, so that the total force on $\mathscr{B}$ is given by (10.9). Now suppose that $\mathscr{B}$ is divided into separate parts $\mathscr{B}_1$ and $\mathscr{B}_2$ so that $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$, as shown in Fig. 10.2a. Let $\mathbf{f}_k$ denote the part of the total force (10.9) that acts on the part $\mathscr{B}_k$, namely,

$$\mathbf{f}_k \equiv \mathbf{f}(\mathscr{B}_k,t) = \int_{\mathscr{B}_k} d\mathbf{F}(P_k,t), \tag{10.12}$$

‡ The theorem on mutual action of forces is due to Noll. See the sources listed in the chapter references.



Total force and torque acting on the free body $\mathscr{B}=\mathscr{B}_1 \cup \mathscr{B}_2$.

Forces and torques acting on the separate bodies $\mathscr{B}_1$ and $\mathscr{B}_2$.

Figure 10.2. Schema for the law of mutual action.

where P_k is a material point belonging to $\mathscr{B}_k$. Then by (10.9) and (10.12), the total force that acts on $\mathscr{B}$ may be written as

$$\mathbf{F}(\mathscr{B}, t) = \mathbf{f}_1 + \mathbf{f}_2. \tag{10.13}$$

Further, let $\mathbf{b}_{kj} = \mathbf{b}_j(\mathscr{B}_k, t)$ represent the *resultant mutual force* exerted on the part $\mathscr{B}_k$ by the part $\mathscr{B}_j$, as shown in Fig. 10.2b. This force may consist of a mutual body force distribution and a contact force distribution over the mutual boundary separating $\mathscr{B}_1$ and $\mathscr{B}_2$. Then the total force on the separate free body $\mathscr{B}_k$ is

$$\hat{\mathbf{F}}_k \equiv \hat{\mathbf{F}}(\mathscr{B}_k, t) = \mathbf{f}_k + \mathbf{b}_{kj}, \qquad k \neq j = 1, 2. \tag{10.14}$$

Since Euler's first law of motion holds for the total force on each separated part in Fig. 10.2b, application of (10.2) to the part $\mathscr{B}_k$ gives

$$\hat{\mathbf{F}}_k = \frac{d\mathbf{p}_k}{dt}, \tag{10.15}$$

where $\mathbf{p}_k \equiv \mathbf{p}(\mathscr{B}_k, t)$. On the other hand, application of (10.2) to the entire body $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ in Fig. 10.2a yields

$$\mathbf{F}(\mathscr{B}, t) = \frac{d}{dt}\mathbf{p}(\mathscr{B}_1 \cup \mathscr{B}_2, t) = \frac{d}{dt}\mathbf{p}(\mathscr{B}_1, t) + \frac{d}{dt}\mathbf{p}(\mathscr{B}_2, t); \tag{10.16}$$

therefore, by (10.15),

$$\mathbf{F}(\mathscr{B}, t) = \hat{\mathbf{F}}_1 + \hat{\mathbf{F}}_2. \tag{10.17}$$

Upon summing the forces (10.14) and recalling (10.13), however, we find

$$\hat{\mathbf{F}}_1 + \hat{\mathbf{F}}_2 = \mathbf{F}(\mathscr{B}, t) + \mathbf{b}_{12} + \mathbf{b}_{21};$$

and hence, in view of (10.17), we have $\mathbf{b}_{12} + \mathbf{b}_{21} = \mathbf{0}$. In sum, *the resultant mutual force exerted on the body $\mathscr{B}_1$ by the body $\mathscr{B}_2$ is the opposite of the resultant mutual force vector exerted on the body $\mathscr{B}_2$ by the body $\mathscr{B}_1$.*

Similarly, introduce a total torque $\boldsymbol{\tau}_k \equiv \boldsymbol{\tau}_O(\mathscr{B}_k, t) = \int_{\mathscr{B}_k} d\mathbf{M}_O(P_k, t)$ that acts on the part $\mathscr{B}_k$ so that the total torque about O is $\mathbf{M}_O(\mathscr{B}, t) = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2$ for $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ in Fig. 10.2a. Now, consider torque abstractly and let $\mathbf{c}_{kj} = \mathbf{c}_j(\mathscr{B}_k, t)$ denote the *resultant mutual torque* about a fixed point O exerted on the part $\mathscr{B}_k$ by the part $\mathscr{B}_j$ in Fig. 10.2b. Then by a parallel argument that uses Euler's second law (10.3) for the moment of momentum about O, namely, $\mathbf{M}_O(\mathscr{B}, t) = d\mathbf{h}_O(\mathscr{B}_1 \cup \mathscr{B}_2)/dt = d\mathbf{h}_O(\mathscr{B}_1)/dt + d\mathbf{h}_O(\mathscr{B}_2)/dt$, it follows that $\mathbf{c}_{12} + \mathbf{c}_{21} = \mathbf{0}$. Thus, *the resultant mutual torque exerted on the body $\mathscr{B}_1$ by the body $\mathscr{B}_2$ is the opposite of the resultant mutual torque exerted on the body $\mathscr{B}_2$ by the body $\mathscr{B}_1$.* Notice here that restriction to torques that are moments of force is not required.

Collecting the results, we have the two balance equations, $\mathbf{b}_{12} + \mathbf{b}_{21} = \mathbf{0}$, $\mathbf{c}_{12} + \mathbf{c}_{21} = \mathbf{0}$. These express the principle of mutual action for all bodies, including deformable bodies.

The law of mutual action: *The resultant mutual actions of two bodies one upon the other are oppositely directed, equal vectors*:

$$\mathbf{b}_{12} = -\mathbf{b}_{21}, \qquad \mathbf{c}_{12} = -\mathbf{c}_{21}. \tag{10.18}$$

Because the body and contact forces and torques that act on a body and its parts may act separately, the relations (10.18) may be applied separately to mutual contact forces, body forces, contact torques, and body torques. If a body presses normally against a surface, the surface presses with an equal but oppositely directed normal reaction on the body, as we know. Similarly, if a disk applies a contact torque to a shaft, the shaft exerts an equal but oppositely directed contact torque on the disk. When the disk is suddenly released, the torque due to the shaft acts to restore the disk to its primary state and induces its oscillation.

Mutual actions are internal actions, and hence the law of mutual action shows in (10.18) that the total internal force and the total internal torque acting on a body vanish. Therefore, *only external forces and external torques influence the motion of a body, and hence any part of a body.* Of course, when the part $\mathscr{B}_1$ alone is considered, the resultant mutual force $\mathbf{b}_{12}$ and the resultant mutual torque $\mathbf{c}_{12}$ are now external actions, and the respective total external force and total external torque that act on $\mathscr{B}_1$, for example, are respectively given by $\mathbf{F}(\mathscr{B}_1, t) = \mathbf{f}_1 + \mathbf{b}_{12}$ and $\mathbf{M}_O(\mathscr{B}_1, t) = \boldsymbol{\tau}_1 + \mathbf{c}_{12}$. Truesdell's *Essays* remind us that this all seemed rather apparent to Euler who asserts that "since a body does not spontaneously assume any motion in virtue of whatever internal forces there may be within it, these do not contribute to its motion as a whole." Clearly, internal forces, torques and motion at the atomic or molecular levels are of no concern here.

The first rule in (10.18) shows that the resultant mutual internal forces occur in equal, oppositely directed pairs. The second rule in (10.18) shows that the resultant mutual internal torques behave similarly. Recall, however, that a pair of equal and oppositely directed forces might constitute a couple whose torque does not vanish and is independent of the choice of reference point. This torque can be zero if and only if the forces are directed along their mutual line. The rule on the vanishing of the total internal torque in (10.18), interpreted as the moment of the mutual internal forces about a fixed point O reveals that the resultant mutual internal forces must be collinear.

To prove this, let us consider the moment about a fixed point O of all forces that act on $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ and its separate parts $\mathscr{B}_1$ and $\mathscr{B}_2$. For the body $\mathscr{B}$ in Fig. 10.2a, by (10.3) and (10.10),

$$\mathbf{M}_O(\mathscr{B}, t) = \int_{\mathscr{B}} \mathbf{x}_O(P, t) \times d\mathbf{F}(P, t) = \frac{d\mathbf{h}_O(\mathscr{B}, t)}{dt}. \tag{10.19}$$

In addition, since $\mathbf{f}_1$ and $\mathbf{f}_2$ are those portions of the total force $\mathbf{F}(\mathscr{B}, t)$ that act on the contiguous parts $\mathscr{B}_1$ and $\mathscr{B}_2$, respectively, we have

$$\mathbf{M}_O(\mathscr{B}, t) = \int_{\mathscr{B}_1} \mathbf{x}_O(P_1, t) \times d\mathbf{f}_1(P_1, t) + \int_{\mathscr{B}_2} \mathbf{x}_O(P_2, t) \times d\mathbf{f}_2(P_2, t), \tag{10.20}$$

where P_k denotes a material point of the part $\mathscr{B}_k$. And for the separate parts $\mathscr{B}_1$ and $\mathscr{B}_2$ in Fig. 10.2b,

$$\mathbf{M}_O(\mathscr{B}_1, t) = \int_{\mathscr{B}_1} \mathbf{x}_O(P_1, t) \times d\mathbf{f}_1(P_1, t) + \int_{\mathscr{B}_1} \mathbf{x}_O(P_1, t) \times d\mathbf{b}_{12}(P_1, t) = \frac{d\mathbf{h}_O(\mathscr{B}_1, t)}{dt},$$

$$\mathbf{M}_O(\mathscr{B}_2, t) = \int_{\mathscr{B}_2} \mathbf{x}_O(P_2, t) \times d\mathbf{f}_2(P_2, t) + \int_{\mathscr{B}_2} \mathbf{x}_O(P_2, t) \times d\mathbf{b}_{21}(P_2, t) = \frac{d\mathbf{h}_O(\mathscr{B}_2, t)}{dt},$$

wherein $d\mathbf{b}_{jk}$ is the elemental mutual force exerted on the disjoint part $\mathscr{B}_j$ by $\mathscr{B}_k$. Adding the last two equations and introducing (10.20), we obtain

$$\begin{aligned}\mathbf{M}_O(\mathscr{B}_1, t) + \mathbf{M}_O(\mathscr{B}_2, t) &= \mathbf{M}_O(\mathscr{B}, t) + \textstyle\int_{\mathscr{B}_1} \mathbf{x}_O(P_1, t) \times d\mathbf{b}_{12}(P_1, t) \\ &+ \textstyle\int_{\mathscr{B}_2} \mathbf{x}_O(P_2, t) \times d\mathbf{b}_{21}(P_2, t) = \dfrac{d\mathbf{h}_O(\mathscr{B}_1, t)}{dt} + \dfrac{d\mathbf{h}_O(\mathscr{B}_2, t)}{dt}.\end{aligned} \tag{10.21}$$

Since $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$, also $\mathbf{h}_O(\mathscr{B}_1, t) + \mathbf{h}_O(\mathscr{B}_2, t) = \mathbf{h}_O(\mathscr{B}_1 \cup \mathscr{B}_2, t) = \mathbf{h}_O(\mathscr{B}, t)$. Hence, using this relation in (10.21) and recalling (10.19), we conclude that *the* *resultant moment about O of the mutual forces vanishes*:

$$\int_{\mathscr{B}_1} \mathbf{x}_O(P_1, t) \times d\mathbf{b}_{12}(P_1, t) + \int_{\mathscr{B}_2} \mathbf{x}_O(P_2, t) \times d\mathbf{b}_{21}(P_2, t) = \mathbf{0}. \quad (10.22)$$

This coincides with the second rule in (10.18) for the resultant mutual torques.

Finally, we introduce the following equipollent moments about O defined by

$$\mathbf{x}_O(\mathscr{B}_k, t) \times \mathbf{b}_{kj}(\mathscr{B}_k, t) \equiv \int_{\mathscr{B}_k} \mathbf{x}_O(P_k, t) \times d\mathbf{b}_{kj}(P_k, t), \qquad j \neq k = 1, 2. \quad (10.23)$$

Given the resultant mutual force $\mathbf{b}_{kj}(\mathscr{B}_k, t) = \int_{\mathscr{B}_k} d\mathbf{b}_{kj}(P_k, t) \neq \mathbf{0}$, a vector $\mathbf{x}_O(\mathscr{B}_k, t) \equiv \mathbf{r}_k(t)$ for which (10.23) holds at each time t can be found. The vectors $\mathbf{r}_k$ are position vectors from O to distinct points on the lines of application of the resultant mutual forces $\mathbf{b}_{kj}$, positioned so that their respective moment about O produces the same torque determined by the right-hand side in (10.23). Thus, using (10.23) to rewrite (10.22) and introducing the law of mutual action for the resultant mutual forces, we obtain

$$(\mathbf{r}_2(t) - \mathbf{r}_1(t)) \times \mathbf{b}_{21} = \mathbf{0}. \quad (10.24)$$

Herein $\mathbf{r}_2(t) - \mathbf{r}_1(t)$ is the vector connecting distinct points on the lines of application of the resultant mutual forces and to which, by (10.24), the forces $\mathbf{b}_{12} = -\mathbf{b}_{21}$ must be parallel. *Consequently, the resultant mutual internal forces are oppositely directed collinear vectors.*

This concludes the discussion of the law of mutual action. We now return to Euler's first law and present some general auxiliary principles on the motion of the center of mass.

10.4. Euler's First Principle and Motion of the Center of Mass

In view of the result on external forces, a more complete statement of (10.2) in the form of (10.11) now reads as follows.

Euler's first principle of motion: *The total external force that acts on a body $\mathscr{B}$ is equal to the time rate of change of the momentum of its center of mass in the inertial frame Φ:*

$$\mathbf{F}(\mathscr{B}, t) = \dot{\mathbf{p}}^*(\mathscr{B}, t), \quad (10.25)$$

or, equivalently, the product of its mass times the acceleration of its center of mass in Φ:

$$\mathbf{F}(\mathscr{B}, t) = m(\mathscr{B})\mathbf{a}^*(\mathscr{B}, t). \quad (10.26)$$

From (10.25), we derive immediately the following familiar conservation law.

Principle of conservation of momentum of a body: *The total external force component in a fixed direction* **e** *in the inertial frame* Φ *vanishes if and only if the corresponding scalar component of the momentum of the center of mass, hence also that of the body, is constant in* Φ:

$$\mathbf{F}(\mathscr{B}, t) \cdot \mathbf{e} = 0 \quad \Leftrightarrow \quad \mathbf{p}^*(\mathscr{B}, t) \cdot \mathbf{e} = \text{const.} \tag{10.27}$$

Thus, in the absence of any external force, the momentum of the center of mass, hence the momentum of the body, is a constant vector in Φ. In this case, the center of mass moves uniformly on a straight line, or if at rest, it remains so; but this does not preclude the body's having motion relative to its center of mass.

Integration of (10.25) with respect to time yields the additional familiar result.

The impulse–momentum principle for a body: *The impulse*

$$\mathscr{I}(t; t_0) \equiv \int_{t_0}^{t} \mathbf{F}(\mathscr{B}, t)\, dt, \tag{10.28}$$

of the total external force acting on a body over the time interval $[t_0, t]$ *is equal to the change in the momentum of its center of mass, hence also the total momentum of the body, during that time*:

$$\mathscr{I}(t; t_0) = \Delta\mathbf{p}^*(\mathscr{B}, t). \tag{10.29}$$

An instantaneous impulse $\mathscr{I}^*$, as in (7.7) for a particle, is defined by

$$\mathscr{I}^* \equiv \lim_{t \to t_0} \mathscr{I}(t; t_0). \tag{10.30}$$

Thus, with (10.29), the instantaneous impulse is equal to the instantaneous change in the momentum of the center of mass. In an instantaneous impulse there is an instantaneous change in the velocity of the center of mass of the body, but there is no change in its position at the impulsive instant. Moreover, finite-valued external forces do not contribute to the instantaneous impulse.

The basic equations (10.25) and (10.26) clearly show that our earlier use of Newton's laws for a particle, a center of mass object, in applications to bodies of finite size was correct and well-modeled; but for bodies there is more—the effects of external torques must be described. Similar principles for the external torque, impulsive torque, and moment of momentum of a body are presented later. The quantities used in the proof of the first equation in (10.18) are illustrated next.

Example 10.1. A 400 lb crate shown in Fig. 10.3 is lifted in an elevator that weighs 2000 lb. The tension in the hoisting cable is 3000 lb. (i) What total force **N** acts on the bottom of the crate? (ii) Relate the various forces here to those identified in the construction of the mutual action principle. Ignore the mutual gravitational body force between $\mathscr{B}_1$ and $\mathscr{B}_2$, and assume that $g = 32$ ft/sec^2.

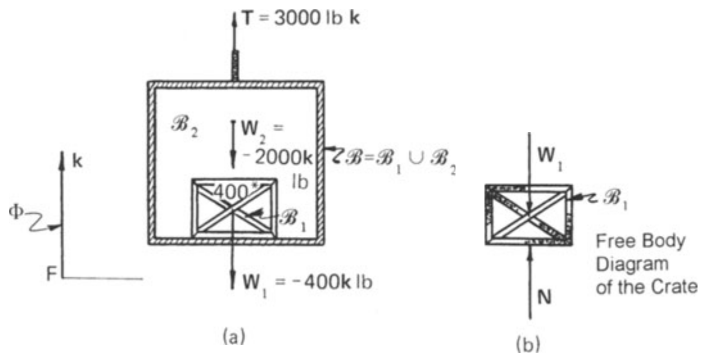


Figure 10.3. Forces on a rigid crate in a moving elevator.

Solution of (i). First consider the crate $\mathscr{B}_1$ alone; its free body diagram is shown in Fig. 10.3b. Thus, with the total external force $\mathbf{F}(\mathscr{B}_1, t) = \mathbf{N} + \mathbf{W}_1$ acting on $\mathscr{B}_1$, (10.26) yields the resultant force $\mathbf{N} = m(\mathscr{B}_1)\mathbf{a}^*(\mathscr{B}_1, t) - \mathbf{W}_1$ exerted on the crate by the elevator floor. With $\mathbf{W}_1 = -400\mathbf{k}$ lb and $m(\mathscr{B}_1) = W_1/g = 400/32$ slug,

$$\mathbf{N} = \tfrac{25}{2}\mathbf{a}^*(\mathscr{B}_1, t) + 400\mathbf{k} \text{ lb}. \tag{10.31a}$$

Since the system moves only in pure translation, every particle has the same acceleration. Therefore, the acceleration $\mathbf{a}^*(\mathscr{B}_1, t)$ of the center of mass of the crate $\mathscr{B}_1$ is the same as the acceleration $\mathbf{a}^*(\mathscr{B}, t)$ of the center of mass of the body $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ consisting of the crate $\mathscr{B}_1$ and the elevator $\mathscr{B}_2$. This acceleration is determined by (10.26). The total external force $\mathbf{F}(\mathscr{B}, t)$ acting on $\mathscr{B}$ is the sum of its weight $\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2 = -(400 + 2000)\mathbf{k}$ lb $= -2400\mathbf{k}$ lb and the cable tension $\mathbf{T} = 3000\mathbf{k}$ lb, as indicated in Fig. 10.3a. Hence, $\mathbf{F}(\mathscr{B}, t) = 600\mathbf{k}$ lb; and with $m(\mathscr{B}) = W/g = 2400/32 = 75$ slug, (10.26) yields $\mathbf{a}^*(\mathscr{B}, t) = \mathbf{F}(\mathscr{B}, t)/m(\mathscr{B}) = 600/75\mathbf{k} = 8\mathbf{k}$ ft/sec^2. Therefore, with $\mathbf{a}^*(\mathscr{B}_1, t) = \mathbf{a}^*(\mathscr{B}, t)$ in (10.31a), we obtain $\mathbf{N} = 500\mathbf{k}$ lb.

Solution of (ii). The bodies $\mathscr{B}$, $\mathscr{B}_1$, and $\mathscr{B}_2$ may be identified with those in the construction of the mutual action principle. The total force (10.9) acting on $\mathscr{B}$ in Fig. 10.3a is

$$\mathbf{F}(\mathscr{B}, t) = \mathbf{W}_1 + \mathbf{W}_2 + \mathbf{T}. \tag{10.31b}$$

Notice that $\mathbf{f}_1 = \mathbf{W}_1$ is that part of the total force (10.31b) that acts on the part $\mathscr{B}_1$, while $\mathbf{f}_2 = \mathbf{W}_2 + \mathbf{T}$ is that part of the total force (10.31b) that acts on the part $\mathscr{B}_2$. Hence, $\mathbf{F}(\mathscr{B}, t) = \mathbf{f}_1 + \mathbf{f}_2$ in (10.13) is equivalent to (10.31b). We shall ignore the mutual body force between $\mathscr{B}_1$ and $\mathscr{B}_2$. Then $\mathbf{N} = \mathbf{b}_{12}$ is the mutual contact force exerted on $\mathscr{B}_1$ by $\mathscr{B}_2$; and in accordance with the law of mutual action, we wish to demonstrate that $\mathbf{b}_{21} = -\mathbf{b}_{12} = -\mathbf{N}$ for the mutual contact force exerted on $\mathscr{B}_2$ by $\mathscr{B}_1$. (The reader should now draw the free body diagrams suggested in Fig. 10.2. for the problem in Fig. 10.3a.) Therefore, with the aid of (10.15) and the foregoing identification of terms, the total force (10.14) on the free body $\mathscr{B}_1$, the crate alone, is

$$\hat{\mathbf{F}}_1 = \mathbf{W}_1 + \mathbf{N} = \frac{d\mathbf{p}_1}{dt}, \tag{10.31c}$$

and on $\mathscr{B}_2$, the elevator alone, is

$$\hat{\mathbf{F}}_2 = \mathbf{W}_2 + \mathbf{T} + \mathbf{b}_{21} = \frac{d\mathbf{p}_2}{dt}. \tag{10.31d}$$

Adding (10.31c) and (10.31d) and noting (10.31b), we reach

$$\hat{\mathbf{F}}_1 + \hat{\mathbf{F}}_2 = \mathbf{F}(\mathscr{B}, t) + \mathbf{N} + \mathbf{b}_{21} = \frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2). \tag{10.31e}$$

However, by (5.11), $\mathbf{p}(\mathscr{B}, t) \equiv \mathbf{p}(\mathscr{B}_1 \cup \mathscr{B}_2, t) = \mathbf{p}(\mathscr{B}_1, t) + \mathbf{p}(\mathscr{B}_2, t)$, and hence the far right-hand side of (10.31e) is the total force $\mathbf{F}(\mathscr{B}, t)$ on $\mathscr{B}$. Therefore, $\mathbf{N} + \mathbf{b}_{21} = \mathbf{0}$, i.e. $\mathbf{b}_{21} = -\mathbf{N}$.

In conclusion, note also that $d\mathbf{p}(\mathscr{B}, t)/dt = d\mathbf{p}(\mathscr{B}_1, t)/dt + d\mathbf{p}(\mathscr{B}_2, t)/dt = m(\mathscr{B}_1)\mathbf{a}^*(\mathscr{B}_1, t) + m(\mathscr{B}_2)\mathbf{a}^*(\mathscr{B}_2, t)$. For the current case, since $\mathbf{a}^*(\mathscr{B}_1, t) = \mathbf{a}^*(\mathscr{B}_2, t) = \mathbf{a}^*(\mathscr{B}, t)$ and $m(\mathscr{B}) = m(\mathscr{B}_1) + m(\mathscr{B}_2)$, we confirm that $\mathbf{F}(\mathscr{B}, t) = d\mathbf{p}(\mathscr{B}, t)/dt = m(\mathscr{B})\mathbf{a}^*(\mathscr{B}, t)$.

This example illustrates in specific terms the analysis used in the construction of the mutual action principle for bodies. Obviously, there is no need to repeat these details in each problem solution. □

Example 10.2. A 2 ft diameter log weighing 3220 lb is moved steadily on a large conveyor belt shown in Fig. 10.4. At shutdown the belt speed decreases at the rate of 2 ft/sec^2, and the log is observed to roll without slipping. At an instant of interest t_o, the log has an angular speed ω which is increasing at the rate $\dot{\omega} = 1$ rad/sec^2 relative to the belt, and $g = 32.2$ ft/sec^2. Find the total force acting on the log at the instant t_o.

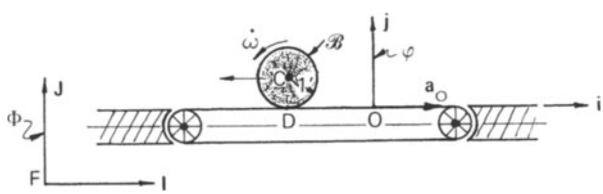


Figure 10.4. Motion of a rigid log on a conveyor belt.

Solution. The total force acting on the log is determined by (10.26). The mass of the log is $m(\mathscr{B}) = 3220/g = 100$ slug; therefore,

$$\mathbf{F}(\mathscr{B}, t) = 100\mathbf{a}^*(\mathscr{B}, t). \tag{10.32a}$$

We choose a reference frame φ fixed in the belt, as shown in Fig. 10.4. Since there is no rotation of φ, the absolute acceleration of the center of mass of $\mathscr{B}$ in the ground frame Φ is determined by

$$\mathbf{a}^* \equiv \mathbf{a}_{CF} = \mathbf{a}_{CO} + \mathbf{a}_{OF}, \tag{10.32b}$$

wherein $\mathbf{a}_{OF} = -2\mathbf{i}$ ft/sec^2. Since the log rolls on the belt without slipping, with $\boldsymbol{\omega} = \omega\mathbf{k}$ and $\mathbf{x} = 1\mathbf{j}$ ft, we find $\mathbf{v}_{CO} = \mathbf{v}_{CD} = \boldsymbol{\omega} \times \mathbf{x} = -\omega\mathbf{i}$ for all t. Therefore, at the moment of interest, $\mathbf{a}_{CO} = -\dot{\omega}\mathbf{i} = -1\mathbf{i}$ ft/sec^2 relative to the belt frame, and (10.32b) yields $\mathbf{a}^* = -1\mathbf{i} - 2\mathbf{i} = -3\mathbf{i}$ ft/sec^2. Hence, by (10.32a), the total force acting on the log at the time t_o is $\mathbf{F}(\mathscr{B}, t) = -300\mathbf{i}$ lb. □

10.5. Moment of Momentum Transformation Equations

We now turn to Euler's second law in (10.3) and recall that this principle holds only for a point O fixed in the inertial frame Φ. Our target in Section 10.6 is to derive a similar form of Euler's second law valid for a moving reference point and to determine any restrictions on its use. With this objective in mind, two transformation rules for the moment of momentum of a body are derived—one is a point transformation relation, the other a velocity transformation relation. The results obtained do not require that the body be rigid.

10.5.1. Point Transformation Rule for the Moment of Momentum

The moment of momentum of a body about any point O in an inertial frame Φ is given by (5.33). If another reference point is used, say Q shown in Fig. 10.5, the moment of momentum about Q is

$$\mathbf{h}_Q(\mathscr{B}, t) = \int_{\mathscr{B}} \mathbf{r}(P, t) \times \mathbf{v}(P, t)\, dm(P), \tag{10.33}$$

in which $\mathbf{r}(P, t)$ is the position vector of the particle P from Q and $\mathbf{v}(P, t)$ is the velocity of P in Φ. The first rule relating $\mathbf{h}_O$ and $\mathbf{h}_Q$ is obtained from (5.33) by use of the point transformation for $\mathbf{x}_O(P, t) \equiv \mathbf{x}(P, t) = \mathbf{B}(Q, t) + \mathbf{r}(P, t)$ evident in Fig. 10.5. With the aid of (5.16) and (10.33), this leads to the *general point transformation relation for the moment of momentum vector*:

$$\mathbf{h}_O(\mathscr{B}, t) = \mathbf{h}_Q(\mathscr{B}, t) + \mathbf{B}(Q, t) \times \mathbf{p}^*(\mathscr{B}, t). \tag{10.34}$$

With $\mathbf{B}(Q, t) \equiv \mathbf{r}_{OQ}(t)$, (10.34) is similar to the point transformation rule for a particle given below (5.31). See also the general rule (5.19).

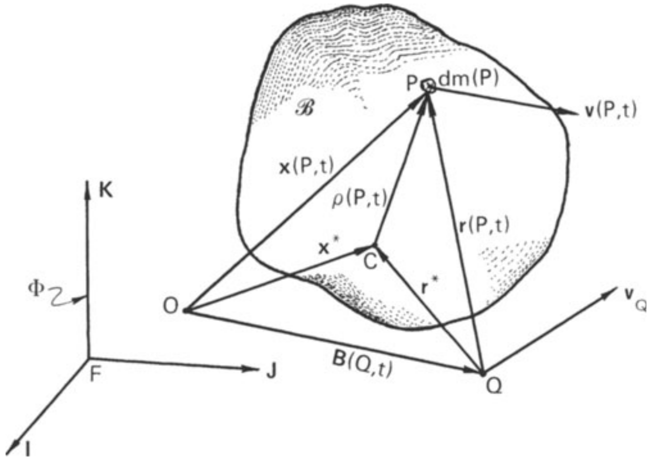


Figure 10.5. Schema for a change of reference point for the moment of momentum.

In the special case when point Q is the center of mass C of $\mathscr{B}$, we have $\mathbf{B}(Q,t) = \mathbf{x}^*(\mathscr{B},t)$ and $\mathbf{r}(P,t) = \boldsymbol{\rho}(P,t)$, where $\boldsymbol{\rho}(P,t)$ is the position vector of P from C in Fig. 10.5; and hence (10.34) becomes

$$\mathbf{h}_O(\mathscr{B},t) = \mathbf{h}_O^*(\mathscr{B},t) + \mathbf{h}_C(\mathscr{B},t), \tag{10.35}$$

wherein $\mathbf{h}_O^*(\mathscr{B},t) \equiv \mathbf{x}^*(\mathscr{B},t) \times \mathbf{p}^*(\mathscr{B},t)$ is the moment about O of the momentum of the center of mass of $\mathscr{B}$ in Φ, and

$$\mathbf{h}_C(\mathscr{B},t) \equiv \int_{\mathscr{B}} \boldsymbol{\rho}(P,t) \times \mathbf{v}(P,t)\, dm(P), \tag{10.36}$$

is the moment about C of the momentum of $\mathscr{B}$ in Φ. It follows from both (10.34) and (10.35) that *if the center of mass is fixed in Φ, the moment of momentum of the body about every point O in Φ is the same as the moment of momentum about its fixed center of mass, i.e.* $\mathbf{h}_O(\mathscr{B},t) = \mathbf{h}_C(\mathscr{B},t)$, *and conversely.*

10.5.2. Velocity Transformation Rule for the Moment of Momentum

The moment about any point Q of the momentum of $\mathscr{B}$ relative to Q, called briefly the *moment of momentum relative to Q*, is defined by

$$\mathbf{h}_{rQ}(\mathscr{B},t) \equiv \int_{\mathscr{B}} \mathbf{r}(P,t) \times \dot{\mathbf{r}}(P,t)\, dm(P), \tag{10.37}$$

in which $\mathbf{r}(P,t)$ in Fig. 10.5 is the position vector of P from Q and $\dot{\mathbf{r}}(P,t)$ is the velocity of P relative to Q. Notice that the absolute velocity of P appears in (10.33). Thus, to relate $\mathbf{h}_Q$ and $\mathbf{h}_{rQ}$, introduce the velocity transformation

$\mathbf{v}(P, t) = \mathbf{v}_Q + \dot{\mathbf{r}}(P, t)$ in (10.33), expand the result, and recall (10.37) and (5.12) applied to $\mathbf{r}$. This yields *the general rule relating the moment of momentum about a point Q to the moment of momentum relative to Q*:

$$\mathbf{h}_Q(\mathscr{B}, t) = \mathbf{h}_{rQ}(\mathscr{B}, t) + m(\mathscr{B})\mathbf{r}^*(\mathscr{B}, t) \times \mathbf{v}_Q, \tag{10.38}$$

wherein $\mathbf{r}^*$ *is the position of the center of mass from Q*. (See Fig. 10.5.) This is similar to the rule (8.24) for a system of particles.

It follows from (10.38) that $\mathbf{h}_Q(\mathscr{B}, t) = \mathbf{h}_{rQ}(\mathscr{B}, t)$, if and only if $m\mathbf{r}^* \times \mathbf{v}_Q = \mathbf{0}$ for all t. This condition may be satisfied (i) trivially, if Q is a fixed point so that $\mathbf{v}_Q = \mathbf{0}$, or (ii) when $\mathbf{v}_Q$ is parallel to $\mathbf{r}^*$ so that $\mathbf{v}_Q = k\mathbf{r}^*$ for constant k, that is, the point Q is moving on a line through the center of mass, or (iii) if point Q is the center of mass C, in which case $\mathbf{r}^* = \mathbf{0}$ and (10.38) reduces to $\mathbf{h}_C(\mathscr{B}, t) = \mathbf{h}_{rC}(\mathscr{B}, t)$, the moment of momentum relative to C. *Consequently, the moment about C of the momentum of a body in the inertial frame Φ is equal to its moment of momentum relative to C in Φ; that is,*

$$\mathbf{h}_C(\mathscr{B}, t) = \int_{\mathscr{B}} \boldsymbol{\rho}(P, t) \times \dot{\boldsymbol{\rho}}(P, t)\, dm(P) = \mathbf{h}_{rC}(\mathscr{B}, t), \tag{10.39}$$

where $\boldsymbol{\rho}(P, t)$ is the position vector of P from C and $\dot{\boldsymbol{\rho}}(P, t) = \mathbf{v}_P - \mathbf{v}_C$ is the velocity of P relative to C. This is parallel to the rule (8.22) for a system of particles.

In view of (10.39), we may rewrite (10.35):

$$\mathbf{h}_O(\mathscr{B}, t) = \mathbf{h}_O^*(\mathscr{B}, t) + \mathbf{h}_{rC}(\mathscr{B}, t). \tag{10.40}$$

That is, *the total moment of momentum of a body about a point O in an inertial reference frame is equal to the moment about O of the momentum of the center of mass plus the moment of momentum relative to the center of mass.* This is the same rule recorded in (8.27) for a system of particles.

The foregoing concepts for the moment of momentum $\mathbf{h}_Q$ about a point Q and the moment of momentum $\mathbf{h}_{rQ}$ relative to Q applied to a simple rigid body in motion in an inertial reference frame are illustrated in an example.

Example 10.3. A connecting rod QR of a simple machine shown in Fig. 10.6 is modeled as a homogeneous thin rod of length ℓ, uniform cross section, and mass σ per unit length. The rod is hinged at Q at a distance α from the center O of a flywheel that turns with a constant angular velocity $\boldsymbol{\Omega} = \Omega\mathbf{k}$, as shown, in the inertial ground frame $0 = \{O; \mathbf{I}, \mathbf{J}, \mathbf{k}\}$ fixed at O. Determine (i) the moment of momentum of the rod relative to Q and (ii) its moment of momentum about Q.

Solution of (i). The moment of momentum $\mathbf{h}_{rQ}$ of the rod relative to Q is determined by (10.37), in which the rigid body velocity of a rod particle P relative to Q in the ground frame $0 = \{O; \mathbf{I}, \mathbf{J}, \mathbf{k}\}$ is given by

$$\dot{\mathbf{r}}(P, t) = \mathbf{v}(P, t) - \mathbf{v}_Q = \boldsymbol{\omega} \times \mathbf{r}(P, t), \tag{10.41a}$$

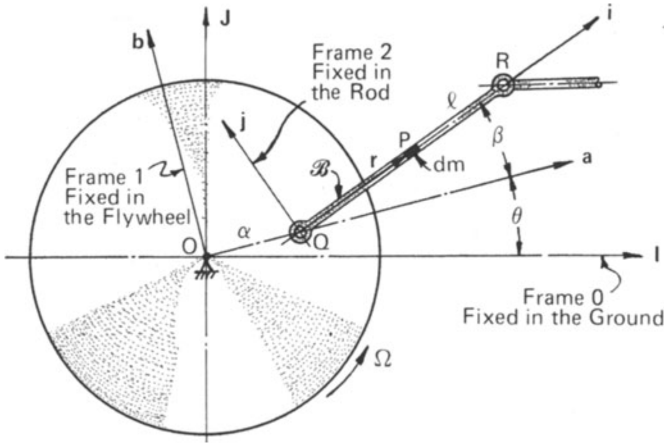


Figure 10.6. Moment of momentum about a moving point Q in a simple machine.

where $\mathbf{r}(P,t) = r\mathbf{i}$, referred to the rod frame $2 = \{Q; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, and the total angular velocity of the rod frame 2 in the ground frame 0 is $\boldsymbol{\omega} \equiv \boldsymbol{\omega}_{20}$. Let $\boldsymbol{\omega}_{21} = \dot{\beta}\mathbf{k}$ denote the angular velocity of the rod frame 2 relative to the flywheel frame $1 = \{O; \mathbf{a}, \mathbf{b}, \mathbf{k}\}$ whose angular velocity is $\boldsymbol{\omega}_{10} = \boldsymbol{\Omega}$ relative to frame 0. Then $\boldsymbol{\omega} = \boldsymbol{\omega}_{21} + \boldsymbol{\omega}_{10}$, that is,

$$\boldsymbol{\omega} = (\dot{\beta} + \Omega)\mathbf{k}, \tag{10.41b}$$

and (10.41a) gives the relative rigid body velocity $\dot{\mathbf{r}}(P,t) = r(\dot{\beta} + \Omega)\mathbf{j}$. Therefore, with $dm = \sigma dr$, where dr is the elemental length of the rod, (10.37) yields

$$\mathbf{h}_{rQ}(\mathscr{B}, t) = \sigma(\dot{\beta} + \Omega)\mathbf{k} \int_0^{\ell} r^2 dr = \sigma(\dot{\beta} + \Omega)\frac{\ell^3}{3}\mathbf{k}. \tag{10.41c}$$

The total mass of the rod is $m(\mathscr{B}) = \sigma\ell$, and hence the moment of momentum of the rod relative to Q is

$$\mathbf{h}_{rQ}(\mathscr{B}, t) = \frac{m\ell^2}{3}(\dot{\beta} + \Omega)\mathbf{k}. \tag{10.41d}$$

Solution of (ii). Now consider the moment of momentum $\mathbf{h}_Q$ of the rod about Q defined by (10.33) in which $\mathbf{v}_Q = \boldsymbol{\Omega} \times \mathbf{x} = \Omega\mathbf{k} \times \alpha\mathbf{a}$ yields

$$\mathbf{v}_Q = \alpha\Omega\mathbf{b}, \tag{10.41e}$$

referred to frame 1. Use of (10.41e) in (10.41a) gives $\mathbf{v}(P,t) = \alpha\Omega(\sin\beta\mathbf{i} + \cos\beta\mathbf{j}) + r(\dot{\beta} + \Omega)\mathbf{j}$, the velocity of P in frame 0 but referred to the rod frame 2;

and with $\mathbf{r}(P, t) = r\mathbf{i}$, (10.33) becomes

$$\mathbf{h}_Q(\mathscr{B}, t) = \int_0^\ell \sigma[\alpha\Omega r \cos\beta + r^2(\dot{\beta} + \Omega)]dr\mathbf{k}. \tag{10.41f}$$

An easy integration yields

$$\mathbf{h}_Q(\mathscr{B}, t) = \left[\frac{m\ell}{2}\alpha\Omega\cos\beta + \frac{m\ell^2}{3}(\dot{\beta} + \Omega)\right]\mathbf{k}. \tag{10.41g}$$

See Problem 10.4. □

Exercise 10.1. Begin with (10.38) and derive (10.41g). □

Although use of the relation (10.38) is demonstrated in the exercise and appears in an application below, it is enough that the reader focus only on the definitions in (10.33) and (10.37). Observe that (10.33) involves the *absolute velocity* of a material point in the inertial frame, while (10.37) entails use of the *relative velocity* of that point in the inertial frame.

10.6. The Second Law of Motion for a Moving Reference Point

The transformation rules (10.34) and (10.38) for the moment of momentum vector are applied here to derive two forms of Euler's second law for a moving reference point, one relating $\mathbf{M}_Q$ to $\dot{\mathbf{h}}_Q$, the other to $\dot{\mathbf{h}}_{rQ}$. The objective is to identify *all* reference points Q for which Euler's second law retains the same basic form (10.3) valid for a point O fixed in the inertial frame.

10.6.1. The First Form of Euler's Law for a Moving Point

Let O be a point fixed in Φ and let Q be any other point in motion with velocity $\mathbf{v}_Q = \dot{\mathbf{B}}(Q, t)$ in Φ, as shown in Fig. 10.5. Then, with Euler's law (10.3) for the fixed point O in mind, we differentiate (10.34) with respect to time t and recall (10.11) to obtain

$$\mathbf{M}_O(\mathscr{B}, t) = \dot{\mathbf{h}}_O(\mathscr{B}, t) = \dot{\mathbf{h}}_Q(\mathscr{B}, t) + \mathbf{v}_Q \times \mathbf{p}^*(\mathscr{B}, t) + \mathbf{B}(Q, t) \times \mathbf{F}(\mathscr{B}, t).$$

With $\mathbf{r}_{OQ} = \mathbf{B}(Q, t)$ in (5.24), we obtain *the first form of Euler's second law of motion for an arbitrary moving point Q*:

$$\mathbf{M}_Q(\mathscr{B}, t) = \dot{\mathbf{h}}_Q(\mathscr{B}, t) + \mathbf{v}_Q \times \mathbf{p}^*(\mathscr{B}, t). \tag{10.42}$$

Notice that this relation is similar to the rule (6.80) for a particle. It follows from (10.42) that Euler's second law in the form (10.3), namely,

$$\mathbf{M}_Q(\mathscr{B}, t) = \dot{\mathbf{h}}_Q(\mathscr{B}, t), \tag{10.43}$$

holds for the reference point Q, if and only if $\mathbf{v}_Q \times \mathbf{p}^* = \mathbf{0}$ for all time t. This is satisfied

(i) trivially, if either Q or the center of mass is fixed in Φ, or
(ii) when $\mathbf{v}_Q$ is parallel to $\mathbf{p}^*$, hence to the velocity $\mathbf{v}^*$, or
(iii) if point Q is the center of mass, for then $\mathbf{v}_Q = \mathbf{p}^*/m$.

Case (iii) thus provides the following important general rule. In addition to a fixed point in the inertial frame, Euler's second law holds also with respect to the moving center of mass:

$$\mathbf{M}_C(\mathscr{B}, t) = \dot{\mathbf{h}}_C(\mathscr{B}, t). \tag{10.44}$$

The reader may confirm in a similar way that (10.44) also follows directly from the special rule (10.35).

10.6.2. The Second Form of Euler's Law for a Moving Point

It follows from (10.39) that (10.44) also may be written as $\mathbf{M}_C = \dot{\mathbf{h}}_{rC}$. This rule, however, derives from a more general equation relating $\mathbf{M}_Q$ to $\mathbf{h}_{rQ}$, the moment of momentum relative to an arbitrary point Q, a formulation that leads to a somewhat different conclusion on the main result for the center of mass point.

We differentiate (10.38) with respect to time, observe from Fig. 10.5 that $m\dot{\mathbf{r}}^* = m\dot{\mathbf{x}}^* - m\mathbf{v}_Q = \mathbf{p}^* - m\mathbf{v}_Q$, and thus obtain

$$\dot{\mathbf{h}}_Q(\mathscr{B}, t) + \mathbf{v}_Q \times \mathbf{p}^*(\mathscr{B}, t) = \dot{\mathbf{h}}_{rQ}(\mathscr{B}, t) + \mathbf{r}^*(\mathscr{B}, t) \times m(\mathscr{B})\mathbf{a}_Q, \tag{10.45}$$

in which $\mathbf{a}_Q = \dot{\mathbf{v}}_Q$ is the acceleration of Q in Φ. Use of (10.42) in (10.45) leads to the *second form of Euler's second law for a moving reference point Q*:

$$\mathbf{M}_Q(\mathscr{B}, t) = \dot{\mathbf{h}}_{rQ}(\mathscr{B}, t) + \mathbf{r}^*(\mathscr{B}, t) \times m(\mathscr{B})\mathbf{a}_Q. \tag{10.46}$$

Hence, there exists a point Q with respect to which Euler's second law in the form (10.3), namely,

$$\mathbf{M}_Q(\mathscr{B}, t) = \dot{\mathbf{h}}_{rQ}(\mathscr{B}, t) \tag{10.47}$$

holds in an inertial reference frame, when and only when $\mathbf{r}^* \times m\mathbf{a}_Q = \mathbf{0}$ for all t. That is,

(i) trivially, when Q is fixed or has a uniform motion in Φ so that $\mathbf{a}_Q = \mathbf{0}$, or
(ii) if the acceleration of Q is directed through the center of mass so that $\mathbf{r}^*$ and $\mathbf{a}_Q$ are parallel vectors, or
(iii) if Q is the center of mass point so that $\mathbf{r}^* = \mathbf{0}$.

Case (i) shows that Euler's second law holds in all uniformly translating, nonrotating frames, i.e. in all inertial frames in which Q is fixed or has a uniform motion. Case (ii) is the least general of the possible cases. It applies, for example, to the point of contact Q of a homogeneous wheel that rolls without slipping on a fixed surface. In this special event, the acceleration, as shown in Chapter 2, page 109, is directed through the mass center of the wheel. It is wrong, however, to assume that Euler's law necessarily holds for a contact point of rolling without slip. (See Problem 10.39 illustrating a case where this fails; here $\mathbf{v}_Q = \mathbf{0}$, while $\mathbf{a}_Q$ is not directed through the center of mass. In general, therefore, use of an instantaneous center for which $\mathbf{v}_Q = \mathbf{0}$ as a reference point in application of Euler's second law is not recommended.) In the most general and nontrivial Case (iii), (10.47) becomes

$$\mathbf{M}_C(\mathscr{B}, t) = \dot{\mathbf{h}}_{rC}(\mathscr{B}, t). \qquad (10.48)$$

The reader may confirm in a similar way that (10.48) also follows from the special rule (10.40). In fact, by (10.39) and (10.44), $\mathbf{M}_C(\mathscr{B}, t) = \dot{\mathbf{h}}_C(\mathscr{B}, t) = \dot{\mathbf{h}}_{rC}(\mathscr{B}, t)$, which is (10.48).

Suppose C is fixed and Q is any other point in the inertial frame. Then, by (10.35), $\mathbf{h}_Q = \mathbf{h}_C$ for all time; and by (10.43) and (10.44), evident also from (5.25), $\mathbf{M}_Q = \dot{\mathbf{h}}_Q = \dot{\mathbf{h}}_C = \mathbf{M}_C$ for an arbitrary point Q, where $\dot{\mathbf{h}}_Q(\mathscr{B}, t) = \dot{\mathbf{h}}_{rQ}(\mathscr{B}, t) + \mathbf{r}^*(\mathscr{B}, t) \times m(\mathscr{B})\mathbf{a}_Q$, by (10.45). (See Problems 10.24 and 10.25.) Because Euler's laws involve only external forces and external torques acting on a body, henceforward we shall refer to these briefly as forces and torques.

Finally, it is important to bear in mind that the moment of momentum vector $\mathbf{h}_Q$ or $\mathbf{h}_{rQ}$ may be referred to a moving frame φ having an angular velocity $\boldsymbol{\omega}_f$ relative to Φ. Therefore, recalling the rule (4.11) for the time derivative of a vector referred to a moving frame, Euler's second law of motion (10.47) valid for a fixed or an appropriate moving reference point Q characterized above, becomes

$$\mathbf{M}_Q(\mathscr{B}, t) = \frac{d\mathbf{h}_{rQ}(\mathscr{B}, t)}{dt} = \frac{\delta \mathbf{h}_{rQ}(\mathscr{B}, t)}{\delta t} + \boldsymbol{\omega}_f \times \mathbf{h}_{rQ}(\mathscr{B}, t). \qquad (10.49)$$

The same derivative rule applies to $\dot{\mathbf{h}}_Q$ in (10.43), and also to $\dot{\mathbf{h}}_C = \dot{\mathbf{h}}_{rC}$ in either (10.44) or (10.48). We shall return to this major rule later. We conclude with an illustration involving (10.42), (10.46), and (10.48) applied to a rigid rod.

Example 10.4. Use the results for Example 10.3, page 424, to determine (i) the total torque about the hinge Q, and (ii) the total torque about the center of mass C, required to sustain the motion of the thin rod.

Solution of (i). The total torque about the moving point Q in Fig. 10.6 can be found from either (10.42) or (10.46). Let us consider (10.46), note that

$$\mathbf{r}^*(\mathscr{B}, t) \times m\mathbf{a}_Q = -\frac{\ell}{2}\mathbf{i} \times m\alpha\Omega^2\mathbf{a} = \frac{m\alpha\ell}{2}\Omega^2 \sin\beta\mathbf{k}, \qquad (10.50a)$$

and recall $\mathbf{h}_{rQ} = (m\ell^2/3)(\dot\beta + \Omega)\mathbf{k}$ from (10.41d), which is a vector referred to the moving rod frame 2 whose total angular velocity is $\boldsymbol{\omega}_f = (\Omega + \dot\beta)\mathbf{k}$ in (10.41b). Because $\boldsymbol{\omega}_f$ is parallel to $\mathbf{h}_{rQ}$, (10.49) simplifies to

$$\dot{\mathbf{h}}_{rQ}(\mathscr{B}, t) = \frac{\delta \mathbf{h}_{rQ}(\mathscr{B}, t)}{\delta t} = \frac{m\ell^2}{3}\ddot\beta\mathbf{k}. \tag{10.50b}$$

Hence, use of (10.50a) and (10.50b) in (10.46) delivers the total torque about Q required to sustain the motion of the connecting rod:

$$\mathbf{M}_Q(\mathscr{B}, t) = \left(\frac{m\ell^2}{3}\ddot\beta + \frac{m\alpha\ell}{2}\Omega^2 \sin\beta\right)\mathbf{k}. \tag{10.50c}$$

The reader may show that (10.42) delivers the same result.

Solution of (ii). Now let us consider Euler's simple rule (10.48) for the moving center of mass and recall the definition (10.39) in which $\boldsymbol{\rho}(P, t) = \rho\mathbf{i}$ denotes the position vector of a material point P at ρ from the center of mass C. Then, with the aid of (10.41b), the relative rigid body velocity of P is $\dot{\boldsymbol{\rho}}(P, t) = \boldsymbol{\omega} \times \boldsymbol{\rho} = \rho(\dot\beta + \Omega)\mathbf{j}$, $dm = \sigma d\rho$, and by (10.39), the moment of momentum relative C may be written as

$$\mathbf{h}_{rC} = \sigma(\dot\beta + \Omega)\int_{-\ell/2}^{\ell/2} \rho^2 d\rho\mathbf{k} = m(\dot\beta + \Omega)\frac{\ell^2}{12}\mathbf{k}. \tag{10.50d}$$

This vector is parallel to $\boldsymbol{\omega}_f = \boldsymbol{\omega}$, and hence (10.48) and (10.49) yield the total torque about C needed to sustain the rod's motion in the machine frame, i.e.

$$\mathbf{M}_C(\mathscr{B}, t) = \dot{\mathbf{h}}_{rC}(\mathscr{B}, t) = \frac{m\ell^2}{12}\ddot\beta(t)\mathbf{k}. \tag{10.50e}$$

□

Exercise 10.2. Show that the required torque about the hinge Q is given by $\mathbf{M}_Q(\mathscr{B}, t) = \mathbf{M}_C(\mathscr{B}, t) + \mathbf{x}_Q^*(\mathscr{B}, t) \times m(\mathscr{B})\mathbf{a}^*(\mathscr{B}, t)$, where $\mathbf{x}_Q^*(\mathscr{B}, t) = (\ell/2)\mathbf{i}$ is the vector of C from Q, and thus derive (10.50c) with the aid of (10.50e). What is the total force $\mathbf{F}(\mathscr{B}, t)$ acting to sustain the motion of the connecting rod?

□

10.7. Euler's Second Principle of Motion and Related First Integrals

The general form of Euler's second law that has the same classical form as (10.3) and holds for an appropriate moving reference point in an inertial reference frame is best described by (10.47), abbreviated as follows.

Euler's second principle of motion: *Let Q be a reference point which is either fixed, has uniform motion, or coincides with the moving center of mass of a body $\mathscr{B}$ in an inertial reference frame Φ. Then the total torque acting on $\mathscr{B}$ about Q is equal to the time rate of change of its moment of momentum relative to Q in Φ*:

$$\mathbf{M}_Q(\mathscr{B}, t) = \dot{\mathbf{h}}_{rQ}(\mathscr{B}, t). \tag{10.51}$$

If the point Q does not satisfy these or other less relevant conditions identified earlier, then either (10.42) or (10.46) must be used; fortunately, this is seldom necessary. Two familiar basic principles follow immediately as first integrals of (10.51).

Principle of conservation of moment of momentum of a body: *The total torque about a fixed direction* $\mathbf{e}$ *in the inertial frame Φ at Q vanishes when and only when the corresponding component of the moment of the momentum relative to Q is constant in Φ*:

$$\mathbf{M}_Q(\mathscr{B}, t) \cdot \mathbf{e} = 0 \iff \mathbf{h}_{rQ}(\mathscr{B}, t) \cdot \mathbf{e} = \text{const.} \tag{10.52}$$

Moreover, the total torque about Q vanishes if and only if the moment of momentum relative to Q is a constant vector in Φ.

The torque–impulse principle for a body: *The impulse*

$$\mathscr{M}_Q(t; t_0) \equiv \int_{t_0}^{t} \mathbf{M}_Q(\mathscr{B}, t)\, dt, \tag{10.53}$$

of the total torque acting on a body about a point Q over the interval $[t_0, t]$ is equal to the change in moment of momentum of the body relative to Q, during that time:

$$\mathscr{M}_Q(t; t_0) = \Delta\mathbf{h}_{rQ}(\mathscr{B}, t). \tag{10.54}$$

An instantaneous torque–impulse for a body is defined parallel to (7.17) for a particle, namely,

$$\mathscr{M}^*_Q \equiv \operatorname*{limit}_{t \to t_0} \mathscr{M}_Q(t; t_0). \tag{10.55}$$

Therefore, the instantaneous torque–impulse about point Q is equal to the instantaneous change in the moment of momentum relative to Q. In an instantaneous torque–impulse there is an instantaneous change in the moment of momentum relative to Q, but no change in the body's spatial configuration at the impulsive instant. Of course, in the limit as $\Delta t \to 0$, *finite* torques, like those due to the weight of a body, the action of a spring, or a harmonic time varying force with constant amplitude, for example, do not contribute to the instantaneous torque–impulse in (10.55).

Moreover, by an argument parallel to that leading to (7.18), it can be shown that *the instantaneous torque–impulse due to a suddenly applied force* $\mathbf{F}$ *acting on a body is equal the moment about* Q *of the instantaneous impulse due to* $\mathbf{F}$, *that is,*

$$\mathscr{M}_Q^* = \mathbf{x}_Q \times \mathscr{I}^*. \tag{10.56}$$

Here $\mathbf{x}_Q$ is the instantaneous position vector from Q to the point of application of the instantaneous impulse $\mathscr{I}^*$ defined in (10.30). See Problems 10.22, 10.23, and 10.25 through 10.31.

Except for several previous illustrative examples involving the plane motion of a simple rigid rod, none of the results requires that the body be rigid; Euler's principles and first integrals hold for any sort of body. And Euler's moment of momentum principle for a moving reference point has its simplest general form with respect to the center of mass. Nevertheless, this rule generally is not useful for a deformable body, because the center of mass will vary with the extent and nature of the deformation, and also because of other considerations beyond our concern here. A rigid body, on the other hand, does not suffer this difficulty, and, from now on, our studies focus on Euler's theory for the general motion of a rigid body.

10.8. Moment of Momentum of a Rigid Body

We now develop the general equation for the moment of momentum of a rigid body. The velocity of any point P of a rigid body relative to a base point Q, i.e. another *body point*, whose velocity is $\mathbf{v}_Q(t)$ in the inertial frame Φ, in the notation of Fig. 10.5, page 423, is given by $\dot{\mathbf{r}}(P,t) \equiv \mathbf{v}_P(t) - \mathbf{v}_Q(t) = \boldsymbol{\omega}(t) \times \mathbf{r}(P,t)$, wherein $\mathbf{v}_P(t)$ is the total velocity of P in Φ, $\mathbf{r}(P,t)$ is the position vector of P from Q, and $\boldsymbol{\omega}(t)$ is the body's total angular velocity in Φ. Hence, by (10.37), the rigid body moment of momentum relative to Q is given by

$$\mathbf{h}_{rQ} = \int_{\mathscr{B}} \mathbf{r}(P,t) \times [\boldsymbol{\omega}(t) \times \mathbf{r}(P,t)]\, dm(P). \tag{10.57}$$

We expand the integrand, recall (3.26) and recast (10.57) in the tensorial form

$$\mathbf{h}_{rQ} = \int_{\mathscr{B}} [(\mathbf{r}\cdot\mathbf{r})\mathbf{1} - \mathbf{r}\otimes\mathbf{r}]\boldsymbol{\omega}\, dm. \tag{10.58}$$

Because $\boldsymbol{\omega}(t)$ depends only on time, it can be removed from the integral. Then, recalling (9.10) in which $\mathbf{x}$ is replaced with $\mathbf{r}$, we obtain the following major result.

Moment of momentum of a rigid body: *The moment of momentum of a rigid body* $\mathscr{B}$ *relative to any specified body point* Q *is a linear transformation of* *the angular velocity of the body*:

$$\mathbf{h}_{rQ}(\mathscr{B}, t) = \mathbf{I}_Q(\mathscr{B}, t)\boldsymbol{\omega}(t), \tag{10.59}$$

where, with $\mathbf{r} = \mathbf{r}(P, t)$, *the position vector of* P *from* Q,

$$\mathbf{I}_Q(\mathscr{B}, t) \equiv \int_{\mathscr{B}} [(\mathbf{r} \cdot \mathbf{r})\mathbf{1} - \mathbf{r} \otimes \mathbf{r}]\, dm, \tag{10.60}$$

is the moment of inertia tensor relative to Q.

With the aid of (9.11) and (3.21), (10.59) may be written in its Cartesian component form as $\mathbf{h}_{rQ} = (I^Q_{jk}\mathbf{e}_j \otimes \mathbf{e}_k)\boldsymbol{\omega} = I^Q_{jk}\mathbf{e}_j(\mathbf{e}_k \cdot \boldsymbol{\omega})$ referred to a reference frame $\psi = \{Q; \mathbf{e}_k\}$. For convenience, the subscripts r and Q are here written temporarily as superscripts. Therefore, $\mathbf{h}_{rQ} = h^{rQ}_j \mathbf{e}_j = I^Q_{jk}\omega_k \mathbf{e}_j$, with scalar components $h^{rQ}_j = I^Q_{jk}\omega_k$, repeated indices indicating summation in the right-hand side of these relations. Thus, in expanded notation, (10.59) becomes

$$\begin{aligned} \mathbf{h}_{rQ} = \mathbf{I}_Q\boldsymbol{\omega} = &\left(I^Q_{11}\omega_1 + I^Q_{12}\omega_2 + I^Q_{13}\omega_3\right)\mathbf{e}_1 \\ &+ \left(I^Q_{21}\omega_1 + I^Q_{22}\omega_2 + I^Q_{23}\omega_3\right)\mathbf{e}_2 \\ &+ \left(I^Q_{31}\omega_1 + I^Q_{32}\omega_2 + I^Q_{33}\omega_3\right)\mathbf{e}_3. \end{aligned} \tag{10.61}$$

Here we recall the symmetry of the moment of inertia tensor: $I^Q_{jk} = I^Q_{kj}$. The same result can be expressed in the matrix form $[\mathbf{h}_{rQ}] = [\mathbf{I}_Q][\boldsymbol{\omega}]$ referred to the orthonormal basis $\mathbf{e}_k$ at the base point Q.

In general, the time dependence of $\mathbf{I}_Q(\mathscr{B}, t)$ in (10.60) is determined by the choice of the reference frame to which the motion is referred. The importance of our choosing a suitable body reference frame at Q to remove this time dependence was emphasized earlier. We shall say more about this in a moment. Also, at each base point Q there exists a principal vector basis $\hat{\mathbf{e}}_k$, say, of a body reference frame $\hat{\varphi} = \{Q; \hat{\mathbf{e}}_k\}$ with respect to which the products of inertia vanish. Thus, (10.61) simplifies to the important result that *the rigid body moment of momentum relative to* Q, *referred to a principal body reference frame* $\hat{\varphi} = \{Q; \hat{\mathbf{e}}_k\}$, *is determined by*

$$\mathbf{h}_{rQ} = \mathbf{I}_Q\boldsymbol{\omega} = \hat{I}^Q_{11}\hat{\omega}_1\hat{\mathbf{e}}_1 + \hat{I}^Q_{22}\hat{\omega}_2\hat{\mathbf{e}}_2 + \hat{I}^Q_{33}\hat{\omega}_3\hat{\mathbf{e}}_3. \tag{10.62}$$

Here $\hat{I}^Q_{ij}$ *and* $\hat{\omega}_k$ *are the principal scalar components of* $\mathbf{I}_Q$ *and* $\boldsymbol{\omega}$ *in* $\hat{\varphi}$.

In the principal basis only the body's geometry and mass distribution determine $\mathbf{I}_Q$, and hence also the principal directions $\hat{\mathbf{e}}_k$, both being independent of time in $\hat{\varphi}$. Henceforward, in applications where it is clear that a particular principal reference system is used, the hat notation and the superscript Q may be discarded to simplify expressions like (10.62).

Example 10.5. Apply the foregoing results to find the moment of momentum relative to Q for the connecting rod of the simple machine in Fig. 10.6, page 425.

Solution. The rod frame $2 = \{Q; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ in Fig. 10.6 is a principal reference frame for which $\hat{\mathbf{e}}_k \equiv \mathbf{i}_k$ and whose total angular velocity referred to frame 2 is given in (10.41b). Hence, $\omega_1 = \omega_2 = 0, \omega_3 = \dot{\beta} + \Omega$ are the principal components $\hat{\omega}_k \equiv \omega_k$. The principal components of the moment of inertia about the rod's end point Q, bearing in mind the basis directions, may be read from (9.46d) or from Fig. D.7 of Appendix D. For a uniform thin rod of mass m and length ℓ, however, it is instructive to show directly that $I_{22}^Q = I_{33}^Q = \int_{\mathscr{B}} r^2 dm = (m/\ell) \int_0^\ell r^2 dr$, so

$$I_{22}^Q = I_{33}^Q = \tfrac{1}{3} m\ell^2. \tag{10.63a}$$

Thus, (10.62) yields $\mathbf{h}_{rQ} = I_{33}^Q \omega_3 \mathbf{i}_3$, that is,

$$\mathbf{h}_{rQ} = \frac{m\ell^2}{3} \left(\dot{\beta} + \Omega\right) \mathbf{k}. \tag{10.63b}$$

This is the same as (10.41d) obtained earlier by direct integration. Let the reader show that the nontrivial principal moments of inertia about the center of mass are $I_{22}^C = I_{33}^C = m\ell^2/12$, and thus confirm that the moment of momentum relative to the center of mass is $\mathbf{h}_{rC} = \left(m\ell^2/12\right)\left(\dot{\beta} + \Omega\right) \mathbf{k}$, as shown differently in (10.50d). □

10.9. Euler's Equations of Motion for a Rigid Body

So far, the body point Q in (10.51) may be the origin of some moving reference frame $\varphi = \{Q; \mathbf{e}_k\}$ having angular velocity $\boldsymbol{\omega}_f$ relative to the inertial frame Φ. If $\mathbf{h}_{rQ}$ in (10.59) is then referred to φ, we need to use the time derivative rule (10.49) to write Euler's second law for a rigid body in the form

$$\mathbf{M}_Q(\mathscr{B}, t) = \frac{\delta \mathbf{I}_Q}{\delta t} \boldsymbol{\omega} + \mathbf{I}_Q \frac{\delta \boldsymbol{\omega}}{\delta t} + \boldsymbol{\omega}_f \times \mathbf{I}_Q \boldsymbol{\omega}. \tag{10.64}$$

In general, $\mathbf{I}_Q(\mathscr{B}, t)$ will vary when the geometry is referred to the moving frame φ, an undesirable situation that considerably complicates matters through the term $\delta \mathbf{I}_Q(\mathscr{B})/\delta t$. When φ is fixed in the body, however, we have $\boldsymbol{\omega}_f = \boldsymbol{\omega}$, $\delta \boldsymbol{\omega}/\delta t = \dot{\boldsymbol{\omega}}$, and $\delta \mathbf{I}_Q(\mathscr{B})/\delta t = \mathbf{0}$ in (10.64). Thus, by using axes imbedded in the body, we obtain *the classical form of Euler's second law of motion for a rigid body*:

$$\mathbf{M}_Q(\mathscr{B}, t) = \mathbf{I}_Q \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}_Q \boldsymbol{\omega}. \tag{10.65}$$

In applications of Euler's law for the motion of a rigid body in an inertial frame Φ, bear in mind that all quantities in (10.65) are referred to a body reference frame $\varphi = \{Q; \mathbf{e}_k\}$, and in the most general case the base point Q must be either fixed, in uniform motion in Φ, or at the moving center of mass. The hinge point Q in Fig. 10.6, for example, violates these conditions on Q, so (10.65) cannot be used to determine $\mathbf{M}_Q$. One procedure for this case, based on (10.46), is described in Example 10.4, page 428. The same result, however, may be obtained by a point transformation of the center of mass result based on (10.65), as described in Exercise 10.2, page 429.

The component form of Euler's law (10.65) comprises a formidable coupled system of three ordinary nonlinear differential equations for the angular motion of a rigid body. This system is somewhat simplified, though still coupled and nonlinear, in a principal body reference frame $\varphi = \{Q; \mathbf{e}_k\}$ for which (10.59) has the simpler principal component form (10.62). Thus, using (10.62) in (10.65) and discarding the circumflex notation, we obtain the scalar component form of Euler's second law for a rigid body, referred to a principal body basis $\mathbf{e}_k$ at Q:

$$\begin{aligned} M_1^Q &= I_{11}^Q\dot{\omega}_1 + \omega_2\omega_3\left(I_{33}^Q - I_{22}^Q\right), \\ M_2^Q &= I_{22}^Q\dot{\omega}_2 + \omega_3\omega_1\left(I_{11}^Q - I_{33}^Q\right), \\ M_3^Q &= I_{33}^Q\dot{\omega}_3 + \omega_1\omega_2\left(I_{22}^Q - I_{11}^Q\right). \end{aligned} \tag{10.66}$$

These are known as *Euler's equations* for the rotational motion of a rigid body. Their vector form in (10.65), however, is easy to remember.

Euler's system of differential equations for the general motion of a rigid body are thus given by the two vector differential equations (10.26) and (10.65). Determination of the motion, however, does not always split neatly into a translational part for the motion of the center of mass and a rotational part for the motion about the center of mass. Complications arise from circumstances that couple these motions. Euler's complex system, however, may be greatly simplified in special cases. One of these is a pure translation for which $\boldsymbol{\omega} \equiv \mathbf{0}$ in (10.65), and for which the total torque about the center of mass C vanishes. Thus, for a pure rigid body translation, Euler's laws reduce to $\mathbf{F} = m\mathbf{a}^*$ and $\mathbf{M}_C = \mathbf{0}$. Both equations must be satisfied, and, in rough terms, the first law determines the translational motion of the center of mass and hence the body in this case, and the second law provides relations among the forces that act on it. (See Problems 10.7 through 10.9.) Other special cases involving rotation are discussed below. First, we revisit the problem of equilibrium.

10.10. Equilibrium of a Rigid Body

The principle of determinism implies that for every point Q in an inertial reference frame both

$$\mathbf{F}(\mathscr{B}, t) = \mathbf{0}, \qquad \mathbf{M}_Q(\mathscr{B}, t) = \mathbf{0}, \tag{10.67}$$

are *necessary* conditions for equilibrium of an *arbitrary* body, that is, the system of forces must be equipollent to zero. With Euler's equations of motion for a rigid body in hand, we can show that subject to auxiliary initial data, (10.67) also suffice for equilibrium of a rigid body, exactly as described heuristically in Chapter 5. In accordance with (10.59), the equations of motion (10.26) and (10.51) for a rigid body in an inertial frame Φ, for an appropriate body point Q, may be written as

$$\mathbf{F}(\mathscr{B}, t) = m(\mathscr{B})\mathbf{a}^*(\mathscr{B}, t), \qquad \mathbf{M}_Q(\mathscr{B}, t) = \frac{d}{dt}(\mathbf{I}_Q(\mathscr{B})\boldsymbol{\omega}(t)). \tag{10.68}$$

Now suppose, conversely, that for a rigid body both relations in (10.67) hold and that *initially*, at an instant t_0, the body is either at rest or has a uniform motion in Φ. The first law shows that $\mathbf{F}(\mathscr{B}, t) = \mathbf{0}$ for all time if and only if the center of mass C has a uniform motion, or, trivially, if C is at rest in Φ. Consequently, the initial condition is trivially satisfied and we may take the point Q at C, the origin of a new inertial frame, and focus only on the second law referred to a body frame at C. The second law in (10.68) then yields $\mathbf{I}_C\boldsymbol{\omega}(t) = \mathbf{I}_C\boldsymbol{\omega}_0$ in Φ, where $\boldsymbol{\omega}_0 \equiv \boldsymbol{\omega}(t_0)$ is the initial angular velocity of the body, and therefore $\boldsymbol{\omega}(t) = \boldsymbol{\omega}_0$, a constant for all t. Initially, however, $\boldsymbol{\omega}_0 = \mathbf{0}$; hence, $\boldsymbol{\omega}(t) = \mathbf{0}$ for all t. The transformation rule (5.24) confirms that $\mathbf{M}_O = \mathbf{0}$ for every point O in Φ. Consequently, the conditions (10.67) for a rigid body that *initially* is either at rest or in uniform motion in an inertial frame Φ suffice for the body to continue in that state until compelled by additional force or torque to alter it.[§]

In sum, *for equilibrium of a rigid body initially at rest or in uniform motion in an inertial frame it is necessary and sufficient that both conditions in* (10.67) *hold for all time.* The initial data are key to this result. In the absence of these data, a rigid body can have a torque-free motion with constant angular velocity $\boldsymbol{\omega}(t) = \boldsymbol{\omega}_0$ about its center of mass, at rest or in uniform motion, for which the equations in (10.67) hold for all time in Φ.

10.11. Rotation about a Fixed Body Axis and Plane Motion

There are numerous engineering applications in which a rigid body is constrained to move parallel to a plane and to rotate about a body axis having a fixed direction in space. For this important class of problems, Euler's equation (10.65) is greatly simplified, especially when the body axis is a principal axis of inertia.

[§] In the development in Chapter 5, it is evident that one might start with a weak form of Newton's first law for a particle, as merely a necessary null condition on the applied force when the motion is uniform in an inertial frame Φ, the special frame for which the law holds. It would then follow from Newton's second law that, in fact, the null force condition is both necessary and sufficient for a uniform motion. It is clear, however, if only intuitively, given that the force acting on a particle is zero, the motion must be along a straight path in Φ. It is easy to fabricate examples to show that in order to move a particle along any curved path requires a push or pull action, if only to continuously change its direction of motion. Accepting this, it is equally simple to argue that, unless a force acts on the particle, it cannot accelerate along its straight trajectory, since some sort of push or pull action is needed to continuously alter its velocity in Φ. Hence, it seems most natural to adopt Newton's universal principle that the total force acting on a particle vanishes when and only when its motion in Φ is uniform, or one of ease.

10.11.1. Rotation about a Fixed Body Axis

First, let us consider a rigid body constrained to rotate about a fixed body axis $\mathbf{e}_3$ so that $\boldsymbol{\omega}(t) = \omega(t)\mathbf{e}_3$ and $\dot{\boldsymbol{\omega}}(t) = \dot{\omega}(t)\mathbf{e}_3$. Choose a fixed origin Q at any point on the axis of rotation, which need not pass through the center of mass. The moment of momentum $\mathbf{h}_{rQ}$ relative to Q may be read from (10.61):

$$\mathbf{h}_{rQ} = \mathbf{I}_Q\boldsymbol{\omega} = (I_{13}\mathbf{e}_1 + I_{23}\mathbf{e}_2 + I_{33}\mathbf{e}_3)\,\omega(t), \tag{10.69}$$

referred to a body frame $\varphi = \{Q; \mathbf{e}_k\}$. Use of (10.69) in Euler's equation (10.65) yields

$$\mathbf{M}_Q = (I_{13}\dot{\omega} - I_{23}\omega^2)\mathbf{e}_1 + (I_{23}\dot{\omega} + I_{13}\omega^2)\mathbf{e}_2 + I_{33}\dot{\omega}\mathbf{e}_3. \tag{10.70}$$

The axial component,

$$M_3 \equiv \mathbf{M}_Q \cdot \mathbf{e}_3 = I_{33}\dot{\omega}(t), \tag{10.71}$$

relates the rotational motion to the external applied torque M_3 about the fixed axis. In general, this rule may be used to determine the rotational motion of the body. The remaining components

$$M_1 \equiv \mathbf{M}_Q \cdot \mathbf{e}_1 = I_{13}\dot{\omega} - I_{23}\omega^2, \qquad M_2 \equiv \mathbf{M}_Q \cdot \mathbf{e}_2 = I_{23}\dot{\omega} + I_{13}\omega^2, \tag{10.72}$$

determine the *restraining torques* about the body axes $\mathbf{e}_1$ and $\mathbf{e}_2$ required to control the rotation about the fixed direction $\mathbf{e}_3$. These torques, which are due to asymmetrical distributions of mass about the 12-plane, are perpendicular to the axis of rotation and rotate with the body.

In addition, Euler's first law of motion for the center of mass also must be satisfied. If the center of mass C is on the fixed axis of rotation, then $\mathbf{a}^* = \mathbf{0}$ and the total force in (10.26) vanishes: $\mathbf{F} = \mathbf{0}$. Otherwise, when C is not on the axis of rotation, its acceleration is given by

$$\mathbf{a}^* = \mathbf{a}_Q + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}^*) + \dot{\boldsymbol{\omega}} \times \mathbf{x}^*, \tag{10.73}$$

where $\mathbf{x}^*(t) = x^*(t)\mathbf{e}_1 + y^*(t)\mathbf{e}_2 + z^*(t)\mathbf{e}_3$ is the vector of C from Q, and $\mathbf{a}_Q = \mathbf{0}$. With the aid of (10.73), the Newton–Euler law (10.26) relates the motion of the center of mass to the total force acting on the body, which also depends on its rotation:

$$\mathbf{F} = -m(\dot{\omega}y^* + \omega^2 x^*)\mathbf{e}_1 + m(\dot{\omega}x^* - \omega^2 y^*)\mathbf{e}_2. \tag{10.74}$$

If the fixed axis $\mathbf{e}_3$ is a principal axis at Q, the products of inertia vanish: $I_{13} = I_{23} = 0$. In this case, no restraining torques (10.72) are required, and (10.69) and (10.70) simplify to

$$\mathbf{h}_{rQ} = I_{33}\omega(t)\mathbf{e}_3 = I_Q\boldsymbol{\omega}, \qquad \mathbf{M}_Q = I_{33}\dot{\omega}\mathbf{e}_3 = I_Q\dot{\boldsymbol{\omega}}, \tag{10.75}$$

where $I_Q \equiv I_{33}$ is the moment of inertia about the fixed principal axis $\mathbf{e}_3$ at Q. These equations also follow immediately from (10.62) and Euler's equations (10.66) for principal axes. It is seen from (10.74) and (10.75) that no forces or torques are needed to sustain a *constant* angular velocity about a fixed principal axis through the center of mass, for in this case both $\mathbf{F} = \mathbf{0}$ and $\mathbf{M}_Q = \mathbf{M}_C = \mathbf{0}$. The system of forces is then equipollent to zero, but the rigid body is not in equilibrium.

Similar equations hold for the general spatial motion of a body rotating about a body axis in a fixed spatial direction, provided that the base point Q is taken at the center of mass C, for which $\mathbf{a}^*$ in the spatial motion must be determined.

Example 10.6. A homogeneous mechanical system consists of a flywheel and clutch plate. The flywheel rotates in an inertial frame Φ with angular speed ω_f about its axis of symmetry with respect to which its moment of inertia is I_f. The clutch plate, initially at rest in Φ, has coaxial symmetry about which its moment of inertia is I_c. The flat surface of the clutch plate suddenly engages the plane surface of the flywheel and immediately adheres to it without slipping. Ignore frictional and other extraneous loads due to bearings and other attachments. Determine the angular speed of the clutch and flywheel assembly at the moment after engagement.

Solution. The clutch and flywheel rotate about a common axis of symmetry $\mathbf{e}_3$, fixed in an inertial frame. We neglect extraneous loads and frictional effects of attached devices, and recall (10.75) and the instantaneous torque-impulse principle (10.55). The clutch plate is at rest initially; hence, the instantaneous torque–impulse exerted on the clutch c by the flywheel f about a point Q on the axis and in their plane of contact is $\mathscr{M}^*_{Qcf} = \Delta\mathbf{h}_{Qc} = I_c\omega_c\mathbf{e}_3$, where $\boldsymbol{\omega}_c = \omega_c\mathbf{e}_3$ is the angular velocity of the clutch after its engagement.The instantaneous torque–impulse exerted on the flywheel by the clutch is $\mathscr{M}^*_{Qfc} = \Delta\mathbf{h}_{Qf} = I_f(\Omega_f - \omega_f)\mathbf{e}_3$. The adherence condition requires that the final angular speed of the flywheel $\Omega_f = \omega_c$, as the two plates instantaneously rotate together. Because the torques are mutual internal torques, $\mathscr{M}^*_{Qcf} = -\mathscr{M}^*_{Qfc}$. Therefore, at the impulsive instant, $I_c\omega_c = -I_f(\omega_c - \omega_f)$, and hence the angular speed of the clutch and flywheel assembly is

$$\omega_c = \frac{I_f}{I_c + I_f}\omega_f.$$

Clearly, $\omega_c < \omega_f$, as one may expect intuitively. The flywheel, however, because of its greater mass and size, usually has a much greater moment of inertia so that $I_c/I_f \ll 1$, and hence $\omega_c = \omega_f$, very nearly.

Alternatively, because there are no external impulsive forces or torques, the total torque–impulse $\mathscr{M}^*_Q$ about the axis of rotation must vanish, and hence $\Delta\mathbf{h}_Q = \mathbf{0}$ at the impulsive instant, that is, the total moment of momentum about the axis $\mathbf{e}_3$ is constant in the inertial frame. Thus, the final moment of momentum of the system equals its initial moment of momentum: $(I_f + I_c)\omega_c\mathbf{e}_3 = I_f\omega_f\mathbf{e}_3$. This yields the same result given above. □

10.11.2. Plane Motion of a Rigid Body

Equations (10.71) through (10.75) also apply to the *plane motion* of a rigid body, a motion for which every material point of the body moves parallel to a specified plane. Consequently, choose the body plane through the center of mass C as the plane $z = 0$ to apply (10.71) and (10.72) in a body frame at C. The restraining moments (10.72) vanish when the plane of the motion is a principal body plane so that $I_{13} = I_{23} = 0$, then (10.75) holds for $Q = C$. Recall also the Newton–Euler equation (10.26) for the center of mass motion in the plane inertial frame $\Phi = \{O; \mathbf{I}, \mathbf{J}\}$, namely, $\mathbf{F} = m(\ddot{X}^*\mathbf{I} + \ddot{Y}^*\mathbf{J})$, where $\mathbf{X}^* = X^*\mathbf{I} + Y^*\mathbf{J}$ describes the position vector of C in Φ.

Exercise 10.3. (a) Prove that the restraining moments (10.72) vanish, if and only if the axis of rotation $\mathbf{e}_3$ is a principal axis of $\mathbf{I}_Q$. (b) Show that $\mathbf{h}_{rQ}$ may be parallel to $\boldsymbol{\omega}$, if and only if $\boldsymbol{\omega}$ is a rotation about a principal axis of $\mathbf{I}_Q$. Hence, $\mathbf{h}_{rQ} = I_Q\boldsymbol{\omega}$ and $\mathbf{M}_Q = I_Q\dot{\boldsymbol{\omega}}$, where $I_Q \equiv I_{33}^Q$. □

10.11.3. Dynamic Balance of a Rotor

The general equations for rotation about a fixed body axis may be applied to study the static and dynamic balance of a machine rotor that turns about its fixed body axis $\mathbf{e}_3$, say. Let $\mathbf{x}^*$ denote the position vector of the center of mass from a point on the rotor axis. The rotor is said to be *statically balanced* when its center of mass is on the rotor axis, so that $\mathbf{x}^* = \mathbf{0}$ when the machine is at rest; otherwise, the rotor is statically unbalanced. For a statically balanced rotor, by (10.74), $x^* = y^* = 0$ and hence the total of the static bearing reaction forces and the rotor's weight must vanish. When the rotor is statically balanced, these loads are of no further concern in effecting the dynamic balance of the rotor.

By (10.72), however, a statically balanced rotor still requires restraining torques supplied by dynamic reaction forces at each bearing. Since the body axes to which the dynamic bearing forces and torques are referred are rotating about the rotor axis, these actions are alternating periodically in direction, the period being the time of one revolution of the rotor. This action can induce a forced vibration of the machine structure that ultimately may lead to its severe damage, even its destruction. To assure smooth operation and greater bearing longevity, the rotor also must be *dynamically balanced* so that its bearings will experience no dynamic loads whatsoever. Dynamic balance, therefore, additionally requires that the restraining torques vanish in (10.72). This is possible if and only if the axis of rotation is a principal axis for the rotor. Hence, static and dynamic balance may require the addition of balance weights appropriately located in two arbitrarily assigned planes perpendicular to the rotor axis and situated on each side of the normal plane through the adjusted center of mass on the rotor axis, so that both

$I_{13} = I_{23} = 0$ and $x^* = y^* = 0$. *Consequently, for static and dynamic balance of the machine rotor, its axis of rotation must be a principal axis through the center of mass.*

To apply this rule, let us consider an unbalanced rotor of mass m with its center of mass situated at (x_C, y_C) from a point Q on the rotor axis. Locate two *correction planes* A and B at $z = z_A$ and $z = z_B$, respectively, from Q, and let m_A and m_B denote respective balance masses to be situated in these planes. The balance problem then is to determine the masses m_A, m_B and their respective locations (x_A, y_A), (x_B, y_B) so that the center of mass of the entire system, the rotor and the balance masses together, lies at Q and the axis $\mathbf{e}_3$ is a principal axis. The balance masses are assumed to be small enough that these may be modeled as particles. From (9.10), the moment of inertia about Q of a particle of mass m_A at $\mathbf{x}_A$ from Q is defined by $\mathbf{I}_Q(m_A) = m_A[(\mathbf{x}_A \cdot \mathbf{x}_A)\mathbf{1} - \mathbf{x}_A \otimes \mathbf{x}_A]$, and hence its products of inertia are given by $I_{ij}^Q(m_A) = -m_A x_i^A x_j^A$. With the aid of this relation and (5.5) for the center of mass of a system of total mass $M = m + m_A + m_B$, the rotor will be dynamically balanced provided that

$$M\mathbf{x}^* = m(x_C, y_C) + m_A(x_A, y_A) + m_B(x_B, y_B) = \mathbf{0},$$
$$I_{ij}^Q(M) = I_{ij}^Q(m) + I_{ij}^Q(m_A) + I_{ij}^Q(m_B) = 0, \qquad ij = 13, 23. \tag{10.76a}$$

These constitute a system of four equations for the four unknown products $m_A x_A$, $m_A y_A$, $m_B x_B$, and $m_B y_B$, namely,

$$\begin{aligned} m_A x_A + m_B x_B &= -m x_C, \\ m_A y_A + m_B y_B &= -m y_C, \\ m_A x_A z_A + m_B x_B z_B &= I_{13}^Q(m), \\ m_A y_A z_A + m_B y_B z_B &= I_{23}^Q(m), \end{aligned} \tag{10.76b}$$

where the quantities on the right-hand sides of these equations are assumed to be known, usually from experiments. Since only the products are relevant, the values of the balance masses or their squared distance from the rotor axis can be arbitrarily assigned. Then (10.76b) solves the rotor balance problem. An example of the static balance and dynamic imbalance of a rotating rectangular plate will be illustrated later on.

10.12. Further Applications of Euler's Laws

Euler's laws in (10.26) and (10.65) are the fundamental equations for study of all rigid body dynamics problems. Their solution for several special problems of practical interest are explored below.

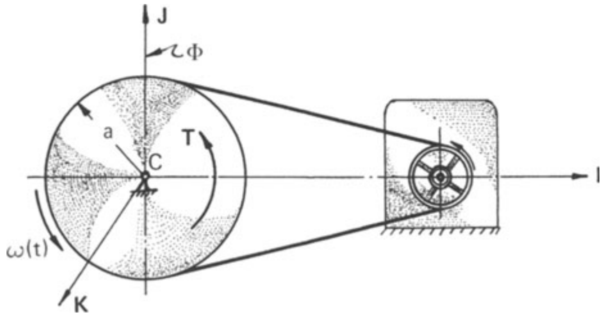


Figure 10.7. A belt driven pulley system.

10.12.1. Application to a Belt Driven Machine

Example 10.7. A homogeneous cylindrical pulley of radius a and mass m, initially at rest, is driven by a belt of negligible mass. During the start-up period, the drive belt exerts a constant torque $\mathbf{T}$ on the pulley about its axle at C, as shown in Fig. 10.7. Determine the pulley's angular speed $\omega(t)$.

Solution. The homogeneous pulley rotates with angular velocity $\boldsymbol{\omega}(t) = \omega(t)\mathbf{e}_3$ about a fixed principal body axis $\mathbf{e}_3 = \mathbf{K}$ at its center of mass C in Fig. 10.7. The constant belt torque about C is $\mathbf{M}_C = \mathbf{T} = T\mathbf{K}$, and Euler's equation in (10.75) requires $\mathbf{M}_C = I_C\dot{\boldsymbol{\omega}} = I_C\dot{\omega}\mathbf{K}$. Hence, equating components, the angular speed $\omega(t)$ during startup is determined by

$$I_C\dot{\omega} = T. \tag{10.77a}$$

Integration of (10.77a) with $\omega(0) = \omega_0$ initially yields the angular speed,

$$\omega(t) = \omega_0 + \frac{T}{I_C}t. \tag{10.77b}$$

Since the pulley starts from rest, we set $\omega_0 = 0$ to obtain $\omega(t) = Tt/I_C$. □

The result shows that the angular speed varies inversely with the pulley properties through I_C. Suppose the pulley is modeled as a homogeneous thin disk for which, by (9.33), $I_C = I_{33}^C = ma^2/2$. Then, (10.77b) yields $\omega(t) = \omega_0 + 2Tt/ma^2$ in terms of the assigned quantities. Therefore, as the size and mass of the pulley increase, it takes a longer startup time to reach the same angular speed. This discussion is next extended to study the effect of a flywheel in controlling the fluctuating motion of machinery.

10.12.2. Application to a Flywheel

The uniform pulley results (10.77a) and (10.77b) also hold for a homogeneous flywheel subjected to a constant torque. Generally, however, the driving torque of an automobile engine and other kinds of reciprocating machines is not steady. There may be a drag torque on the drive shaft, perhaps due to a pump or a brake device, that varies with the angular speed, for example. A flywheel, therefore, is used to control and steady the fluctuating motions of machinery. To illustrate this useful property, let us consider a homogeneous flywheel of moment of inertia $I_C = I$ about its axle, and subjected to a fluctuating torque $T_1 \cos pt$, a drag torque $-T_2\omega$, and a steady torque T_0 about its axis. We thus introduce the total torque $T = T_0 + T_1 \cos pt - T_2\omega$ in (10.77a) to obtain the equation of motion for the flywheel:

$$I\dot{\omega} + T_2\omega = T_0 + T_1 \cos pt, \tag{10.77c}$$

in which T_0, T_1, T_2, and p are appropriate constants.

The general solution of the homogeneous equation is $\omega_H = \omega_0 e^{-\alpha t}$, where $\alpha \equiv T_2/I$ and ω_0 is a constant of integration. A particular solution of (10.77c) is $\omega_p = \beta + \gamma \cos(pt + \varepsilon)$, where $\beta \equiv T_0/T_2$, and $\gamma \equiv T_1/I(\alpha^2 + p^2)^{1/2}$ is the amplitude of the fluctuation with phase $\varepsilon = \tan^{-1}(p/\alpha)$, the last being relatively unimportant. Hence, the general solution of (10.77c) is

$$\omega(t) = \omega_0 e^{-\alpha t} + \beta + \gamma \cos(pt + \varepsilon). \tag{10.77d}$$

The first term decays to zero as t increases, and the ratio of the amplitudes of the remaining terms may be written as

$$\frac{\gamma}{\beta} = \frac{T_1}{T_0} \cdot \frac{T_2}{\sqrt{T_2^2 + p^2 I^2}}. \tag{10.77e}$$

Thus, by our increasing the moment of inertia, that is, by increasing the size and mass of the flywheel, the fluctuation amplitude ratio γ/β can be made sufficiently small so that the constant term β in (10.77d) is dominant, even though the torque amplitude T_1 of the fluctuation may be greater than the steady torque T_0. In this way, the fluctuation is diminished and the rotation of the flywheel is rendered very nearly steady with angular speed $\omega(t) = \beta = T_0/T_2$. Notice the importance in (10.77e) of our accounting for the drag torque, i.e. the load, on the system. □

Exercise 10.4. A horizontal force $\mathbf{P} = -P\mathbf{i}$ is applied at $d = 3/4$ ft above the center of the log whose motion at an instant of interest is described in Example 10.2, page 421. Model the log as a homogeneous circular cylinder, and thus determine all of the forces that act on the log at that instant. □

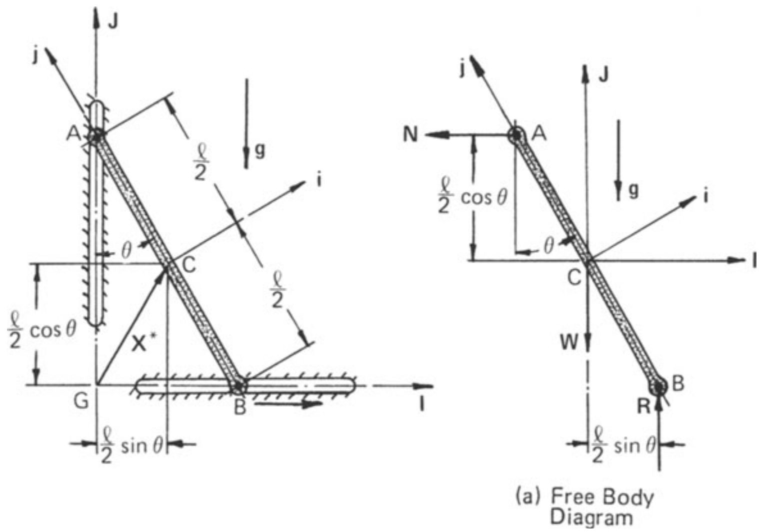


Figure 10.8. Plane motion of a slender rigid link.

10.12.3. Application to the Plane Motion of a Rigid Link

Example 10.8. A homogeneous slender link of length ℓ shown in Fig. 10.8 is constrained to move in lubricated slots A and B. The link starts from rest at an angle θ_0 in the vertical plane. Find the forces that act on the link as functions of the angle θ. Model the link as a uniform thin rod.

Solution. The free body diagram in Fig. 10.8(a) shows the equipollent normal reaction forces $\mathbf{N} = -N\mathbf{I}$ and $\mathbf{R} = R\mathbf{J}$ exerted on the link by the smooth slots, and the weight $\mathbf{W} = -W\mathbf{J} = m\mathbf{g}$ of the homogeneous link acting at its center of gravity, which coincides with the center of mass. Other resultant contact forces $\mathbf{A} = A\mathbf{K}$ and $\mathbf{B} = B\mathbf{K}$ perpendicular to the plane of motion are represented by heavy dots at A and B. Hence, the total force acting on the link is $\mathbf{F} = \mathbf{A} + \mathbf{B} + \mathbf{N} + \mathbf{R} + \mathbf{W} = m\mathbf{a}^*$, in accordance with Euler's first law (10.26). We wish to determine the unknown forces $\mathbf{A}$, $\mathbf{B}$, $\mathbf{N}$, and $\mathbf{R}$.

The motion of the center of mass of the link in the inertial frame $\Phi = \{G; \mathbf{I}_k\}$ is defined by $\mathbf{X}^* = \ell/2(\sin\theta\mathbf{I} + \cos\theta\mathbf{J})$, the path being a circle of radius $\ell/2$ centered at G. Then, with $\mathbf{a}^* = \ddot{\mathbf{X}}^*$, Euler's first law yields

$$-N\mathbf{I}+(R-W)\mathbf{J}+(A+B)\mathbf{K}=\frac{m\ell}{2}[(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta)\mathbf{I}-(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta)\mathbf{J}],$$

from which

$$N = -\frac{m\ell}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta), \qquad R = W - \frac{m\ell}{2}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta),$$

$$A + B = 0. \tag{10.78a}$$

The forces N and R, therefore, are determined once $\theta(t)$ is known.

To find $\theta(t)$, we introduce the principal body frame $\varphi = \{C; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ at the center of mass C and recall Euler's law (10.75) for a plane motion: $\mathbf{M}_C = I_C\dot{\boldsymbol{\omega}}$. With $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$, $\mathbf{M}_C = I_C\ddot{\theta}\mathbf{k}$, where for a uniform slender rod, $I_C = I_{33}^C = m\ell^2/12$ (cf. Fig. D.7 of Appendix D). With reference to Fig. 10.8(a), the total moment about C of the forces acting on the link is given by

$$\mathbf{M}_C = \frac{\ell}{2}(A - B)\mathbf{i} + \left(\frac{\ell}{2}N\cos\theta + \frac{\ell}{2}R\sin\theta\right)\mathbf{k}. \tag{10.78b}$$

Upon equating corresponding components, we find the additional scalar equations

$$N\cos\theta + R\sin\theta = \frac{m\ell}{6}\ddot{\theta}, \qquad A - B = 0. \tag{10.78c}$$

The last relations in (10.78a) and (10.78c) show that $\mathbf{A} = \mathbf{B} = \mathbf{0}$, an anticipated result in as much as the rod has no normal motion and no other forces act on the rod normal to the plane. The remaining system of three equations in (10.78a) and (10.78c) determine the three unknowns $N(\theta)$, $R(\theta)$, and $\theta(t)$. To solve the equations, substitute the first two relations in (10.78a) into (10.78c) and simplify the result to obtain

$$\ddot{\theta} = \frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right) = \frac{3g}{2\ell}\sin\theta. \tag{10.78d}$$

An easy integration with the initial condition $\theta(0) = \theta_0$ then yields

$$\dot{\theta}^2 = \frac{3g}{\ell}(\cos\theta_0 - \cos\theta), \tag{10.78e}$$

in which $\theta(t) \geq \theta_0$. Substitution of (10.78d) and (10.78e) into the first pair of equations in (10.78a) delivers the forces that act on the link as functions of $\theta(t)$:

$$\mathbf{N}(\theta) = \tfrac{3}{4}W\sin\theta(3\cos\theta - 2\cos\theta_0)\mathbf{I}, \tag{10.78f}$$

$$\mathbf{R}(\theta) = \frac{W}{4}[1 + 3\cos\theta(3\cos\theta - 2\cos\theta_0)]\mathbf{J} = \left[\frac{W}{4} + N(\theta)\cot\theta\right]\mathbf{J}. \tag{10.78g}$$

Finally, integration of (10.78e) yields

$$\sqrt{3\frac{g}{\ell}}\,t = \int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{\cos\theta_0 - \cos\theta}}. \tag{10.78h}$$

This may be used to determine $\theta(t)$, and hence the motion $\mathbf{X}^*(t)$ of the center of mass and the reaction forces $\mathbf{N}(t)$ and $\mathbf{R}(t)$ as functions of time. □

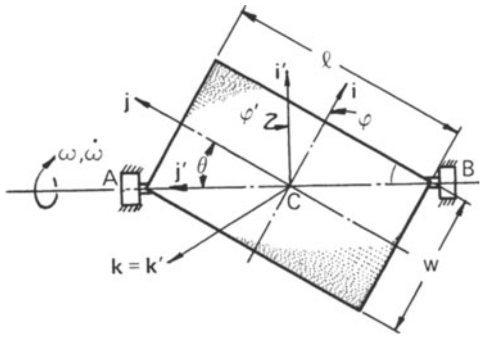


Figure 10.9. Rotation about an asymmetric body axis fixed in space.

10.12.4. Analysis of Dynamical Bearing Reaction Forces

Example 10.9. A thin homogeneous rectangular plate in Fig. 10.9 rotates about a diagonal axis with angular velocity $\boldsymbol{\omega}$ and angular acceleration $\dot{\boldsymbol{\omega}}$. (i) What is the total torque exerted on the plate about C, referred to the principal body frame $\varphi = \{C; \mathbf{i}_k\}$? (ii) Determine the total torque about C referred to the body frame $\varphi' = \{C; \mathbf{i}'_k\}$, and identify the restraining torques acting on the plate. (iii) Discuss the role of the static bearing reaction forces, derive an equation relating the drive torque $\mathbf{T}$ to the rotation, and determine the dynamic bearing reaction forces. (iv) Suppose that $\mathbf{T}(t)$ is specified; find the angular speed.

Solution of (i). The total torque $\mathbf{M}_C$ acting on the plate about C is determined by Euler's equations (10.66). With this in mind, select the principal body frame $\varphi = \{C; \mathbf{i}_k\}$ at the center of mass and, for convenience, write

$$\sigma \equiv \sin\theta = \frac{w}{\sqrt{\ell^2 + w^2}}, \qquad \gamma \equiv \cos\theta = \frac{\ell}{\sqrt{\ell^2 + w^2}}. \tag{10.79a}$$

Referred to φ, the angular velocity and angular acceleration are given by

$$\boldsymbol{\omega} = \omega(-\sigma\mathbf{i} + \gamma\mathbf{j}), \qquad \dot{\boldsymbol{\omega}} = \dot{\omega}(-\sigma\mathbf{i} + \gamma\mathbf{j}); \tag{10.79b}$$

and from (9.27), with $(\mathbf{i}_1^*, \mathbf{i}_2^*, \mathbf{i}_3^*) = (\mathbf{j}, -\mathbf{i}, \mathbf{k})$, the principal moments of inertia about C for the homogeneous thin plate are

$$\mathbf{I}_C = \frac{m\ell^2}{12}\mathbf{i} \otimes \mathbf{i} + \frac{mw^2}{12}\mathbf{j} \otimes \mathbf{j} + \frac{m}{12}(\ell^2 + w^2)\mathbf{k} \otimes \mathbf{k}. \tag{10.79c}$$

The total torque on the plate is now given by Euler's principal axis equations (10.66) for $Q = C$:

$$\mathbf{M}_C = \frac{\dot{\omega} m}{12}(-\sigma\ell^2\mathbf{i} + \gamma w^2\mathbf{j}) + \frac{\omega^2 m\sigma\gamma}{12}(\ell^2 - w^2)\mathbf{k}. \tag{10.79d}$$

This result shows that a moment about the **k**-axis perpendicular to the plane of the plate is required to sustain the motion even when the angular velocity is constant. This torque rotates with the **k**-axis normal to plate and produces alternating reactions at the support bearings. A square plate ($\ell = w$), however, can spin at a constant angular speed without application of any torque whatsoever. □

Exercise 10.5. Apply (10.65) to confirm (10.79d). □

Solution of (ii). To relate $\mathbf{M}_C$ to the plate frame $\varphi' = \{C; \mathbf{i}'_k\}$ for which $\mathbf{j}'$ is the axis of rotation and $\mathbf{k}' = \mathbf{k}$ in Fig. 10.9, we use (10.79a) and the vector transformation law (3.107a): $M'_C = AM_C$, where $A = [A_{jk}] = [\cos\langle \mathbf{i}'_j, \mathbf{i}_k \rangle]$, to obtain

$$M'_C = \begin{bmatrix} \gamma & \sigma & 0 \\ -\sigma & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} \gamma M_1 + \sigma M_2 \\ -\sigma M_1 + \gamma M_2 \\ M_3 \end{bmatrix}, \tag{10.79e}$$

in which M_k are the components in (10.79d). Thus, referred to φ', the total torque exerted on the plate is

$$\mathbf{M}_C = \frac{mw\ell(\ell^2 - w^2)}{12(\ell^2 + w^2)}(-\dot{\omega}\mathbf{i}' + \omega^2\mathbf{k}') + \frac{\dot{\omega}mw^2\ell^2}{6(\ell^2 + w^2)}\mathbf{j}'. \tag{10.79f}$$

The torque about the bearing axle $\mathbf{j}'$ is related to the applied drive torque **T**, and the remaining components, which arise from the asymmetrical distribution of mass about the plate diagonal, are restraining torques supplied by the bearings at A and B. □

Exercise 10.6. Determine the moment of inertia tensor components referred to φ', which is not a principal reference frame. Note that $\boldsymbol{\omega} = \omega\mathbf{j}'$, $\dot{\boldsymbol{\omega}} = \dot{\omega}\mathbf{j}'$, and apply (10.65) to derive (10.79f). □

Solution of (iii). We next explore the role of the static bearing reaction forces. The free body diagram of the plate is shown in Fig. 10.10. When the plate is at rest, each shaft bearing support exerts an equal force $\mathbf{A}_S = \mathbf{B}_S = -(\mathbf{W})/2$ on the plate at A and B, equal to one-half its weight **W**. These static loads are equipollent to zero, and therefore they contribute nothing to the total force or to the total torque about the center of mass C in the dynamics problem. Henceforward, these statically balanced forces may be ignored.

Now let us relate the drive torque $\mathbf{T} = T\mathbf{j}'$ to the rotational motion and determine the resultant dynamic bearing reaction forces $\mathbf{A} = A_k\mathbf{i}'_k$ and $\mathbf{B} = B_k\mathbf{i}'_k$, exerted by the shaft at A and B, respectively, and which we shall suppose, for simplicity, act at the corners of the plate in Fig. 10.10. Euler's first law (10.26) requires that $\mathbf{A} + \mathbf{B} = m\mathbf{a}^* = \mathbf{0}$; thus, $\mathbf{A} = -\mathbf{B}$, so the bearing reaction force system forms a couple with moment arm $2\mathbf{x} = (\ell^2 + w^2)^{1/2}\mathbf{j}'$, where $\mathbf{x}$ is the position

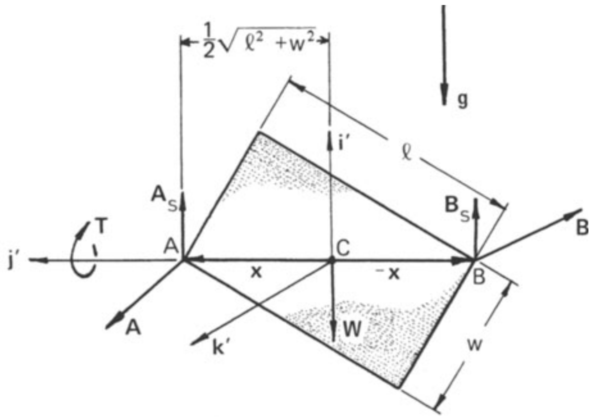


Figure 10.10. Free body diagram of the rotating plate.

vector of **A** from C. Therefore, the total applied torque on the plate about C is

$$\mathbf{M}_C = \mathbf{T} + 2\mathbf{x} \times \mathbf{A} = T\mathbf{j}' + \sqrt{\ell^2 + w^2}(A_3\mathbf{i}' - A_1\mathbf{k}'). \tag{10.79g}$$

Equating the corresponding components in (10.79f) and (10.79g), we obtain an equation relating the drive torque to the rotation and two relations for the dynamic bearing reaction force components:

$$T = \frac{\dot{\omega} m w^2 \ell^2}{6(\ell^2 + w^2)}, \qquad B_1 = -A_1 = \frac{m\omega^2 w\ell(\ell^2 - w^2)}{12(\ell^2 + w^2)^{3/2}},$$
$$B_3 = -A_3 = \frac{T(\ell^2 - w^2)}{2w\ell(\ell^2 + w^2)^{1/2}}. \tag{10.79h}$$

The axial components of the forces exerted by the bearings must satisfy $B_2 = -A_2$; otherwise, this axial force is undetermined by this analysis, and nothing is lost by putting it equal to zero. Usually, however, thrust bearings are used at the shaft ends to secure any axle drift of the drive shaft. Since drag torques due to friction in the bearings are ignored, when the drive torque is removed, the plate will spin indefinitely with constant angular speed and the only nonzero bearing reaction force components in (10.79h) are $A_1 = -B_1$.

For a square plate ($w = \ell$) all dynamic bearing reaction forces in (10.79h) vanish. Therefore, when the drive torque is removed, the force system is equipollent to zero: $\mathbf{F}(\mathscr{B}, t) = \mathbf{0}$ and $\mathbf{M}_C(\mathscr{B}, t) = \mathbf{0}$, but the square plate is not in equilibrium; it continues to spin with a constant angular speed.

Solution of (iv). Suppose the applied driving torque $T(t)$ is given and $\omega(0) = \omega_0$ initially. Then integration of the first equation in (10.79h) yields the angular speed

$$\omega(t) = \omega_0 + \frac{6(\ell^2 + w^2)}{m w^2 \ell^2} \int_0^t T(t)\, dt. \tag{10.79i}$$

The dynamic bearing reactions are then determined by their equations in (10.79h). □

Exercise 10.7. Describe in analytical terms how you would go about correcting for the dynamic imbalance of the plate. Hint: Derive a system of equations of the type (10.76b) and recall results of Exercise 10.6, page 445. □

10.12.5. Bearing Reaction Torque and Stability of Relative Equilibrium

Example 10.10. A homogeneous thin rod of mass m and length ℓ is connected to a vertical shaft S by a smooth hinge bearing at Q. The shaft rotates with a constant angular velocity $\mathbf{\Omega}$, as shown in Fig. 10.11. (i) Derive the equation of motion of the rod. (ii) Determine as a function of θ the hinge bearing reaction torque exerted on the rod at Q, for the initial data $\dot{\theta}(0) = 0$ at $\theta(0) = \theta_0$. (iii) Analyze the infinitesimal stability of the relative equilibrium states of the rod.

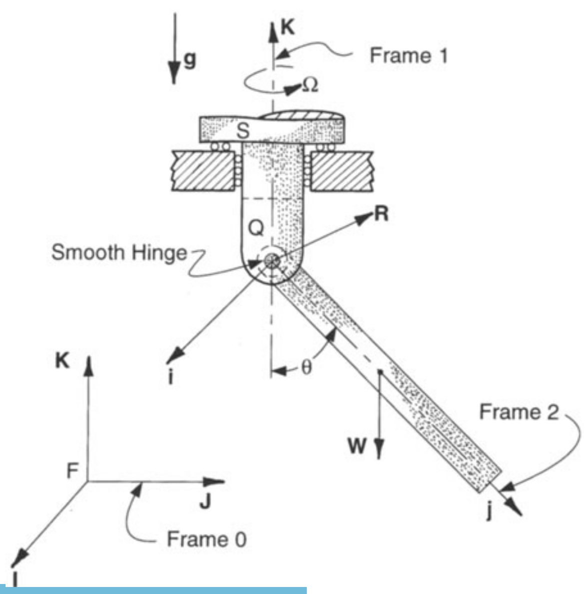


Figure 10.11. Spatial motion of a rigid rod hinged to a rotating shaft.

Solution of (i). The central point Q at the hinge being fixed in the inertial (machine) frame $0 = \{F; \mathbf{I}_k\}$, the equation of motion for the rod is obtained from Euler's law (10.66) in the principal body frame $2 = \{Q; \mathbf{i}_k\}$. We need $\boldsymbol{\omega}$, $\mathbf{I}_Q$, and $\mathbf{M}_Q$.

Let $\mathbf{k} = \mathbf{i}_3$ denote the hinge axis at Q. Then the total angular velocity $\boldsymbol{\omega} \equiv \boldsymbol{\omega}_{20}$ of the rod (frame 2) relative to the machine (frame 0) is the sum of the angular velocity $\boldsymbol{\omega}_{21} = \dot{\theta}\mathbf{k}$ of the rod relative to the vertical shaft (frame 1) and the angular velocity $\boldsymbol{\omega}_{10} = \boldsymbol{\Omega} = \Omega\mathbf{K}$ of the shaft relative to the machine. Thus, referred to the rod frame 2,

$$\boldsymbol{\omega} = -\Omega(\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) + \dot{\theta}\mathbf{k}, \tag{10.80a}$$

and hence the total angular acceleration of the rod is

$$\dot{\boldsymbol{\omega}} = -\Omega\dot{\theta}(\cos\theta\mathbf{i} - \sin\theta\mathbf{j}) + \ddot{\theta}\mathbf{k}. \tag{10.80b}$$

The rod frame at Q is a principal body frame relative to which

$$\mathbf{I}_Q = \tfrac{1}{3}m\ell^2(\mathbf{i}_{11} + \mathbf{i}_{33}), \tag{10.80c}$$

in accordance with (9.46d), with a change of axes in mind. Finally, the moment of the forces and torques acting on the rod about Q is

$$\mathbf{M}_Q = \mu_1\mathbf{i} + \mu_2\mathbf{j} - mg\frac{\ell}{2}\sin\theta\mathbf{k}. \tag{10.80d}$$

Here μ_1 and μ_2 are unknown components of the torque exerted on the rod by the smooth hinge bearing for which $\mu_3 = 0$, and the weight $\mathbf{W}$ of the homogeneous rod acts at its center of mass at $\mathbf{x}^* = \ell/2\mathbf{j}$.

Use of (10.80a) through (10.80d) in Euler's equations (10.66) yields the components of the smooth hinge bearing reaction torque exerted on the rod at Q,

$$\mu_1 = -\tfrac{2}{3}m\ell^2\Omega\dot{\theta}\cos\theta, \qquad \mu_2 = \mu_3 = 0, \tag{10.80e}$$

and the differential equation of motion of the rod,

$$\ddot{\theta} + (p^2 - \Omega^2\cos\theta)\sin\theta = 0, \qquad p^2 \equiv \frac{3g}{2\ell}. \tag{10.80f}$$

With the aid of this result, the reader may now determine from Euler's first law the resultant hinge reaction force $\mathbf{R}$ at Q.

When $\Omega = 0$, $\mu_1 = 0$ and (10.80f) reduces to the familiar pendulum equation. In this case, the circular frequency p for the rod is the same as that of a simple pendulum of length $L = 2\ell/3$. Therefore, the exact and approximate solutions for the vibration of the rod when $\Omega = 0$ may be read from those for the simple pendulum. We next consider the case when $\Omega \neq 0$.

Solution of (ii). To determine the hinge bearing reaction torque (10.80e) as a function of θ, we need $\dot{\theta}(\theta)$, the first integral of (10.80f). With $\ddot{\theta} = d(\frac{1}{2}\dot{\theta}^2)/d\theta$, separation of the variables and use of the initial data $\dot{\theta}(0) = 0$ at $\theta(0) = \theta_0$ yields

$$\dot{\theta} = \pm\sqrt{\Omega^2(\cos^2\theta_0 - \cos^2\theta) + 3\frac{g}{\ell}(\cos\theta - \cos\theta_0)}. \tag{10.80g}$$

Substitution of this result into (10.80e) delivers as a function of θ the hinge bearing reaction torque exerted on the rod at Q:

$$\mu_1 = \mp\frac{2}{3}m\ell^2\Omega\cos\theta\sqrt{\Omega^2(\cos^2\theta_0 - \cos^2\theta) + 3\frac{g}{\ell}(\cos\theta - \cos\theta_0)}. \tag{10.80h}$$

Notice that the reaction torque $\mu_1(\theta)$ vanishes when $\theta = \theta_0$ or $n\pi/2$ (n odd) and, of course, also when $\Omega = 0$.

Solution of (iii). The relative equilibrium states θ_s of the rod, by (10.80f), are given by

$$(p^2 - \Omega^2\cos\theta_s)\sin\theta_s = 0, \tag{10.80i}$$

which yields three distinct states

$$\theta_s = 0, \pi, \qquad \theta_s = \cos^{-1}\left(\frac{p^2}{\Omega^2}\right). \tag{10.80j}$$

To examine the infinitesimal stability of these states, introduce $\theta = \theta_s + \delta$, where δ is an infinitesimal angular disturbance from θ_s. Then recalling (10.80i) and retaining only terms of the first order in δ, we obtain from (10.80f),

$$\ddot{\delta} + \{[p^2 - \Omega^2\cos\theta_s]\cos\theta_s + \Omega^2\sin^2\theta_s\}\delta = 0. \tag{10.80k}$$

Therefore, the disturbance is bounded and simple harmonic, and hence infinitesimally stable, if and only if the coefficient

$$\omega^2 \equiv [p^2 - \Omega^2\cos\theta_s]\cos\theta_s + \Omega^2\sin^2\theta_s > 0. \tag{10.80l}$$

We now recall (10.80j). First consider $\theta_s = 0$. Then (10.80l) requires $\omega^2 = p^2 - \Omega^2 > 0$. Hence, the relative equilibrium state $\theta_s = 0$ is infinitesimally stable if and only if $\Omega < p = \Omega_c$, the critical angular speed of the vertical shaft:

$$\Omega_c \equiv p = \sqrt{\frac{3g}{2\ell}}, \tag{10.80m}$$

which is independent of the mass of the rod. In this case, the small amplitude circular frequency of the rod oscillations about the vertical state $\theta_s = 0$ is $\omega_v \equiv (p^2 - \Omega^2)^{1/2}$.

Of course, the inverted vertical configuration of the rod $\theta_s = \pi$ is impractical in respect of the suggested design in Fig. 10.11. Even so, (10.80l) fails for $\theta_s = \pi$, and hence the inverted relative equilibrium state of the rod is inherently unstable for all angular speeds of the vertical shaft.

Notice that when $\Omega = \Omega_c = p$, $\omega^2 \leq 0$ for all positions (10.80j). Hence, no infinitesimally stable relative equilibrium states of the rod exist at the critical angular speed $\Omega = \Omega_c$.

Finally, consider the case $\cos\theta_s = p^2/\Omega^2$. This requires $p/\Omega < 1$; hence (10.80l) is satisfied, and the relative equilibrium state $\theta_s = \cos^{-1}(p^2/\Omega^2)$, regardless of the mass of the rod, is infinitesimally stable if and only if $\Omega > p = \Omega_c$. The small amplitude vibrational frequency of the rod about this displaced relative equilibrium state is $\omega_d \equiv \Omega(1 - p^4/\Omega^4)^{1/2}$.

In sum, if $\Omega < \Omega_c$, the vertical relative equilibrium position of the rod is its only infinitesimally stable relative equilibrium state. When the vertical shaft attains its critical angular speed, $\Omega = \Omega_c$, however, no infinitesimally stable relative equilibrium positions of the rod exist. Afterwards, when $\Omega > \Omega_c$, the displaced configuration of the rod at $\theta_s = \cos^{-1}(p^2/\Omega^2)$ is its only infinitesimally stable relative equilibrium position. □

10.13. The Gyrocompass

Some subtle Coriolis effects of the Earth's rotation on the motion of a particle were described in Chapter 6. Here we study an important practical effect of the Earth's rotation on the motion of a *gyrocompass*, a gyroscopic instrument that uses the rotation of the Earth to determine the direction of true north.

A model of the gyrocompass shown in Fig. 10.12 consists of a massive cylindrical rotor R that spins about a smooth horizontal bearing axis $\mathbf{j}$ with an angular speed $\dot{\theta}$ relative to its rigid supporting gimbal F, whose mass we shall ignore. The gimbal is free to rotate in smooth bearings about the vertical, skyward directed $\mathbf{k}$-axis fixed relative to the Earth at latitude λ. No torques are applied about the rotor and vertical gimbal bearing axes at the center of mass C, so

$$\mathbf{M}_C \cdot \mathbf{j} = 0, \qquad \mathbf{M}_C \cdot \mathbf{k} = 0. \tag{10.81a}$$

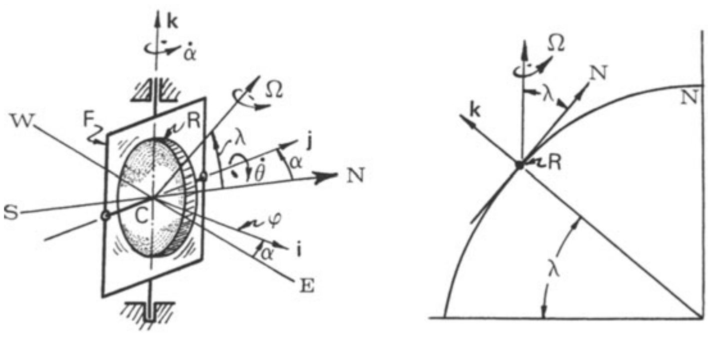


Figure 10.12. Schematic model of a gyrocompass.

We are going to show for these conditions that when the gyroscope is given a small angular disturbance $\alpha(t)$ from its initial north direction shown in Fig. 10.12, it oscillates about that direction. Afterwards, the applied torque is modified to include a small viscous damping couple about the vertical axis. In consequence, the damped oscillation fixes the rotor axis on the true north position and the device thus performs the primary function of a compass.

10.13.1. General Formulation of the Problem

It is convenient to consider the motion referred to the frame $\varphi = \{C; \mathbf{i}_k\}$ fixed in the gimbal F. In this case, however, Euler's equations (10.66) cannot be applied directly because φ is not a principal body reference frame fixed in the rotor at C. So, we need to consider a change of basis to refer the moment of momentum vector¶ to the gimbal frame. Because of the symmetry of the homogeneous circular rotor, however, the components I_{jk} of the inertia tensor $\mathbf{I}_C$ in the gimbal frame $\varphi = \{C; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ have precisely the same constant principal values as its corresponding components I'_{jk} in a rotor reference frame $\psi = \{C; \mathbf{i}', \mathbf{j}, \mathbf{k}'\}$. While this is intuitively evident, the result follows from the tensor transformation law $I = A^T I' A$, and the fact that every axis in the plane of the rotor is a principal axis for which $I_{11} = I_{33}$. Consequently, with $\mathbf{I}_C$ now referred to the moving frame φ, we have $\delta\mathbf{I}_C/\delta t = \mathbf{0}$. Thus, as a first step toward finding the total torque about C, we determine $\mathbf{h}_C = \mathbf{h}_{rC} = \mathbf{I}_C\boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is the total angular velocity of the rotor referred to φ.

The rotor (frame 3) has angular velocity $\boldsymbol{\omega}_{32} = \dot{\theta}\mathbf{j}$ relative to the gimbal F (frame 2). The gimbal has an angular velocity $\boldsymbol{\omega}_{21} = \dot{\alpha}\mathbf{k}$ relative to the Earth (frame 1), whose angular velocity relative to the distant stars (frame 0) is $\boldsymbol{\omega}_{10} = \boldsymbol{\Omega} = \Omega\cos\lambda(\sin\alpha\mathbf{i} + \cos\alpha\mathbf{j}) + \Omega\sin\lambda\mathbf{k}$, all referred to $\varphi = \{C; \mathbf{i}_k\}$ in Fig. 10.12. Hence, the total angular velocity $\boldsymbol{\omega} \equiv \boldsymbol{\omega}_{30}$ of the rotor referred to φ is given by

$$\boldsymbol{\omega} = \Omega\cos\lambda\sin\alpha\mathbf{i} + (\dot{\theta} + \Omega\cos\lambda\cos\alpha)\mathbf{j} + (\dot{\alpha} + \Omega\sin\lambda)\mathbf{k}. \quad (10.81\text{b})$$

The angular velocity $\boldsymbol{\omega}_f = \boldsymbol{\omega}_{20}$ of the moving frame φ is

$$\boldsymbol{\omega}_f = \Omega\cos\lambda(\sin\alpha\mathbf{i} + \cos\alpha\mathbf{j}) + (\dot{\alpha} + \Omega\sin\lambda)\mathbf{k}. \quad (10.81\text{c})$$

Since the gimbal frame φ is a principal frame with $I_{11} = I_{33}$, use of (10.81b) in (10.62) gives the moment of momentum of the rotor about the fixed center of mass C, but referred to the moving frame φ:

$$\mathbf{h}_C = \mathbf{I}_C\boldsymbol{\omega} = I_{11}\Omega\cos\lambda\sin\alpha\mathbf{i} + I_{22}(\dot{\theta} + \gamma)\mathbf{j} + I_{11}(\dot{\alpha} + \Omega\sin\lambda)\mathbf{k}, \quad (10.81\text{d})$$

¶ Clearly, we have $\mathbf{h}_{rC} = \mathbf{I}_C\boldsymbol{\omega} = I_{jk}\omega_k\mathbf{i}_j = I'_{pq}\omega'_q\mathbf{i}'_p$, where the primed quantities are the components of $\mathbf{I}_C$ and $\boldsymbol{\omega}$ referred to the body frame ψ, and the unprimed components are referred to the gimbal frame φ. The inertia tensor components, as noted however, have the same values so that $[I_{jk}] = [I'_{jk}] = \operatorname{diag}\{I_{11}, I_{22}, I_{11}\}$, which are constant principal values in both frames.

wherein, for convenience,

$$\gamma \equiv \Omega \cos\lambda \cos\alpha. \tag{10.81e}$$

Caution: The vanishing torques in (10.81a) do not imply that the corresponding components of the moment of momentum vector are constants. (Why?)

The total torque on the rotor about C must satisfy Euler's general equation (10.64) or, equivalently, (10.49), the latter being more direct. From (10.81c) and (10.81d), we thereby obtain the total torque on the rotor about C, referred to the gimbal frame φ:

$$\begin{aligned}\mathbf{M}_C &= \{I_{11}\dot{\alpha}\gamma + (\dot{\alpha} + \Omega\sin\lambda)[I_{11}\gamma - I_{22}(\dot{\theta} + \gamma)]\}\mathbf{i} + I_{22}(\ddot{\theta} + \dot{\gamma})\mathbf{j}\\ &\quad + \{I_{11}\ddot{\alpha} + \Omega\cos\lambda\sin\alpha[I_{22}(\dot{\theta} + \gamma) - I_{11}\gamma]\}\mathbf{k}.\end{aligned} \tag{10.81f}$$

The zero torque conditions (10.81a) require

$$I_{22}(\dot{\theta} + \gamma) = k, \text{ a constant,} \tag{10.81g}$$

$$I_{11}\ddot{\alpha} + \Omega\cos\lambda\sin\alpha(k - I_{11}\gamma) = 0. \tag{10.81h}$$

Equation (10.81g) shows that the spin $s \equiv \dot{\theta} + \gamma$ is constant. With the aid of these results in (10.81f), the torque applied by the rotor bearing reaction forces, which prohibits rotation about the **i**-axis and thus maintains the planar motion of the rotor axis, is given by

$$\mathbf{M}_C = [I_{11}\dot{\alpha}\gamma + (\dot{\alpha} + \Omega\sin\lambda)(I_{11}\gamma - k)]\mathbf{i}. \tag{10.81i}$$

The effect of the Earth's rotation toward orienting the rotor axis along the true north direction is evident in (10.81h). The directional effect in α is lost altogether when $\Omega = 0$, for then (10.81h) shows that the initial disturbance produces a constant rotational speed $\dot{\alpha}$ about the vertical gimbal bearing axis.

Exercise 10.8. (a) Let $\mathbf{e}(t)$ be a unit vector fixed in frame φ having angular velocity $\boldsymbol{\omega}_f$. Show that, referred to φ, $\mathbf{M}_C \cdot \mathbf{e} = d(\mathbf{h}_C \cdot \mathbf{e})/dt + \boldsymbol{\omega}_f \times \mathbf{h}_C \cdot \mathbf{e}$; hence, in general, $\mathbf{M}_C \cdot \mathbf{e} = 0$ does not imply that $\mathbf{h}_C \cdot \mathbf{e}$ is constant, nor conversely. (b) Consider the matrices I_C and I'_C of $\mathbf{I}_C$ respectively referred to the gimbal frame $\varphi = \{C; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and the rotor frame $\psi = \{C; \mathbf{i}', \mathbf{j}, \mathbf{k}'\}$. Let $A = [\cos\langle \mathbf{i}'_k, \mathbf{i}_l\rangle]$, note that $I'_C = \operatorname{diag}\{I_{11}, I_{22}, I_{11}\}$, and apply the tensor transformation law to find I_C. □

10.13.2. Small Angular Displacement Solution

So far our model results are exact. Certainly, Ω is very small. So ignoring terms in Ω^2, we may simplify (10.81h) to obtain $\ddot{\alpha} + p^2\sin\alpha = 0$, the equation of motion of a simple pendulum with circular frequency given below. Depending on the initial data, it is possible, therefore, that the rotor axis may trace a circle in the horizontal plane. For small α and $\dot{\alpha}$, however, this reduces to the familiar equation

$$\ddot{\alpha} + p^2\alpha = 0, \qquad p^2 \equiv \frac{I_{22}\dot{\theta}\Omega\cos\lambda}{I_{11}}. \tag{10.81j}$$

Therefore, the motion $\alpha(t)$ is a small oscillation about the north direction. Although the Earth's rotational rate Ω is small, its product with a large angular speed $\dot{\theta}$ of the rotor can contribute significantly to the small angular rate $\dot{\alpha}$ of the instrument.

The effect of the Earth's rotation on the frequency, and hence the period of the motion, is evident in (10.81j). The frequency of the oscillations vanishes at the poles and is greatest at the equator. For illustration, introduce $I_{22} = 2I_{11}$ for a circular disk. Then the frequency of the small oscillation is given by $f = (1/2\pi)(2\dot{\theta}\Omega\cos\lambda)^{1/2}$; and for a gyrocompass having an angular speed of $10,000$ rpm, the smallest period of the oscillation is $\tau = 16.1$ sec at the equator $\lambda = 0$, and near the pole at $\lambda = 89.9°$, say, $\tau = 384.9$ sec.

Finally, for small α, (10.81e) reduces to $\gamma = \Omega\cos\lambda$, and (10.81g) may be written as $I_{22}(\dot{\theta} + \Omega\cos\lambda) = k$. With this relation and $I_{22} = 2I_{11}$, the torque (10.81i) supplied by the rotor bearings in the small angular motion is given by

$$\mathbf{M}_C = -2I_{11}\dot{\theta}(\dot{\alpha} + \Omega\sin\lambda)\mathbf{i}, \tag{10.81k}$$

to the first order in Ω. At the equator $\lambda = 0$, the *gyroscopic moment* is $\mathbf{M}_C = -2I_{11}\dot{\theta}\dot{\alpha}\mathbf{i}$, the value it has when the Earth's rotation is neglected. Notice that its magnitude is not symmetric at the poles $\lambda = \pm\pi/2$, being smallest at the south pole.

10.13.3. Small Angular Motion with Viscous Damping

Finally, let us suppose that a damping couple $M_3 = \mathbf{M}_C \cdot \mathbf{k} = -2I_{11}\nu\dot{\alpha}$, with damping exponent ν, is applied about the vertical $\mathbf{k}$-axis. We then find from (10.81f), with $I_{22} = 2I_{11}$, that the equation of motion for small α becomes

$$\ddot{\alpha} + 2\nu\dot{\alpha} + p^2\alpha = 0, \tag{10.81l}$$

where p^2 is defined in (10.81j). This is the differential equation (6.83) for damped vibrations. For light damping, $\nu < p$, the general solution has the form of (6.86h):

$$\alpha(t) = \alpha_0 e^{-\nu t}\cos(pt + \epsilon), \tag{10.81m}$$

in which α_0 and ϵ are constants. As $t \to \infty$ the oscillations die out in the manner illustrated in Fig. 6.21, page 154, and the gyroscope axis becomes fixed on the true north direction. The instrument thus performs the primary function of a compass.

10.14. Stability of the Torque-free, Steady Rotation About a Principal Axis

Euler's equations (10.66) show that a rotational motion with no applied torque about an admissible reference point Q, e.g., a "body" point fixed or in uniform motion in an inertial reference frame, or at the center of mass, may be possible when and only when the total angular velocity of the rigid body, referred to the principal body frame $\varphi = \{Q; \mathbf{e}_k\}$, satisfies the coupled system of homogeneous equations

$$\begin{aligned} I_{11}\dot{\omega}_1 + \omega_2\omega_3(I_{33} - I_{22}) &= 0, \\ I_{22}\dot{\omega}_2 + \omega_3\omega_1(I_{11} - I_{33}) &= 0, \\ I_{33}\dot{\omega}_3 + \omega_1\omega_2(I_{22} - I_{11}) &= 0. \end{aligned} \tag{10.82}$$

In particular, for an arbitrary rigid body having a constant angular velocity $\mathbf{\Omega} = \omega\mathbf{e}_1$ about the principal axis $\mathbf{e}_1$, say, (10.82) are identically satisfied. But this steady rotation about a principal body axis at Q may not be stable.

To investigate the infinitesimal stability of the torque-free, steady rotation of a rigid body about a principal axis, suppose the body has a constant spin $\mathbf{\Omega} = \omega\mathbf{e}_1$ and is then subjected to an arbitrary infinitesimal disturbance $\hat{\boldsymbol{\omega}}$ for which $|\hat{\boldsymbol{\omega}}| \ll |\mathbf{\Omega}|$. Then the total angular velocity of the perturbed body is

$$\boldsymbol{\omega} = \mathbf{\Omega} + \hat{\boldsymbol{\omega}}(t) = (\omega + \hat{\omega}_1)\mathbf{e}_1 + \hat{\omega}_2\mathbf{e}_2 + \hat{\omega}_3\mathbf{e}_3, \tag{10.83}$$

referred to the principal body frame $\varphi = \{Q; \mathbf{e}_k\}$. Substituting (10.83) into (10.82) and neglecting terms of the second order in the small components of $\hat{\boldsymbol{\omega}}$, we obtain a system of three linearized, homogeneous equations for the components $\hat{\omega}_k$. The first of these equations shows that $\hat{\omega}_1$ is constant. Since $\boldsymbol{\omega} = \mathbf{\Omega}$ initially, without loss of generality, we may take $\hat{\omega}_1 = 0$. The remaining two equations are

$$I_{22}\,\dot{\hat{\omega}}_2 + \omega\hat{\omega}_3(I_{11} - I_{33}) = 0, \tag{10.84}$$

$$I_{33}\,\dot{\hat{\omega}}_3 + \omega\hat{\omega}_2(I_{22} - I_{11}) = 0. \tag{10.85}$$

Differentiation of (10.84) with respect to time and use of (10.85) yields

$$\ddot{\hat{\omega}}_2 + p^2\hat{\omega}_2 = 0, \qquad p^2 \equiv \frac{\omega^2}{I_{22}I_{33}}(I_{11} - I_{33})(I_{11} - I_{22}). \tag{10.86}$$

The same equation in terms of $\hat{\omega}_3$ may be similarly derived starting with (10.85). Thus, we conclude from these equations that the perturbed rotational motion is bounded and simple harmonic with circular frequency p, and hence infinitesimally stable, if and only if $p^2 > 0$, that is, by the second equation in (10.86), if and only if

$$(I_{11} - I_{33})(I_{11} - I_{22}) > 0. \tag{10.87}$$

This holds provided that the principal moments of inertia satisfy

$$I_{11} > I_{33} \quad \text{and} \quad I_{11} > I_{22}, \qquad \text{or } I_{33} > I_{11} \quad \text{and} \quad I_{22} > I_{11}. \quad (10.88)$$

The first pair of these inequalities show that the principal axis of steady rotation $\mathbf{\Omega} = \Omega \mathbf{e}_1$ of the body must be the principal axis for which I_{11} is the greatest of the principal moments of inertia at Q, whereas the second pair show that I_{11} is the smallest of the principal moments of inertia at Q. *Therefore, the torque-free, steady rotation of a rigid body about a principal body axis corresponding to either the greatest or the smallest principal moment of inertia about an admissible reference point Q is infinitesimally stable.*

If, however, $I_{33} > I_{11} > I_{22}$, or $I_{22} > I_{11} > I_{33}$, the stability criterion (10.87) is violated and the perturbed angular motion, as shown by (10.86), grows exponentially in time. *Consequently, the torque-free, steady rotation of a rigid body is unstable when the axis of rotation is the principal axis corresponding to the intermediate-valued principal moment of inertia.*

10.14.1. An Illustration and an Experiment

The special case of a thin rod for which $I_{11} = 0$, the smallest possible principal moment of inertia about an admissible point Q, is universal, because the frequency coefficient in (10.86) is $p^2 = \omega^2$, independent of any other aspects of the body. Hence, the steady rotation of a thin rigid rod about its principal axes at Q, regardless of its other properties, is infinitesimally stable. Also, for an admissible point Q on the axis $\mathbf{i}_1$ of an axisymmetric body for which $I_{22} = I_{33}$, (10.88) holds for the principal axes at Q. In particular, the steady rotation about every principal axis at the center of mass of a body of revolution is infinitesimally stable.

The torque-free rotational instability of a rigid body may be demonstrated physically by carefully tossing a uniform rectangular block into the air while imparting to it a constant spin initially about one of its principal axes at its center of mass, as shown in Fig. 10.13. When the block is rotated about axes $\mathbf{e}_1$ and $\mathbf{e}_2$

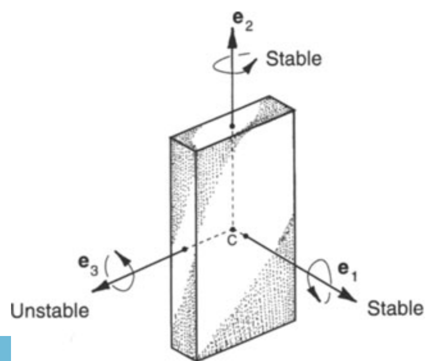


Figure 10.13. Stable and unstable principal axes of inertia for a homogeneous block.

of its smallest and largest principal moments of inertia at its center of mass, the block is observed to rotate in a fairly steady, stable manner. But when the block is set spinning about its intermediate axis $\mathbf{e}_3$ at its center of mass, it is observed that the block wobbles from its initial spin axis. See Problem 10.45.

10.14.2. Discussion of the General Torque-Free Steady Rotation of a Rigid Body

The problem of the stability of the torque-free motion of a rigid body concerns the steady rotation about a principal axis, which is one of several solutions of the homogeneous system (10.82). In particular, each of the three cases $\omega_k = \omega_0$, a constant, $\omega_i = \omega_j = 0$, for $i \neq j \neq k \neq i = 1, 2, 3$, is a solution of (10.82) corresponding to a steady rotation about a principal axis, not all of which need be stable. In fact, *the principal axes are the only axes about which a rigid body can sustain steady rotation under no torques*. To see this, consider a steady rotation with angular velocity $\boldsymbol{\omega}$ about a body point Q fixed in an inertial reference frame or at the center of mass. Then, in the absence of any external torques, Euler's equation (10.65) requires that $\boldsymbol{\omega} \times \mathbf{I}_Q\boldsymbol{\omega} = 0$, and hence $\mathbf{I}_Q\boldsymbol{\omega}$ must be parallel to $\boldsymbol{\omega}$, that is, $\mathbf{I}_Q\boldsymbol{\omega} = \alpha\boldsymbol{\omega}$, α being a scalar. This relation, in accordance with (9.63) or (9.69), shows that $\boldsymbol{\omega}$ must be parallel to a principal axis of the inertia tensor $\mathbf{I}_Q$, and α a corresponding principal value. Thus, a steady rotation about a principal axis is essentially the trivial solution of the homogeneous system (10.82). For a nontrivial rotation of a general rigid body under zero torque about a fixed point, the solution is more complicated, somewhat algebraically tedious, with results for the angular velocity components given in terms of Jacobian elliptic functions. Details may be found among resources listed in the references. A simpler example is provided in Problem 10.44, and an important special case is investigated later on.

10.15. Motion of a Billiard Ball

As a final example in this series of applications of Euler's laws for a rigid body, we study the general motion of a billiard ball of radius R and mass m, initially at rest and struck horizontally by a cue, as shown in Fig. 10.14. Subsequent to the impulse, the ball acquires an instantaneous center of mass velocity $\mathbf{v}_0^*$ and an angular velocity $\boldsymbol{\omega}_0$ in the inertial frame $\Phi = \{O; \mathbf{I}, \mathbf{J}, \mathbf{K}\}$. Nothing is specified about the location of the impact, so these initial values shall remain arbitrary. The ball subsequently slips and rolls on the horizontal surface. The objective is to describe the motion of the ball on both an ideally smooth and on a rough horizontal surface, and in the latter case to (i) determine the slip speed of the contact point of the ball at O, (ii) find the time τ required for slipping to end, (iii) describe the

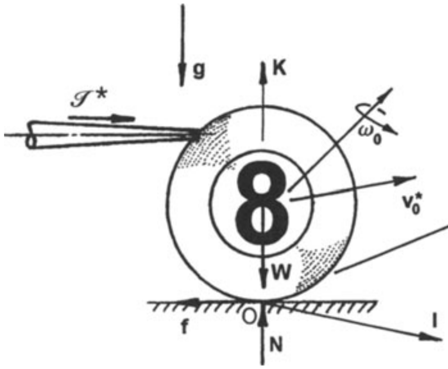


Figure 10.14. Slipping motion of a billiard ball.

motion of the center of mass during slip, and (iv) determine the angular velocity of the ball both during and after the slipping phase.

10.15.1. Motion on a Smooth Surface

First, consider the case when the surface is perfectly smooth. The forces acting on the ball following the instantaneous, horizontal impulsive action $\mathscr{I}^*$ are its weight $\mathbf{W} = -mg\mathbf{K}$ and the surface reaction forces $\mathbf{N} = N\mathbf{K}$ and $\mathbf{f} = -f\mathbf{I}$ shown in Fig. 10.14. All are finite forces that contribute nothing to the instantaneous impulse. There is no vertical motion of the ball, so $\mathbf{N} + \mathbf{W} = \mathbf{0}$, that is, $N = mg$. Therefore, if the surface is perfectly smooth, $\mathbf{f} = \mathbf{0}$ and the motion of the center of mass C is uniform with velocity $\mathbf{v}^* = \mathbf{v}_0^*$ in the inertial frame Φ. In view of (9.34) and, more generally, by Exercise 9.1, page 367, the moment of inertia tensor components for a sphere have the same values for every axis about the center of mass so that $\mathbf{I}_C = I_C\mathbf{1}$. There are no torques about the center of mass, so Euler's equation (10.65) with $\mathbf{M}_C = \mathbf{0}$ shows that $\dot{\boldsymbol{\omega}} = \mathbf{0}$; hence, $\boldsymbol{\omega} = \boldsymbol{\omega}_0$, a constant. Thus, without friction, the ball spins steadily about its initial axis of rotation and its center moves with constant velocity along a straight path in Φ.

10.15.2. Motion on a Rough Surface

Now suppose that the surface is rough with coefficient of dynamic friction ν, and at time t the center of mass has a horizontal velocity $\mathbf{v}^*$ and the ball is rotating with angular velocity $\boldsymbol{\omega}$. These vectors are unknown; in all there are five unknown scalar components. As before, $\mathbf{N} + \mathbf{W} = \mathbf{0}$, and for a sphere $\boldsymbol{\omega} \times \mathbf{I}_C\boldsymbol{\omega} = \boldsymbol{\omega} \times I_C\boldsymbol{\omega} = \mathbf{0}$. Then, with reference to Fig. 10.14, Euler's laws (10.26) and (10.65) in the inertial frame Φ require

$$m\dot{\mathbf{v}}^* = \mathbf{f}, \qquad I_C\dot{\boldsymbol{\omega}} = \mathbf{M}_C = -R\mathbf{K} \times \mathbf{f}. \tag{10.89a}$$

Solution of (i). We first find the slip speed of the contact point at O. We shall assume that the tangential frictional force $\mathbf{f}$ at the contact point O, so long as slip occurs, is determined by Coulomb's law of friction. This force acts in a direction opposite to the slip velocity $\mathbf{v}_s$ of the particle of the ball at O,

$$\mathbf{f} = \mathbf{f}_c = -\nu N\mathbf{v}_s/v_s = -\nu mg\mathbf{v}_s/v_s, \tag{10.89b}$$

where $v_s = |\mathbf{v}_s|$. The constraint equation for the absolute tangential slip velocity $\mathbf{v}_s$ of the body point in contact with the plane at O is $\mathbf{v}_s = \mathbf{v}^* + \boldsymbol{\omega} \times (-R\mathbf{K})$ in Φ, and hence $\dot{\mathbf{v}}_s = \dot{\mathbf{v}}^* - R\dot{\boldsymbol{\omega}} \times \mathbf{K}$. Substituting here both relations in (10.89a) for $\dot{\mathbf{v}}^*$ and $\dot{\boldsymbol{\omega}}$, noting that $\mathbf{f} \cdot \mathbf{K} = 0$, we obtain

$$\dot{\mathbf{v}}_s = \left(1 + \frac{mR^2}{I_C}\right)\frac{\mathbf{f}}{m}. \tag{10.89c}$$

Use of (10.89b) in (10.89c) gives a differential equation for $\mathbf{v}_s$,

$$\dot{\mathbf{v}}_s = -\left(1 + \frac{R^2}{R_C^2}\right)\nu g \cdot \frac{\mathbf{v}_s}{v_s}, \tag{10.89d}$$

where $I_C = mR_C^2$, R_C denoting the radius of gyration, is introduced. Now, write $\mathbf{v}_s = v_s\mathbf{e}_t$, where $\mathbf{e}_t$ is a unit vector tangent to the path of the sliding contact point on the plane. Then $\dot{\mathbf{v}}_s = \dot{v}_s\mathbf{e}_t + v_s\dot{\mathbf{e}}_t$ in Φ. In view of (10.89d), however, this shows that so long as slipping occurs $\dot{\mathbf{e}}_t = \mathbf{0}$, and hence $\mathbf{v}_s$ changes only in magnitude, not direction. We thus obtain an equation for the magnitude of $\dot{\mathbf{v}}_s$;

$$\dot{v}_s = -\nu g\left(1 + \frac{R^2}{R_C^2}\right), \text{ a constant.} \tag{10.89e}$$

Integration of (10.89e) with the initial slip speed $v_s(0) \equiv v_{0s} = |\mathbf{v}_0^* - R\boldsymbol{\omega}_0 \times \mathbf{K}|$ yields the slip speed at the contact point O,

$$v_s = v_{0s} - \nu gt\left(1 + \frac{R^2}{R_C^2}\right). \tag{10.89f}$$

Let the reader show that this result also follows easily from (10.89d) upon forming its scalar product with $\mathbf{v}_s$.

Solution of (ii). We now determine the time for slipping of the ball to end. By (10.89f), slipping ceases when $v_s = 0$ at time $t = \tau$:

$$\tau = \frac{v_{0s}}{\nu g}\left(1 + \frac{R^2}{R_C^2}\right)^{-1}. \tag{10.89g}$$

Subsequently, for time $t \geq \tau$, $\mathbf{v}^* + \boldsymbol{\omega} \times (-R\mathbf{K}) = \mathbf{0}$, the pure rolling constraint.

Hence, $\mathbf{f} = \mathbf{0}$ and the equations (10.89a) are satisfied by constant values of $\mathbf{v}^*$ and $\boldsymbol{\omega}$. The ball thus rolls without slipping, its center moving in a straight line with constant speed $v^* = R\omega$.

Solution of (iii). The motion of the center of mass during the slipping phase is described next. So long as slipping occurs, i.e. for $t < \tau$, $\mathbf{v}_s$ must have a constant, but unknown direction $\mathbf{e}_t$ that makes an angle θ with $\mathbf{I}$, say. The angle θ may be fixed by initial circumstances, currently unspecified. Therefore, it is convenient here to introduce an auxiliary inertial frame $\Psi = \{O; \mathbf{e}_t, \mathbf{j}, \mathbf{K}\}$, where $\mathbf{J} \cdot \mathbf{j} = \mathbf{I} \cdot \mathbf{e}_t = \cos\theta$. Thus, before slipping ceases, by (10.89b) and the first equation in (10.89a), $\dot{\mathbf{v}}^* = -\nu g \mathbf{e}_t$, a constant acceleration vector. It follows that the center of mass has the velocity $\mathbf{v}^* = -\nu g t \mathbf{e}_t + \mathbf{v}_0^*$, where $\mathbf{v}_0^*$ is its initial value. A second integration yields the motion of C; $\mathbf{x}^* = -\nu g t^2/2\mathbf{e}_t + \mathbf{v}_0^* t + \mathbf{x}_0^*$. With $\mathbf{x}_0^* = (0, 0, R)$ and with the initial velocity components $\mathbf{v}_0^* = (\dot{x}_0^*, \dot{y}_0^*, 0)$ in Ψ, the motion of the center of mass in Ψ is thus described by

$$x^* = -\tfrac{1}{2}\nu g t^2 + \dot{x}_0^* t, \qquad y^* = \dot{y}_0^* t, \quad z^* = R. \tag{10.89h}$$

Upon eliminating t, we find that during the slipping phase the path of the center of mass in Ψ is a parabola of the form $x^* = -A(y^*)^2 + By^*$, for $t < \tau$.

Solution of (iv). Finally, we determine the angular velocity of the ball in Ψ during and after the slipping phase. First, recall that during the slipping phase $\mathbf{f} = -\nu m g \mathbf{e}_t$, so the second of (10.89a) may be written as

$$\dot{\boldsymbol{\omega}} = \frac{\nu g R}{R_C^2}\mathbf{j}, \quad \text{a constant,} \tag{10.89i}$$

for all $t \le \tau$. Then the angular velocity of the ball during the slipping phase is given by

$$\boldsymbol{\omega}(t) = \boldsymbol{\omega}_0 + \frac{\nu g R}{R_C^2} t \mathbf{j}, \tag{10.89j}$$

wherein $\boldsymbol{\omega}_0 = \boldsymbol{\omega}(0) = \omega_{01}\mathbf{e}_t + \omega_{02}\mathbf{j} + \omega_{03}\mathbf{K}$ is the constant initial angular velocity of the ball in Ψ. Therefore, $\omega_1 = \omega_{01}$ and $\omega_3 = \omega_{03}$ are constants, and $\omega_2 = \omega_{02} + (\nu g R/R_C^2)t$ in Ψ. The spin (10.89j) continues until slipping ends at $t = \tau$ in (10.89g). The ball then begins to roll with angular velocity $\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}(\tau)$ given by

$$\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}_0 + \frac{v_{0s} R}{R^2 + R_C^2}\mathbf{j}. \tag{10.89k}$$

The foregoing results hold for any sphere for which the mass density varies only with the radial distance from its center. For a homogeneous sphere, by (9.34), $R_C^2 = 2R^2/5$; and, by (10.89g), independent of the physical properties of the sphere, $\tau = 2v_{0s}/7\nu g$ is the time at which slipping ceases. See Problems 10.46 and 10.47. □

Exercise 10.9. (i) Determine the height h above the horizontal plane at which a homogeneous billiard ball, initially at rest, should be struck by a horizontal impulsive force situated in the vertical plane through the center of mass (i.e., without “English”), so that the ball immediately will roll without slipping on a perfectly smooth surface. (ii) Suppose the ball is struck either above or below the height $h > R$, without “English.” Friction then acts on the ball in a direction parallel to the impulse. Find the initial values $\mathbf{v}_0^*$, $\boldsymbol{\omega}_0$, and $\mathbf{v}_{0s}$, and show that $\boldsymbol{\omega}(t) = [(H - R)v_0^* + \nu g R t]/R_C^2\mathbf{j}$ referred to Ψ. Hint: Apply (10.30) and (10.56). □

10.16. Kinetic Energy of a Body

The *(total) kinetic energy* $K(\mathscr{B}, t)$ of a body $\mathscr{B}$ in frame Φ is defined by

$$K(\mathscr{B}, t) \equiv \int_{\mathscr{B}} \frac{1}{2}\mathbf{v}(P, t) \cdot \mathbf{v}(P, t)\, dm(P), \tag{10.90}$$

where $\mathbf{v}(P, t)$ denotes the velocity of a particle of $\mathscr{B}$ in Φ. First, we relate the total kinetic energy of the body to the translational kinetic energy of its center of mass and the kinetic energy of the body relative to the center of mass. A general formula involving the translational motion of an arbitrary point and the motion relative to that point follows. Afterwards, a major equation for the kinetic energy of a rigid body is derived. The results are then illustrated in two examples.

10.16.1. Kinetic Energy Relative to the Center of Mass

The *kinetic energy* $K^*(\mathscr{B}, t)$ *of the center of mass* of $\mathscr{B}$ is defined by,

$$K^*(\mathscr{B}, t) \equiv \tfrac{1}{2} m(\mathscr{B})\mathbf{v}^*(\mathscr{B}, t) \cdot \mathbf{v}^*(\mathscr{B}, t), \tag{10.91}$$

$\mathbf{v}^*(\mathscr{B}, t)$ denoting the velocity of the center of mass. To relate (10.90) and (10.91) in frame Φ, introduce $\mathbf{v}(P, t) = \mathbf{v}^*(\mathscr{B}, t) + \dot{\boldsymbol{\rho}}(P, t)$ in (10.90), in accord with Fig. 10.5, page 423, and thus obtain

$$K(\mathscr{B}, t) = K^*(\mathscr{B}, t) + \mathbf{v}^* \cdot \int_{\mathscr{B}} \dot{\boldsymbol{\rho}}\, dm + \int_{\mathscr{B}} \frac{1}{2}\dot{\boldsymbol{\rho}} \cdot \dot{\boldsymbol{\rho}}\, dm. \tag{10.92}$$

The *kinetic energy of the body relative to the center of mass* is defined by

$$K_{rC}(\mathscr{B}, t) \equiv \int_{\mathscr{B}} \frac{1}{2}\dot{\boldsymbol{\rho}} \cdot \dot{\boldsymbol{\rho}}\, dm. \tag{10.93}$$

Thus, recalling (5.17), we find from (10.92) the major result:

$$K(\mathscr{B}, t) = K^*(\mathscr{B}, t) + K_{rC}(\mathscr{B}, t). \tag{10.94}$$

In words, *the total kinetic energy of a body is equal to the kinetic energy of its center of mass plus the kinetic energy of the body relative to the center of mass.*

10.16.2. Kinetic Energy Relative to an Arbitrary Point

It is useful to have a rule for the total kinetic energy in terms of the motion relative to an arbitrary point Q moving with velocity $\mathbf{v}_Q$ in Φ, as shown in Fig. 10.5. With $\mathbf{v}(P,t) = \mathbf{v}_Q + \dot{\mathbf{r}}(P,t)$ in (10.90), it follows that

$$K(\mathscr{B},t) = \tfrac{1}{2}m(\mathscr{B})\mathbf{v}_Q \cdot \mathbf{v}_Q + \mathbf{v}_Q \cdot m(\mathscr{B})\dot{\mathbf{r}}^*(\mathscr{B},t) + K_{rQ}(\mathscr{B},t), \quad (10.95)$$

where $\dot{\mathbf{r}}^*$ is the velocity of the center of mass relative to Q, and, by definition,

$$K_{rQ}(\mathscr{B},t) \equiv \int_{\mathscr{B}} \frac{1}{2}\dot{\mathbf{r}}(P,t)\cdot\dot{\mathbf{r}}(P,t)\,dm(P). \quad (10.96)$$

This is called the *kinetic energy of the body relative to Q*.

Consequently, when Q is fixed in Φ, the total kinetic energy of the body is equal to its kinetic energy relative to Q:

$$K(\mathscr{B},t) = K_{rQ}(\mathscr{B},t). \quad (10.97)$$

And when Q is the center of mass, (10.95) reduces to (10.94). Otherwise, there are no other simple descriptions of (10.95).

10.16.3. Kinetic Energy of a Rigid Body

So far, the foregoing results hold for a deformable body. Except for the primary definition (10.90), however, these results generally are useful only for a rigid body. For, in this case, the velocity of every body point relative to any assigned base point Q is well defined by $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$. Hence, with the invariance property of $\boldsymbol{\omega}(t)$ in mind, (10.96) may be written as

$$K_{rQ}(\mathscr{B},t) = \int_{\mathscr{B}} \frac{1}{2}\boldsymbol{\omega} \times \mathbf{r} \cdot \dot{\mathbf{r}}\,dm = \frac{1}{2}\boldsymbol{\omega} \cdot \int_{\mathscr{B}} \mathbf{r} \times \dot{\mathbf{r}}\,dm. \quad (10.98)$$

And introduction of (10.37) and (10.59) now yields the following major result for the *kinetic energy of a rigid body relative to an arbitrary base point Q*:

$$K_{rQ}(\mathscr{B},t) = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{h}_{rQ} = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}_Q\boldsymbol{\omega}. \quad (10.99)$$

This rule is especially useful when Q is either a fixed body point or the center of mass. By (10.97), when Q is a fixed body point, (10.99) gives the total kinetic energy of the body. When Q is the center of mass for which $\mathbf{h}_{rC} = \mathbf{h}_C$, (10.99) yields

$$K_{rC} = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{h}_C = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}_C\boldsymbol{\omega}. \quad (10.100)$$

Consequently, in view of (10.91) and (10.100), the principal rule (10.94) yields the following important relation for the *total kinetic energy of a rigid body in terms of center of mass data*:

$$K(\mathscr{B}, t) = \tfrac{1}{2} m \mathbf{v}^* \cdot \mathbf{v}^* + \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}_C \boldsymbol{\omega}. \tag{10.101}$$

The kinetic energy (10.99), of course, has the same value in every reference frame at Q. In a principal reference frame $\hat{\varphi} = \{Q; \hat{\mathbf{e}}_k\}$, however, (10.99) and similarly (10.100), reduces to the simple principal component expression,

$$K_{rQ} = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}_Q \boldsymbol{\omega} = \tfrac{1}{2}\hat{I}^Q_{11}\hat{\omega}_1^2 + \tfrac{1}{2}\hat{I}^Q_{22}\hat{\omega}_2^2 + \tfrac{1}{2}\hat{I}^Q_{33}\hat{\omega}_3^2. \tag{10.102}$$

In particular, for a rotation about a fixed principal axis $\hat{\mathbf{e}}_3$, say, with $\hat{\omega}_3 \equiv \omega$ and $I^Q_{33} \equiv I$, (10.102) yields the familiar elementary relation $K_{rQ} = \frac{1}{2} I \omega^2$ for the kinetic energy relative to Q.

Example 10.11. (i) Apply (10.95) to determine the total kinetic energy of the connecting rod of the simple machine shown in Fig. 10.6, page 425. (ii) Repeat the calculation based on (10.94).

Solution of (i). Since Q in Fig. 10.6 is a moving base point, the total kinetic energy of the rod is obtained from (10.95). With $\boldsymbol{\omega} = (\dot{\beta} + \Omega)\mathbf{k}$ by (10.41b) and $\mathbf{h}_{rQ}$ in (10.41d), (10.99) yields the kinetic energy of the rod relative to Q:

$$K_{rQ}(\mathscr{B}, t) = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{h}_{rQ} = \frac{m\ell^2}{6}(\dot{\beta} + \Omega)^2. \tag{10.103a}$$

The same result also follows easily from (10.102). With $\dot{\mathbf{r}}^* = \boldsymbol{\omega} \times \mathbf{r}^* = \dfrac{\ell}{2}(\dot{\beta} + \Omega)\mathbf{j}$ and use of (10.41e) and (10.103a), (10.95) yields the total kinetic energy of the connecting rod:

$$K(\mathscr{B}, t) = \frac{1}{2} m \left[a^2\Omega^2 + \ell a \Omega(\dot{\beta} + \Omega)\cos\beta + \frac{\ell^2}{3}(\dot{\beta} + \Omega)^2 \right]. \tag{10.103b}$$

Solution of (ii). The solution based on (10.94) is simpler to construct. The kinetic energy of the center of mass of the rod is given by (10.91). Thus, recalling relations introduced above, we obtain

$$\mathbf{v}^* = \mathbf{v}_Q + \boldsymbol{\omega} \times \mathbf{r}^* = a\Omega\mathbf{b} + \frac{\ell}{2}(\dot{\beta} + \Omega)\mathbf{j}, \tag{10.103c}$$

and hence

$$K^*(\mathscr{B}, t) = \frac{1}{2} m \left[a^2\Omega^2 + a\Omega\ell\cos\beta(\dot{\beta} + \Omega) + \frac{\ell^2}{4}(\dot{\beta} + \Omega)^2 \right]. \tag{10.103d}$$

With the aid of (9.28) referred to the principal basis at the center of mass of the rod, (10.102) yields easily the kinetic energy relative to C,

$$K_{rC}(\mathscr{B}, t) = \frac{m\ell^2}{24}(\dot{\beta} + \Omega)^2. \tag{10.103e}$$

Now (10.94) leads again to (10.103b). The reader will appreciate the simplicity of the mnemonic rule (10.94) and the consequent result (10.101) for the moving center of mass, compared with (10.95) for an arbitrary moving point Q. □

Example 10.12. At an instant of interest t_0, the center of mass of a rigid body of mass $m = 20$ kg has an intrinsic velocity $\mathbf{v}^* = 25\mathbf{t}$ m/sec, the moment of momentum about the center of mass is $\mathbf{h}_C = 1100\mathbf{e}_1 - 500\mathbf{e}_2 + 600\mathbf{e}_3$ kg $\cdot$ m^2/sec, and the moment of inertia tensor is $\mathbf{I}_C = 25(\mathbf{e}_{11} + \mathbf{e}_{22}) - 15(\mathbf{e}_{12} + \mathbf{e}_{21}) + 30\mathbf{e}_{33}$ kg $\cdot$ m^2, referred to a body reference frame $\varphi = \{C; \mathbf{e}_k\}$. Find the total kinetic energy of the body.

Solution. The total kinetic energy of the body at the moment of concern may be found from (10.101). An easy calculation gives $K^* = \frac{1}{2}m\mathbf{v}^* \cdot \mathbf{v}^* = 6250\,\mathrm{N \cdot m}$, the kinetic energy of the center of mass. To find $K_{rC} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{h}_C$ in (10.100), bearing in mind that $\mathbf{h}_C$ is given, we must first find $\boldsymbol{\omega} = \omega_k \mathbf{e}_k$ such that $\mathbf{h}_C = \mathbf{I}_C\boldsymbol{\omega}$. For the assigned moment of inertia tensor, the moment of momentum relative to C is given by $\mathbf{h}_C = (25\omega_1 - 15\omega_2)\mathbf{e}_1 + (-15\omega_1 + 25\omega_2)\mathbf{e}_2 + 30\omega_3\mathbf{e}_3$. Equating these scalar components with those given at t_0, we find three equations for the components ω_k:

$$25\omega_1 - 15\omega_2 = 1100, \qquad -15\omega_1 + 25\omega_2 = -500, \qquad 30\omega_3 = 600.$$

These yield $\boldsymbol{\omega} = 50\mathbf{e}_1 + 10\mathbf{e}_2 + 20\mathbf{e}_3$ rad/sec . Hence, with the assigned value for $\mathbf{h}_C$, the kinetic energy relative to the center of mass is $K_{rC} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{h}_C = 31{,}000$ N $\cdot$ m. The total kinetic energy of the body in the inertial frame now follows from (10.101) or (10.94): $K = 37{,}250$ N $\cdot$ m. □

10.17. Torque-Free Rotation of an Axisymmetric Body

The steady rotation about a *principal axis* of a rigid body studied earlier is the simplest of the torque-free rotation problems. Here we extend this study to another simple class of torque-free problems of a general homogeneous and axisymmetric rigid body having a fixed point and rotating about an *arbitrary axis*. We shall see that the kinetic energy and the moment of momentum about the fixed point are central to the general solution.

Consider a homogeneous axisymmetric rigid body with at least two identical orthogonal planes of symmetry and having an angular velocity $\boldsymbol{\omega}$ about an axial body point Q fixed in the inertial frame. This includes a rectangular solid with square cross section and every body of revolution, for example. Then the frame $\varphi = \{Q; \mathbf{e}_k\}$ with $\mathbf{e}_3$ along the axis of symmetry is a principal body frame with respect to which $I_{11}^Q = I_{22}^Q \equiv I_{11}$, and hence Euler's equations (10.82) for the torque-free angular motion of the body about an arbitrary axis may be written as

$$\begin{aligned} I_{11}\dot{\omega}_1 + \omega_2\omega_3(I_{33} - I_{11}) &= 0, \\ I_{11}\dot{\omega}_2 - \omega_3\omega_1(I_{33} - I_{11}) &= 0, \\ I_{33}\dot{\omega}_3 &= 0. \end{aligned} \tag{10.104}$$

Therefore, $\omega_3 = \omega_0$, a constant, and the remaining equations reduce to

$$\dot{\omega}_1 + p\omega_2 = 0, \quad \dot{\omega}_2 - p\omega_1 = 0, \text{ with } p \equiv \frac{I_{33} - I_{11}}{I_{11}}\omega_0, \text{ a constant.} \tag{10.105}$$

Multiply the second equation in (10.105) by $i = \sqrt{-1}$ and add the result to the first equation to obtain

$$\frac{d}{dt}(\omega_1 + i\omega_2) - ip(\omega_1 + i\omega_2) = 0, \tag{10.106}$$

whose general solution, with the aid of Euler's identity (6.49), is

$$\omega_1 + i\omega_2 = Ae^{ipt} = A(\cos pt + i \sin pt), \tag{10.107}$$

A being a constant of integration. Hence, the three angular velocity components are

$$\omega_1 = A\cos pt, \qquad \omega_2 = A\sin pt, \qquad \omega_3 = \omega_0. \tag{10.108}$$

The constants A and ω_0 may be expressed in terms of the total kinetic energy $K = K_{rQ}$, the squared magnitude $h_Q^2 = h_{rQ}^2$ of the moment of momentum relative to Q, and the principal moments of inertia. The total kinetic energy is given by (10.102). Discarding the circumflex notation from here on and using (10.108) in (10.102), we find

$$K = \tfrac{1}{2}I_{11}A^2 + \tfrac{1}{2}I_{33}\omega_0^2, \quad \text{a constant.} \tag{10.109}$$

The squared magnitude of the moment of momentum, namely, $h_Q^2 = \mathbf{h}_{rQ} \cdot \mathbf{h}_{rQ}$ is provided by (10.62); $h_Q^2 = I_{11}^2\omega_1^2 + I_{22}^2\omega_2^2 + I_{33}^2\omega_3^2$. Hence, by (10.108), we have

$$h_Q^2 = I_{11}^2A^2 + I_{33}^2\omega_0^2, \quad \text{a constant.} \tag{10.110}$$

Therefore, the total kinetic energy and the magnitude of the moment of momentum about Q are constants of the motion, and (10.109) and (10.110) thus yield

$$\omega_0 = \sqrt{\frac{h_Q^2 - 2KI_{11}}{I_{33}(I_{33} - I_{11})}}, \qquad A = \sqrt{\frac{2KI_{33} - h_Q^2}{I_{11}(I_{33} - I_{11})}}. \tag{10.111}$$

Use of these in (10.108) completes the solution of the problem of the torque-free rotational motion about a fixed axial point of a homogeneous axisymmetric rigid body with $I_{11}^{Q} = I_{22}^{Q}$.

10.18. Mechanical Power and Work

Let $d\mathbf{F}(P, t)$ be the elemental force acting at a particle P of a body $\mathscr{B}$. Both contact and body forces are included. The *total mechanical power for the body is the rate of working of the force,* defined by

$$\mathscr{P}(\mathscr{B}, t) = \frac{d\mathscr{W}(\mathscr{B}, t)}{dt} \equiv \int_{\mathscr{B}} \mathbf{v}(P, t) \cdot d\mathbf{F}(P, t). \tag{10.112}$$

Consider a rigid body and let $\mathbf{x}_Q(P, t)$ be the position vector of P from an arbitrary base point Q. Now use $\mathbf{v}(P, t) = \mathbf{v}_Q(t) + \boldsymbol{\omega}(t) \times \mathbf{x}_Q(P, t)$ in (10.112), recall (10.9) and (10.10), note that both $\mathbf{v}_Q$ and $\boldsymbol{\omega}$ depend on time alone, and thus deduce the *total mechanical power for a rigid body*:

$$\mathscr{P}(\mathscr{B}, t) = \frac{d\mathscr{W}(\mathscr{B}, t)}{dt} = \mathbf{F}(\mathscr{B}, t) \cdot \mathbf{v}_Q(t) + \mathbf{M}_Q(\mathscr{B}, t) \cdot \boldsymbol{\omega}(t). \tag{10.113}$$

Since the mutual internal forces and torques are equipollent to zero, (10.113) involves only the total of the external forces and their total moment about Q. The first term on the right-hand side in (10.113) is the power $\mathscr{P}_t(\mathscr{B}, t) \equiv d\mathscr{W}_t(\mathscr{B}, t)/dt$ expended in a pure translation for which $\boldsymbol{\omega} = \mathbf{0}$; it is the rate at which work is done by the total force in translation of the body while acting at Q. The last term in (10.113) is the power $\mathscr{P}_r(\mathscr{B}, t) \equiv d\mathscr{W}_r(\mathscr{B}, t)/dt$ expended in a pure rotation about a fixed point Q for which $\mathbf{v}_Q = \mathbf{0}$; it is the rate at which work is done by the torque in turning the body about Q. Therefore, the total mechanical power for a rigid body is equal to the sum of the translational power and the rotational power: $\mathscr{P}(\mathscr{B}, t) = \mathscr{P}_t(\mathscr{B}, t) + \mathscr{P}_r(\mathscr{B}, t)$. *In consequence, the total work done by the forces and torques that act on the body during the time interval* $[t_o, t]$ *is the sum of the translational work* $\mathscr{W}_t(\mathscr{B}, t)$ *and the rotational work* $\mathscr{W}_r(\mathscr{B}, t)$:

$$\mathscr{W}(\mathscr{B}, t) = \mathscr{W}_t(\mathscr{B}, t) + \mathscr{W}_r(\mathscr{B}, t), \tag{10.114}$$

wherein

$$\mathscr{W}_t(\mathscr{B}, t) \equiv \int_{t_0}^{t} \mathbf{F}(\mathscr{B}, t) \cdot \mathbf{v}_Q(t)\, dt, \qquad \mathscr{W}_r(\mathscr{B}, t) \equiv \int_{t_0}^{t} \mathbf{M}_Q(\mathscr{B}, t) \cdot \boldsymbol{\omega}(t)\, dt. \tag{10.115}$$

Clearly, a force perpendicular to the trajectory of point Q does no translational work, and hence no translational power is expended in the motion. Similarly, a torque perpendicular to the axis of rotation does no rotational work, so it contributes nothing to the rotational power expended in the motion.

10.19. The Work–Energy Principle

Euler's first law of motion has a general first integral of the work–energy type (7.36) expressed in terms of the center of mass particle. To see this, start with (10.26), retrace the steps leading to (7.36), introduce the first relation in (10.115) applied to the center of mass, and thus derive the *work–energy principle for the translational motion of a rigid body*:

$$\mathscr{W}_t^*(\mathscr{B}, t) = \int_{t_0}^{t} \mathbf{F}(\mathscr{B}, t) \cdot \mathbf{v}^*(\mathscr{B}, t)\, dt = \Delta K^*(\mathscr{B}, t). \tag{10.116}$$

In words, *the translational work done by the total external force acting on a rigid body at its center of mass is equal to the change in the kinetic energy of its center of mass particle. Consequently, the translational power expended is equal to the time rate of change of the kinetic energy of the center of mass particle*: $\mathscr{P}_t^*(\mathscr{B}, t) \equiv d\mathscr{W}_t^*(\mathscr{B}, t)/dt = dK^*(\mathscr{B}, t)/dt$.

A first integral of Euler's second law of motion for a rigid body, referred to a body reference frame, is derived similarly. Form the scalar product of (10.65) with $\boldsymbol{\omega}(t)$, recall (3.42) and the symmetry of $\mathbf{I}_Q$, and use (10.99) and the second relation in (10.115) to reach the *work–energy principle for the rotational motion of a rigid body*:

$$\mathscr{W}_r(\mathscr{B}, t) = \int_{t_0}^{t} \mathbf{M}_Q(\mathscr{B}, t) \cdot \boldsymbol{\omega}(t) dt = \Delta K_{rQ}(\mathscr{B}, t). \tag{10.117}$$

In summary, *the rotational work done by the total external torque acting on a rigid body in turning it about a base point Q is equal to the change in its kinetic energy relative to Q. Therefore, the rotational power expended is equal to the time rate of change of the kinetic energy relative to Q, as though point Q were fixed*: $\mathscr{P}_r(\mathscr{B}, t) \equiv d\mathscr{W}_r(\mathscr{B}, t)/dt = dK_{rQ}(\mathscr{B}, t)/dt$.

Exercise 10.10. Work through the details leading to (10.117) starting with (10.65), and then with (10.51). □

There are two important special cases to consider for Q, namely, when Q is a fixed point and when Q is the center of mass. First, suppose that Q is a fixed point. Then, by (10.97), $K_{rQ}(\mathscr{B}, t) = K(\mathscr{B}, t) = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}_Q\boldsymbol{\omega}$ is the total kinetic energy of the body, and from (10.114), $\mathscr{W}_r(\mathscr{B}, t) = \mathscr{W}(\mathscr{B}, t)$ is the total work done. Therefore, by (10.117), the *work–energy equation for a rigid body having a fixed point Q in an inertial frame* is

$$\mathscr{W}(\mathscr{B}, t) = \int_{t_0}^{t} \mathbf{M}_Q(\mathscr{B}, t) \cdot \boldsymbol{\omega}(t)\, dt = \Delta K(\mathscr{B}, t). \tag{10.118}$$

Now suppose that Q is the center of mass. Then the total kinetic energy of the body is given by (10.94), and from the sum of (10.116) and (10.117) the result

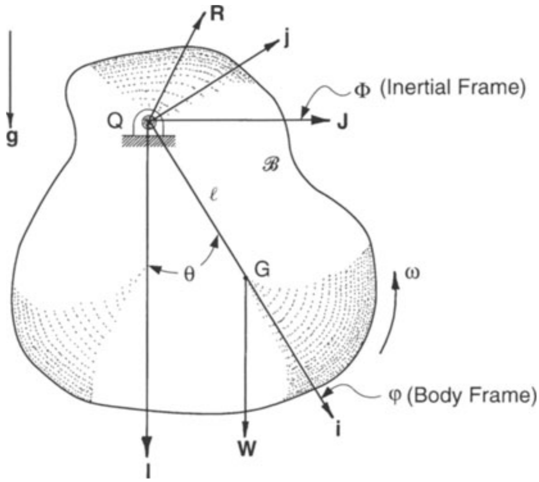


Figure 10.15. Oscillation of a physical pendulum.

(10.114) delivers the fundamental *work–energy equation for a general rigid body motion*:

$$\mathscr{W}(\mathscr{B}, t) = \int_{t_0}^{t} \mathbf{F}(\mathscr{B}, t) \cdot \mathbf{v}^*(\mathscr{B}, t)\, dt + \int_{t_0}^{t} \mathbf{M}_C(\mathscr{B}, t) \cdot \boldsymbol{\omega}(t)\, dt = \Delta K(\mathscr{B}, t). \tag{10.119}$$

Here $K(\mathscr{B}, t)$ has the explicit form (10.101). The principal results (10.118) and (10.119) are summarized in the following rule.

Work–energy principle: *The total work done over the time interval* $[t_0, t]$ *by the external forces and torques acting on a rigid body about a fixed body point* Q *or about the center of mass, in an inertial frame* Φ*, is equal to the change in the total kinetic energy of the body*:

$$\mathscr{W}(\mathscr{B}, t) = \Delta K(\mathscr{B}, t). \tag{10.120}$$

Hence, the corresponding mechanical power is equal to the time rate of change in Φ *of the total kinetic energy*:

$$\mathscr{P}(\mathscr{B}, t) = \frac{d\mathscr{W}(\mathscr{B}, t)}{dt} = \frac{dK(\mathscr{B}, t)}{dt}. \tag{10.121}$$

Example 10.13. A *physical pendulum* shown in Fig. 10.15 swings in its vertical plane of symmetry about a smooth, fixed axle at Q. The pendulum is released from rest at the placement θ_0 with angular speed ω_0. (i) Apply the work–energy principle to derive the equation of motion of the pendulum. Confirm the solution by application of Euler's equation. (ii) Describe the general solution, find the small amplitude circular frequency of the oscillation, and describe a simple pendulum having the same period. (iii) What is the mechanical power expended over the interval $[0, t]$ as a function of θ?

Solution of (i). The idea here is to obtain the work–energy equation for a pivoted rigid body, and then derive from this first integral the differential equation of motion for $\theta(t)$. The center of gravity G of the pendulum is at a distance ℓ from Q, and hence the rotational work done by the torque in turning the pendulum about the smooth hinge Q is determined by (10.118) in which $\boldsymbol{\omega}(t) = \dot{\theta}(t)\mathbf{k}$ and $\mathbf{M}_Q(\mathscr{B}, t) = M_1\mathbf{i} + M_2\mathbf{j} + M_3\mathbf{k} - mg\ell\sin\theta\mathbf{k}$, where M_1 and M_2 are unknown bearing reaction torques, which are workless. Because the hinge is smooth, the bearing reaction component $M_3 = 0$ and the support reaction force $\mathbf{R}$ is workless. Hence, with $\theta(0) = \theta_0$ at $t_0 = 0$,

$$\mathscr{W} = \int_0^t -mg\ell\sin\theta\dot{\theta}\,dt = -mg\ell\int_{\theta_0}^{\theta}\sin\theta\,d\theta. \tag{10.122a}$$

This yields the rotational work done by the applied loads as

$$\mathscr{W} = mg\ell\,(\cos\theta - \cos\theta_0). \tag{10.122b}$$

Notice that this is just the gravitational work done: $\mathscr{W}_g = mgh = mg\ell\,(\cos\theta - \cos\theta_0)$.

We choose a body frame $\varphi = \{Q; \mathbf{i}_k\}$ in the vertical plane of symmetry, as shown in Fig. 10.15, so that $\mathbf{k}$ is a fixed principal axis with $I_{31}^Q = I_{32}^Q = 0$. Then the first relation in (10.75) gives $\mathbf{h}_{rQ} = \mathbf{I}_Q\boldsymbol{\omega} = I\dot{\theta}\mathbf{k}$, and hence the rotational kinetic energy for the body is

$$K(\mathscr{B}, t) = \tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbf{h}_{rQ} = \tfrac{1}{2}I\dot{\theta}^2. \tag{10.122c}$$

With $\dot{\theta}(0) = \omega_0$ at $t_0 = 0$, the change in the kinetic energy is $\Delta K(\mathscr{B}, t) = \frac{1}{2}I[\dot{\theta}^2 - \omega_0^2]$; and, with (10.122b), the work–energy principle (10.118) yields

$$mg\ell\,(\cos\theta - \cos\theta_0) = \tfrac{1}{2}I\left[\dot{\theta}^2 - \omega_0^2\right]. \tag{10.122d}$$

The equation for the motion $\theta(t)$ of the physical pendulum follows by differentiation of (10.122d) with respect to θ:

$$I\ddot{\theta} + mg\ell\sin\theta = 0. \tag{10.122e}$$

This result is readily confirmed by use of Euler's law through the second relation in (10.75). This yields

$$\mathbf{M}_Q(\mathscr{B}, t) = \mathbf{I}_Q\dot{\boldsymbol{\omega}}(t) = I\ddot{\theta}\mathbf{k}, \tag{10.122f}$$

where $\mathbf{M}_Q(\mathscr{B}, t) = M_1\mathbf{i} + M_2\mathbf{j} - mg\ell\sin\theta\mathbf{k}$. We thus recover (10.122e), and find also that the reaction torques vanish: $M_1 = M_2 = 0$. These are the torques identified in (10.72). They vanish because $\mathbf{k}$ is a principal axis.

Solution of (ii). Equation (10.122e) has the same form as (6.67b) for the simple pendulum. Therefore, its exact solution may be expressed in terms of an elliptic integral of the first kind or in terms of a Jacobian elliptic function, as shown in Section 7.10. By (10.122e), the small amplitude motion is described by $\ddot{\theta} + p^2\theta = 0$ with circular frequency $p = (mg\ell/I)^{1/2}$ and period $\tau = 2\pi(I/mg\ell\,)^{1/2}$. The length ℓ_s of a simple pendulum having the same period is $\ell_s = I/m\ell = R^2/\ell$, R denoting the radius of gyration.

Solution of (iii). The mechanical (rotational) power expended is given by (10.121). Hence, by (10.122b), $\mathscr{P}(\mathscr{B}, t) = d\mathscr{W}(\mathscr{B}, t)/dt = -mg\ell\dot{\theta}\sin\theta$. Alternatively, by (10.122c), $\mathscr{P}(\mathscr{B}, t) = dK(\mathscr{B}, t)/dt = I\dot{\theta}\ddot{\theta}$ and use of (10.122e) yields the same result. So, the power expended during $[0, t]$ is given by $\Delta\mathscr{P} \equiv \mathscr{P}(\mathscr{B}, t) - \mathscr{P}(\mathscr{B}, 0) = -mg\ell(\dot{\theta}\sin\theta - \omega_0\sin\theta_0)$; and with (10.122d), we thus find, as a function of $\theta(t)$, that

$$\Delta\mathscr{P} = mg\ell\left(\omega_0\sin\theta_0 \mp \sin\theta\sqrt{2p^2(\cos\theta - \cos\theta_0) + \omega_0^2}\right), \qquad (10.122g)$$

where the sign is chosen accordingly as $\dot{\theta}$ is increasing $(-)$ or decreasing $(+)$ in time. □

Exercise 10.11. Find the bearing reaction force $\mathbf{R}(\theta)$ at the hinge Q. □

10.20. Potential Energy

We now consider the work done on a rigid body by conservative forces and relate this to the total potential energy. Let σ denote the mass, volume, area, or length parameter for a rigid body so that $d\sigma(P)$ is the corresponding time invariant, elemental material entity. Further, let $\mathbf{f}(P, t)$ denote the applied force per unit σ and write the elemental force on a material parcel as $d\mathbf{F}(P, t) = \mathbf{f}(P, t)d\sigma(P)$. By (10.112), the rate of working of the force distribution is then given by

$$\frac{d\mathscr{W}(\mathscr{B}, t)}{dt} = \int_{\mathscr{B}} \mathbf{v}(P, t) \cdot \mathbf{f}(P, t)\, d\sigma(P). \qquad (10.123)$$

Suppose that the applied force density varies only with position $\mathbf{x} = \mathbf{x}(P, t)$ over the body. Then $\mathbf{f}(\mathbf{x})$ is conservative if and only if there exists a potential energy density function $\psi(\mathbf{x})$ per unit σ such that

$$\mathbf{f}(\mathbf{x}) = -\boldsymbol{\nabla}\psi(\mathbf{x}). \qquad (10.124)$$

It follows that

$$\mathbf{v} \cdot \mathbf{f} = -\boldsymbol{\nabla}\psi \cdot \mathbf{v} = -\frac{\partial\psi}{\partial\mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} = -\frac{d\psi(\mathbf{x})}{dt}, \qquad (10.125)$$

and hence (10.123) yields the *rate of working* $d\mathscr{W}_C(\mathscr{B}, t)/dt$ *by the distribution of conservative applied force*:

$$\frac{d\mathscr{W}_C}{dt} = -\frac{d}{dt}\int_{\mathscr{B}} \psi(\mathbf{x})\, d\sigma(P), \tag{10.126}$$

in which time dependence is implicit in $\mathbf{x}(P, t)$. Introducing the *total potential energy* of the body $\mathscr{B}$ defined by

$$V(\mathscr{B}) \equiv \int_{\mathscr{B}} \psi(\mathbf{x})\, d\sigma(P), \tag{10.127}$$

and integrating (10.126), we deduce the familiar basic rule

$$\mathscr{W}_C(\mathscr{B}) = -\Delta V(\mathscr{B}). \tag{10.128}$$

In sum, *the total work done by a conservative force distribution acting on a rigid body is equal to the decrease in the total potential energy.*

Example 10.14. *Gravitational potential energy.* Determine the total gravitational potential energy of a body $\mathscr{B}$ in a uniform gravitational field of strength $\mathbf{g}$ per unit mass σ of $\mathscr{B}$.

Solution. The constant gravitational force per unit mass is a conservative force distribution given by

$$\mathbf{f} \equiv \mathbf{g} = -\nabla\psi, \tag{10.129a}$$

in which ψ is the gravitational potential energy density per unit mass σ. Form the scalar product $\mathbf{g}\cdot d\mathbf{x} = -\nabla\psi\cdot d\mathbf{x} = -d\psi$, which is equivalent to (10.125), and integrate this equation over the path traversed by the body point at $\mathbf{x}$ to obtain the potential energy $\psi(\mathbf{x})$ per unit mass,

$$\psi(\mathbf{x}) = -\mathbf{g}\cdot\Delta\mathbf{x}, \tag{10.129b}$$

where $\Delta\mathbf{x} \equiv \mathbf{x}(P, t) - \mathbf{x}(P, t_0)$. Then for a uniform gravitational field strength $\mathbf{g}$ and with $\sigma \equiv m$ in (10.127), we obtain

$$V(\mathscr{B}) = \int_{\mathscr{B}} \psi(\mathbf{x})\, dm = -\mathbf{g}\cdot\int_{\mathscr{B}} \Delta\mathbf{x}\, dm = -\mathbf{g}\cdot m\Delta\mathbf{x}^*, \tag{10.129c}$$

where, from (5.12), $\Delta\mathbf{x}^*$ is the displacement vector of the center of mass of $\mathscr{B}$. With $\mathbf{g} = -g\mathbf{k}$, this delivers $V = mgh$ in which $h \equiv \mathbf{k}\cdot\Delta\mathbf{x}^* = \Delta z^*$ is the vertical displacement of the center of mass, a rule similar to the familiar particle relation (7.60). *Thus, the total gravitational potential energy of a rigid body in a uniform gravitational field is equal to the gravitational potential energy of its center of mass particle.* □

10.21. The General Energy Principle

The foregoing results on work and energy are now assembled in an alternative general form of the work–energy principle. First, separate the total elemental force $d\mathbf{F}(P,t)$ into the sum of its conservative part $d\mathbf{F}_C(P,t)$ and its nonconservative part $d\mathbf{F}_N(P,t)$. Then the work, $\mathscr{W}(\mathscr{B},t) = \int_{t_0}^{t}(\int_{\mathscr{B}} \mathbf{v}(P,t)\cdot d\mathbf{F}(P,t))dt$, done by the total force acting on a rigid body is the sum of the work $\mathscr{W}_C(\mathscr{B},t)$ done by the conservative part of the force and the work $\mathscr{W}_N(\mathscr{B},t)$ done by the nonconservative part: $\mathscr{W}(\mathscr{B},t) = \mathscr{W}_C(\mathscr{B},t) + \mathscr{W}_N(\mathscr{B},t)$. Therefore, in view of (10.128),

$$\mathscr{W}(\mathscr{B},t) = -\Delta V(\mathscr{B}) + \mathscr{W}_N(\mathscr{B},t), \tag{10.130}$$

where $\mathscr{W}_N(\mathscr{B},t)$ is determined by (10.114) for the nonconservative part of the force, and the potential energy is at most an implicit function of time. Recalling the work–energy principle (10.120) and introducing the *total energy*, $\mathscr{E}(\mathscr{B},t) \equiv K(\mathscr{B},t) + V(\mathscr{B})$, we have the following important first integral for a rigid body.

General energy principle: *Let the motion be referred to a body reference frame $\varphi = \{Q; \mathbf{e}_k\}$ at the center of mass or at a base point Q fixed in an inertial reference frame. Then the change in the total energy of a rigid body is equal to the work done by the nonconservative part of the force*:

$$\Delta\mathscr{E} = \mathscr{W}_N(\mathscr{B},t). \tag{10.131}$$

This leads at once to the following useful corollary, referred to $\varphi = \{Q; \mathbf{e}_k\}$.

Principle of conservation of energy: *The total energy of a rigid body is constant if and only if the nonconservative part of the force does no work in the motion or, trivially, when the total force is conservative*:

$$K(\mathscr{B},t) + V(\mathscr{B}) = E, \text{ a constant.} \tag{10.132}$$

Example 10.15. Apply the energy principle to derive the first integral of the equation of motion for the physical pendulum in Fig. 10.15, page 467.

Solution. The bearing reaction force $\mathbf{R}$ at the smooth support Q in Fig. 10.15 is workless; the bearing reaction torque $\boldsymbol{\mu}_Q \equiv M_1\mathbf{i} + M_2\mathbf{j}$, because there is no rotation of the body about these directions, also is workless, in fact $\boldsymbol{\mu}_Q = \mathbf{0}$; and the gravitational force $\mathbf{W}$ is conservative with potential energy $V = mg\ell\,(1 - \cos\theta)$. Clearly, the system is conservative and (10.132) holds. The rotational kinetic energy is given by $K_{rQ} = \frac{1}{2}I\omega^2$, where $I \equiv I_{33}^{Q}$. With $\omega = \dot\theta$, (10.132) yields the first integral of the equation of motion for the physical pendulum:

$$\tfrac{1}{2}I\dot\theta^2 + mg\ell\,(1 - \cos\theta) = E.$$

For initial data $\theta(0) = \theta_0$ and $\dot\theta(0) = \omega_0$, the constant $E = \frac{1}{2}I\omega_0^2 + mg\ell\,(1 - \cos\theta_0)$, and the last result then agrees with (10.122d). □

Exercise 10.12. A uniform wheel of mass m and radius R rolls without slipping down a hill that resembles a cosine curve on $[0, \pi]$. The center of the wheel has an initial speed v_0^* at the top of the hill. Find the speed of the center of the wheel after it has dropped through a vertical height h from its initial position. □

Example 10.16. Apply the energy method to derive the first integral of the equation of motion of the rod described in Fig. 10.11, page 447.

Solution. The resultant force $\mathbf{R}$ exerted on the rod by the smooth hinge bearing is workless, and the gravitational force on the rod is conservative with total potential energy $V = \frac{1}{2}mg\ell(1 - \cos\theta)$. The body frame $2 = \{Q; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a principal reference frame at Q. Therefore, with the aid of (10.80a) and (10.80c) in (10.102), the total kinetic energy of the rod relative to Q is $K = K_{rQ} = \frac{1}{6}m\ell^2(\dot{\theta}^2 + \Omega^2 \sin^2\theta)$.

The smooth hinge bearing exerts no torque about the hinge axis, the $\boldsymbol{k}$ direction. But the component μ_1 of the bearing reaction torque exerted on the rod about the $\mathbf{i}$-axis does work on the rod to control its spin. Therefore, we shall need to apply the general energy principle (10.131). The total bearing reaction torque exerted on the rod, from (10.80e), is $\mathbf{M}_Q^B = \mu_1\mathbf{i} = -\frac{2}{3}m\ell^2\Omega\dot{\theta}\cos\theta\,\mathbf{i}$. Then with (10.80a) and $\theta_0 = \theta(0)$ at $t = 0$, (10.114) for a fixed point Q gives

$$\mathscr{W}_N = \int_0^t \mathbf{M}_Q^B \cdot \boldsymbol{\omega}\, dt = \frac{2}{3}m\ell^2\Omega^2 \int_{\theta_0}^{\theta} \cos\theta \sin\theta d\theta = \frac{1}{3}m\ell^2\Omega^2(\sin^2\theta - \sin^2\theta_0). \tag{10.133a}$$

Recall the foregoing kinetic and potential energy functions to form $\Delta\mathscr{E}|_0^t = \Delta K|_0^t + \Delta V|_0^t$. Then with (10.133a), the general energy principle (10.131) yields

$$\tfrac{1}{6}m\ell^2\,(\dot{\theta}^2 + \Omega^2\sin^2\theta)\big|_0^t + \tfrac{1}{2}mg\ell\,(1 - \cos\theta)|_0^t = \tfrac{1}{3}m\ell^2\Omega^2(\sin^2\theta - \sin^2\theta_0), \tag{10.133b}$$

wherein $\Delta\mathscr{E}|_0^t \equiv \mathscr{E}(t) - \mathscr{E}(0)$, subject to the initial data $\dot{\theta}(0) = 0$ and $\theta(0) = \theta_0$. We thus obtain the first integral of Euler's equation of motion for the rod:

$$\dot{\theta}^2 = \Omega^2(\sin^2\theta - \sin^2\theta_0) + 3\frac{g}{\ell}(\cos\theta - \cos\theta_0). \tag{10.133c}$$

This is the same as (10.80g) obtained by direct integration of the equation of motion (10.80f). Relative to the vertical shaft, (10.133c) determines the angular speed of the rod as a function of θ. □

Exercise 10.13. A simple brake system for a flywheel of radius R and mass m consists of a brake pad having a coefficient of dynamic friction ν and positioned over a small area at the flywheel's outer circumference. During braking, the pad is pressed suddenly against the rim by a steady hydraulic pressure that generates a resultant normal, central directed force **P** on the flywheel. The flywheel has a constant angular speed Ω about its axle before braking occurs. Apply the general energy principle to determine the number of revolutions of the flywheel in reducing its speed to a value ω in time t, and find the time t required. □

10.22. Motion of Lineal Bodies Subject to a Stokes Retarding Force

A cable, rope, string, wire, stick, a strand of hair, and a rod are familiar examples of thin-structured, essentially one-dimensional bodies that are commonly described by a straight or curved spatial line. Any such one-dimensional material object of this kind is called a *lineal body*. Euler's classical elastica theory of bending of a flexible, inextensible curve, commonly thought of as a slender, straight or curved rod, is a well-known application of the lineal body model. The vibratory motion of an inextensible string or wire subjected to tensile loading is another application. Here we study the motion of a homogeneous, lineal rigid body subjected to a Stokes drag force per unit length of the body, distributed over its entire length.

Consider a lineal rigid body ℓ of uniform mass density $\sigma = dm/ds$ per unit length s moving in an inertial frame $\Phi = \{O; \mathbf{i}_k\}$. Suppose that the body is subjected over all of ℓ to a drag force $\mathbf{f}_d$, per unit length, and to an additional system of forces consisting of a total force $\mathbf{F} = \int_\ell d\mathbf{f}$ and a total torque $\mathbf{M}_Q = \int_\ell \mathbf{x} \times d\mathbf{f}$ about a base point Q that either is fixed in Φ or at the body's center of mass; and assume that ℓ is a sufficiently smooth curve so that the integrations are meaningful. The total drag force $\mathbf{F}_d$ on ℓ and its moment $\mathbf{M}_{dQ}$ about Q may be written as

$$\mathbf{F}_d = \int_\ell \mathbf{f}_d \, ds, \qquad \mathbf{M}_{dQ} = \int_\ell \mathbf{x} \times \mathbf{f}_d \, ds, \tag{10.134}$$

in which $\mathbf{x} = \mathbf{x}(P, t)$ is the position vector of the material parcel P from Q. Then Euler's laws of motion for the momentum $\mathbf{p}^*(\ell\ , t)$ of the center of mass particle and the moment of momentum $\mathbf{h}_{rQ}(\ell\ , t)$ relative to Q yield the differential equations

$$\frac{d\mathbf{p}^*}{dt} = \mathbf{F} + \mathbf{F}_d, \qquad \frac{d\mathbf{h}_{rQ}}{dt} = \mathbf{M}_Q + \mathbf{M}_{dQ}. \tag{10.135}$$

Introducing the Stokes drag force $\mathbf{f}_d = -c\mathbf{v}$, where $c > 0$ is the constant drag coefficient per unit length and $\mathbf{v}(P, t)$ denotes the velocity of the material parcel P of mass $dm(P)$, we obtain from (10.134),

$$\mathbf{F}_d = -\frac{c}{\sigma} \int_\ell \mathbf{v} \, dm = -\beta \mathbf{p}^*, \qquad \mathbf{M}_{dQ} = -\frac{c}{\sigma} \int_\ell \mathbf{x} \times \mathbf{v} \, dm = -\beta \mathbf{h}_{rQ}, \tag{10.136}$$

wherein the constant $\beta \equiv c/\sigma > 0$. Finally, use of (10.136) in (10.135) yields the general equations of interest:

$$\dot{\mathbf{p}}^* + \beta \mathbf{p}^* = \mathbf{F}, \qquad \dot{\mathbf{h}}_{rQ} + \beta \mathbf{h}_{rQ} = \mathbf{M}_Q. \tag{10.137}$$

These are the governing equations of motion for all homogeneous, lineal rigid bodies subjected to a general system of forces that includes a distributed Stokes force over its entire length. Air and water resistance are examples for which Stokes's law holds for bodies moving at low speeds, so let us assume that our lineal rigid body has a slow motion through a Stokes continuum that initially is at rest and remains essentially undisturbed by the motion of the body; otherwise, additional complications enter the analysis. Hence, only the velocity of lineal rigid body contact points is relevant to the discussion.

Example 10.17. Consider a uniform, heavy thin rod of mass m and length a suspended in the vertical plane by a smooth hinge at one end Q, and released with initial angular speed ω_0 at the placement θ_0 in a Stokes medium of negligible buoyancy. The restoring torque is $\mathbf{M}_Q = -\frac{1}{2} mga \sin\theta \mathbf{k}$ and the moment of momentum relative to Q is $\mathbf{h}_{rQ} = I_Q \omega \mathbf{k}$, where $\boldsymbol{\omega} = \omega \mathbf{k}$ is the angular velocity and $I_Q = \frac{1}{3} ma^2$. The first equation in (10.137) determines the support reaction force as a function of the placement $\theta(t)$, and the second equation yields the equation of motion,

$$\dot{\omega} + \beta\omega + p^2 \sin\theta = 0, \qquad p^2 \equiv \frac{3g}{2a}. \tag{10.138a}$$

For small oscillations this reduces to the familiar differential equation of type (6.83) for the free, damped oscillations of the rod: $\ddot{\theta} + \beta\dot{\theta} + p^2\theta = 0$, whose solutions are summarized on page 157.

Notice that the rod geometry enters the result only through the constant p. Therefore, in the absence of gravity, we have $p = 0$, and (10.138a) reduces to $\dot{\omega} + \beta\omega = 0$, which, except for its mass density, is independent of any physical characteristics of the rod. In fact, the same equation holds for any homogeneous rod of arbitrary shape turning about a principal axis at its center of mass. We shall say more about this unusual result in a moment. □

10.22.1. Universal Motions for Equipollent Systems

Suppose that the system of additional forces is equipollent to zero. Then $\mathbf{F} = \mathbf{0}$, $\mathbf{M}_Q = \mathbf{0}$, and (10.137) yields *the universal equations of motion characteristic of all dynamical systems in the class $\mathscr{L}$ of all homogeneous, lineal rigid bodies subject to a linear viscous damping force per unit length, and for which all other forces are equipollent to zero*:

$$\dot{\mathbf{p}}^* + \beta \mathbf{p}^* = \mathbf{0}, \qquad \dot{\mathbf{h}}_{rQ} + \beta \mathbf{h}_{rQ} = \mathbf{0}. \tag{10.139}$$

The first integrals of these equations are given by

$$\mathbf{p}^*(\ell\ ,t) = \mathbf{p}_0 e^{-\beta t}, \qquad \mathbf{h}_{rQ}(\ell\ ,t) = \mathbf{h}_0 e^{-\beta t}, \tag{10.140}$$

in which $\mathbf{p}_0 \equiv \mathbf{p}^*(\ell\ ,0)$ and $\mathbf{h}_0 = \mathbf{h}_{rQ}(\ell\ ,0)$ are certain initial values. Consequently, for all systems in the class $\mathscr{L}$ the momentum of the center of mass and the moment of momentum relative to Q have fixed directions in Φ throughout the motion of the body, which, as may be anticipated intuitively, decays exponentially over time bringing the body eventually to rest.

10.22.2. Universal Motions for Workless Force Systems

Recall that a system of distributed forces $\mathbf{f}_P = \mathbf{f}(P)$, per unit length s, is equipollent to zero if and only if

$$\mathbf{F} = \int_\ell \mathbf{f}_P ds = \mathbf{0}, \qquad \mathbf{M}_Q = \int_\ell \mathbf{x} \times \mathbf{f}_P\, ds = \mathbf{0}, \tag{10.141}$$

where $\mathbf{x}$ is the position vector of the point P from any moment center Q. Now consider a lineal rigid body with base point Q and subjected to any system of distributed forces equipollent to zero. Then $\mathbf{v}_P = \mathbf{v}_Q + \boldsymbol{\omega} \times \mathbf{x}$, and hence the rate $d\mathscr{W}_e/dt$ at which work is done by the additional equipollent system (10.141) is

$$\frac{d\mathscr{W}_e}{dt} = \int_\ell \mathbf{v}_P \cdot \mathbf{f}_P\, ds = \mathbf{v}_Q \cdot \int_\ell \mathbf{f}_P ds + \boldsymbol{\omega} \cdot \int_\ell \mathbf{x} \times \mathbf{f}_P\, ds = 0. \tag{10.142}$$

That is, the additional equipollent system of distributed forces is workless. Therefore, problems in the class $\mathscr{L}$ characterized by the universal equations (10.139) with their first integrals (10.140) are among those in the workless class $\mathscr{C}$ characterized below.

Consider a homogeneous, lineal rigid body ℓ of arbitrary shape subject to a nonconservative force $\mathbf{f}_N$, per unit length s, distributed over its entire length. We shall suppose that all other forces, including any conservative forces, that act on ℓ are workless. Therefore, in accordance with the work–energy principle (10.120) for a rigid body, we have $\Delta K(\ell\ ,t) = \mathscr{W}_N(\ell\ ,t)$. The mechanical power dissipated by the nonconservative force is defined by $\mathscr{P}(\ell\ ,t) = d\mathscr{W}_N/dt$ in (10.112); therefore, we begin with

$$\dot{K}(\ell\ ,t) = \mathscr{P}(\ell\ ,t) = \int_\ell \mathbf{f}_N \cdot \mathbf{v}\, ds, \tag{10.143}$$

where $\mathbf{v} = \mathbf{v}(P,t)$ is the velocity of the material point P on which the nonconservative force is acting on ℓ.

For the Stokes drag force $\mathbf{f}_N = -c\mathbf{v}$ per unit length of ℓ, the total power is given by

$$\mathscr{P}(\ell\,, t) = -\frac{c}{\sigma}\int_{\ell} \mathbf{v}\cdot\mathbf{v}\,dm = -2\beta K(\ell\,, t), \tag{10.144}$$

in terms of the total kinetic energy (10.90). Hence, (10.143) yields *the universal energy equation characteristic of all dynamical systems in the class $\mathscr{C}$ of homogeneous, lineal rigid bodies subject to a linear viscous damping force per unit length, and for which all other forces acting on ℓ are workless, including those that are equipollent to zero*:

$$\dot{K}(\ell\,, t) + 2\beta K(\ell\,, t) = 0, \tag{10.145}$$

with $\beta \equiv c/\sigma > 0$, whose general solution is

$$K(\ell\,, t) = K_0 e^{-2\beta t}, \tag{10.146}$$

in which $K_0 \equiv K(\ell\,, 0)$ is the initial kinetic energy of the body. These rules are mainly useful for any single degree of freedom dynamical system, but they may also be useful in special, constrained higher degree of freedom systems.

Exercise 10.14. Show that the universal energy equation (10.145) may be derived as a first integral of the pair of universal differential equations (10.139) for Q at the center of mass. You will see that the first integral of each vector equation has exactly the same universal form as (10.145). □

10.22.3. Universal Motions of Rotating Rigid Bodies

Many dynamical problems in the workless class $\mathscr{C}$ have the property that the body is turning with angular speed ω about a principal axis so that

$$K = \kappa\omega^2, \tag{10.147}$$

wherein κ is a constant characteristic of the body. For a homogeneous circular ring of radius R, axial moment of inertia I, and mass m, rolling slowly without slip on a fixed horizontal surface, we have $\kappa = \frac{1}{2}(I + mR^2)$, for example. Let ω_0 denote the initial angular speed of the body; then $K_0 \equiv \kappa\omega_0^2$ and (10.146) yields

$$\omega(t) = \omega_0 e^{-\beta t}. \tag{10.148}$$

Integration of (10.148) with $\omega \equiv \dot{\theta}$ and $\theta_0 = \theta(0)$ gives the angular placement of ℓ:

$$\theta(t) = \theta_0 + \frac{\omega_0}{\beta}(1 - e^{-\beta t}). \tag{10.149}$$

Finally, use of (10.147) in (10.145) yields

$$\dot{\omega} + \beta\omega = 0, \tag{10.150}$$

or, in terms of θ,

$$\ddot{\theta} + \beta\,\dot{\theta} = 0. \tag{10.151}$$

The universal equation of motion[‖] (10.150), *or equivalently* (10.151), *is exactly the same for every homogeneous, lineal rigid body having the same mass density, and it is independent of the shape, length, or any other physical characteristics of the body.*

It is not unusual that several dynamical systems may be formally characterized by the same differential equation. Many systems are described by the familiar equation characteristic of a simple harmonic oscillator, for example. But the universal equation of motion (10.151) is unusual because it is identically the same equation for every dynamical system in the workless class $\mathscr{C}$. There is no need to mention any specific mechanical system. The effect is due to the linear nature of Stokes damping and to the special nature of the system of additional forces.

10.22.4. General Motion with Additional Forces

Suppose that the body is subjected to an additional system of forces for which the total mechanical power is $\mathscr{P}_a$, say. This may include additional conservative forces excluded in (10.143). In this case, (10.145) is replaced by the differential equation

$$\dot{K} + 2\beta K = \mathscr{P}_a, \tag{10.152}$$

which generally is not universal. Indeed, *the universal equation* (10.145) *holds if and only if* $\mathscr{P}_a = 0$, *that is, when and only when the additional system of forces is workless.* In particular, the differential equation for the rotation in the vertical plane of a homogeneous, rigid rod about any point other than the center of mass is not in the universal class $\mathscr{C}$, because the conservative gravitational force acting on the lineal body does work.

In the exceptional case for which the extra power may be proportional to the total kinetic energy of the body, so that $\mathscr{P}_a = \gamma K$, where γ is a constant, the motion is characterized by the similar differential equation

$$\dot{K} + (2\beta - \gamma)K = 0, \tag{10.153}$$

with solution $K = K_0 e^{(\gamma - 2\beta)t}$. But this equation is *not exactly the same as* (10.145), and hence this exceptional case is not a member of the universal class $\mathscr{C}$ for which β depends on only the viscosity of the medium and the mass density of the body.

Exercise 10.15. Consider a homogeneous, thin rigid plate or shell of uniform mass density σ per unit area, moving in an inertial frame $\Phi\{O; \mathbf{e}_k\}$. The shell may be open or closed. Suppose that the body is subjected to a Stokes drag force over its

[‖] See Wilms (1995) and Beatty (1997, 2002) in the References.

entire surface. Retrace the foregoing development starting with (10.134) and show that all of the previous results hold for an arbitrary rigid plate or shell structure. See Problems 10.67 and 10.68. □

Exercise 10.16. Consider an open and a closed thin hemispherical shell, a thin spherical shell, and an open thin conical shell, all of radius R and mass density σ spinning with angular speed ω about their fixed, vertical principal axis of symmetry, and each subjected to a Stokes drag force over its surface. All other forces are workless. Find the motion for each of these very different systems. □

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Problems

10.1. Show that equations (10.2) and (10.3) satisfy (10.5) in accordance with the principle of determinism.

10.2. Suppose that the velocity of every particle of a body is the same so that $\mathbf{v}(P,t) = \mathbf{v}(t)$ is a function of only the time. Prove that (10.2) and (10.3) may be written in the form

$$\mathbf{F}(\mathscr{B},t) = m(\mathscr{B})\mathbf{a}(t), \qquad \mathbf{M}_O(\mathscr{B},t) = \mathbf{x}^*_O(\mathscr{B},t) \times \mathbf{F}(\mathscr{B},t).$$

Hence, in this special case the force and moment vanish together, if and only if the motion is uniform. Here $\mathbf{a}(t) = \dot{\mathbf{v}}(t)$ and $\mathbf{x}^*_O(\mathscr{B},t)$ is the position vector of the center mass from a fixed point O in the inertial frame.

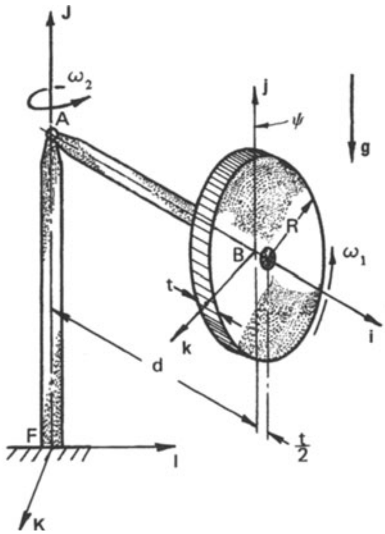
10.3. A truck is moving on a horizontal road with speed $v = 1.5$ m/sec which is decreasing at the rate of 0.5 m/sec^2; and at the instant of interest, a concrete pipe section of mass $m = 1600$ kg and diameter 1.5 m begins to roll without slipping toward the rear of the truck, from one constraining block to another, with an angular speed $\omega = 0.2$ rad/sec, increasing at the rate of 0.1 rad/sec^2 relative to the truck. Derive an equation for the total force acting on the pipe, and determine its value at the moment of interest.

10.4. Use the results in Example 10.3, page 424, and Example 10.4, page 428, to determine by two methods the moment of momentum of the rod about point O fixed in the ground frame. What is the total torque on the rod about O?

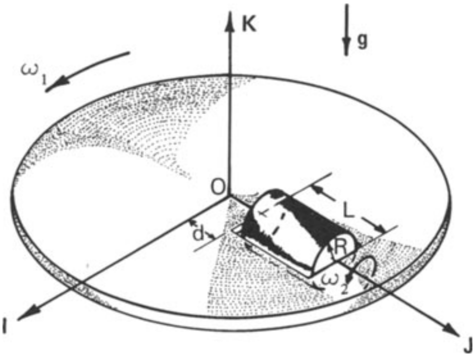
10.5. A thin circular disk of radius R and mass m is spinning with a constant angular velocity ω_1 about its axle AB which is turning about a fixed vertical axis FA with a constant angular velocity ω_2, as shown in the figure. What is the moment of momentum of the disk about points A and B referred to frame $\psi = \{B; \mathbf{i}_k\}$ fixed in the axle? Compute all essential moments of inertia. Find the total torque acting on the disk about A and B.

10.6. A cylinder of radius R, length L, and mass m rotates with a constant angular velocity ω_2 relative to a table on which its bearings are mounted. The table turns with a constant angular velocity ω_1, as shown in the diagram. Find the moment of momentum of the cylinder about its center of mass C. Compute all essential moments of inertia. What is the total torque acting on the cylinder, about C?

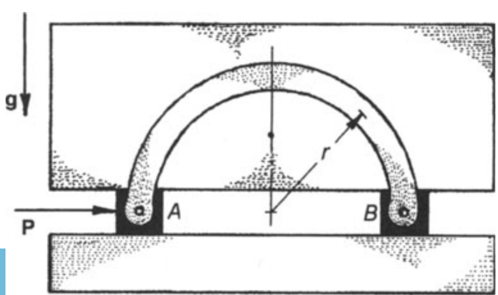
10.7. The motion of a thin semicircular rod of radius r induced by a horizontal force $\mathbf{P}$ applied at A is guided by small blocks that slide in a smooth horizontal track shown in the figure. (a) Find the acceleration of the center of mass of the rod, and determine the resultant surface reaction forces at A and B. (b) What force $\mathbf{P}$ will sustain the rod's motion so that the surface reaction at B vanishes? Find this motion, when the system starts from rest initially.



Problem 10.5.

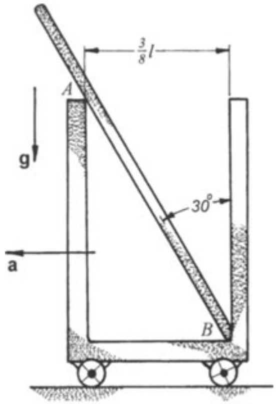


Problem 10.6.



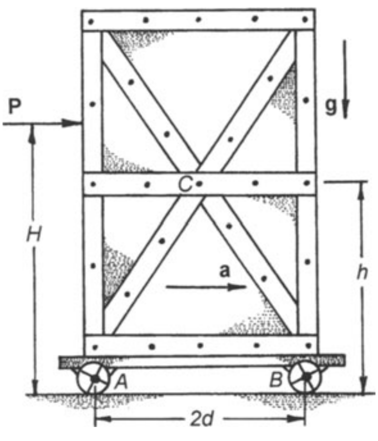
Problem 10.7.

10.8. The figure shows a ladder of length l and mass m transported in a cart that moves from rest with a constant acceleration $\mathbf{a}$. (a) What are the support reaction forces on the ladder at A and B? (b) Determine the maximum acceleration of the cart for which the ladder maintains contact at A, and find the corresponding reaction at B.



Problem 10.8.

10.9. A (roughly) homogeneous and symmetric crate of weight W is mounted on a pallet with casters that roll smoothly, with negligible friction, on a fixed horizontal surface due a force $\mathbf{P}$ applied as shown. Determine the range of values of H for which the crate will not tip at A and B.

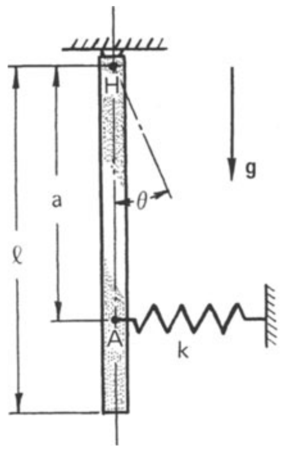


Problem 10.9.

10.10. According to elasticity theory, the end torque T required to twist a uniform circular shaft is proportional to the angle θ of the end twist, that is, $T = k\theta$. The constant k is called the *torsional stiffness*. A rigid circular plate of radius a is centrally welded to the end of a thin, inextensible torsion rod of negligible mass, twisted through a small angle θ_0 and released to perform torsional oscillations about its axis (See Fig. 1.7 in Volume 1). (a) Derive the equation of motion of the plate, and determine the period τ_1 of its oscillation. This period may be found independently by experiment. (b) An axisymmetric body B, a wheel for example, is placed on the plate with its central axis along the rod/plate axis, and the period of the torsional vibration of the system is now τ_2. Derive an equation for the moment of inertia I_B of B in terms of τ_1 and τ_2. (c) Experiments reveal that $\tau_1 = 5$ sec, $\tau_2 = 25$ sec, and the torsional stiffness of the shaft is $k = \pi^2/100$ N·m/rad. Determine I_B for these test data.

10.11. A thin, homogeneous rod of length ℓ and mass m is suspended vertically by a torsion wire attached to its center C. When the rod is displaced through an angle θ in the horizontal plane, the wire exerts on the rod a restoring torque $T = k\theta$, where k is the constant torsional stiffness of the wire. The rod is released from rest initially at an angle θ_0. (a) What is the oscillational frequency of the rod in terms of assigned quantities? (b) Determine the angular motion $\theta(t)$ of the rod. (c) The torsional stiffness of a wire of length h, radius a, and having a shear modulus μ is given by $k = \pi\mu a^4/2h$. Show how your results may be applied to determine by experiment the material constant μ.

10.12. A homogeneous, slender rod of length ℓ and mass m is supported by a smooth hinge at H and connected at A to a linear spring of stiffness k. Apply Euler's law and derive the equation for the circular frequency of small plane oscillations of the rod about its natural state shown in the diagram, expressed in terms of the assigned quantities only.

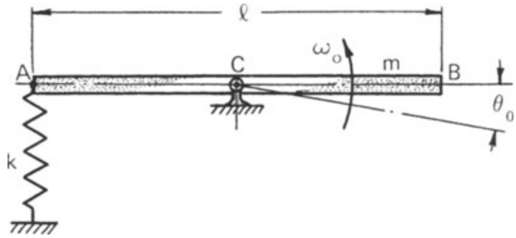


Problem 10.12.

10.13. A homogeneous rectangular plate, 6 ft × 2 ft, swings in the vertical plane about a smooth supporting hinge H located on the lengthwise center line of the plate, 2 ft from its center of mass. Calculate the moment of inertia and determine the radius of gyration R_H about H. Derive an equation for the small amplitude circular frequency in terms of g and R_H, and thus determine the period of the motion in terms of g alone. Notice that the results are independent of the mass of the plate, however great or small.

10.14. A thin, homogeneous rigid rod of length ℓ and mass m is supported by a frictionless hinge at its center of mass C. The rod is connected at A to a spring of stiffness k and is constrained to move only in the horizontal plane. Initially, the rod has a small angular speed ω_0 and a small angular displacement θ_0 shown in the diagram. (a) Determine the small oscillatory motion $\theta(t)$ of the rod. (b) What is the period of the oscillation? (c) Find the reaction exerted by the support. Compute all quantities involved and express the results in terms of only the assigned parameters.

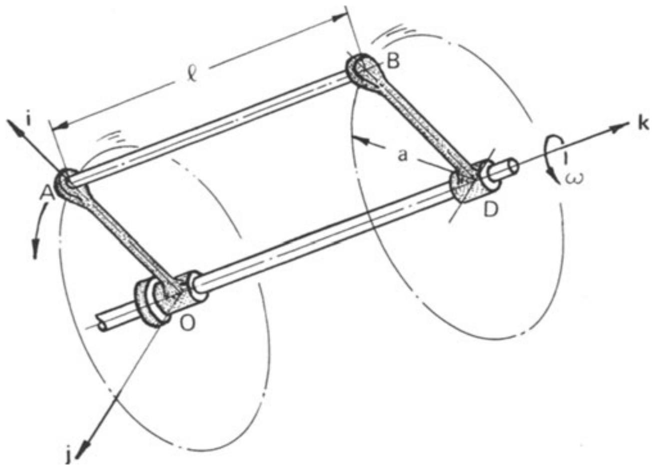
10.15. A nonhomogeneous, slender rigid rod of length ℓ has a mass density σ, per unit length, that varies linearly with the distance from one end O where its value is σ_0 to the value $2\sigma_0$ at the other end A. The rod is suspended by a smooth hinge pin at O, given a small angular



Problem 10.14.

displacement, and released to perform small oscillations in the vertical plane. (a) What is the frequency of the vibration? (b) What is the length of a simple pendulum having the same frequency? (c) Find the radius of gyration of the rod about O.

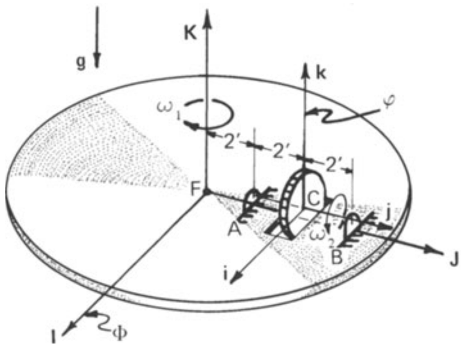
10.16. A machine member AB shown in the diagram is modeled as a homogeneous slender rod of length ℓ and mass m. (a) Find the total force and the total torque about the point O required to sustain the motion of the rod with an angular acceleration $\dot{\boldsymbol{\omega}}$ about the fixed axle OD. (b) Find a relation between the magnitudes of the force and the torque when $\boldsymbol{\omega}$ is constant.



Problem 10.16.

10.17. The figure shows a thin, homogeneous disk of radius 2 ft and mass $m = 10$ slug rotating with an angular speed $\omega_2 = 100$ rad/sec relative to a platform that is spinning with angular speed $\omega_1 = 40$ rad/sec in the ground frame. Neglect the mass of the supporting shaft AB, assume that the roller bearing at A supports the shaft only around its radial direction, normal to AB. Compute the total bearing reactions at A and B, including static contributions.

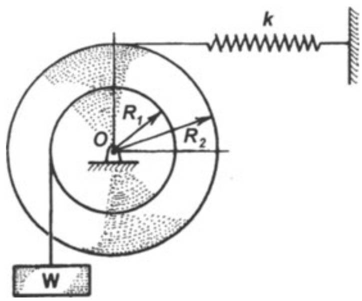
10.18. A homogeneous circular disk of radius r has a small circumferential groove around which a long inextensible cord of negligible mass is wound. The end point O of the cord is held fixed, and the disk is released from rest in the vertical plane, the cord unwinding as the disk falls (like a yo-yo toy). Find the velocity of the center of mass C and the angular velocity of the disk after it has fallen a distance d. Explain why the point of contact of the cord with the disk, say Q, may be used as the moment center in Euler's law, and solve the problem with the use of Q. Solve the problem again with the use of C as the moment center. Are the results equivalent? Express the results in terms of only the assigned quantities.



Problem 10.17.

10.19. The normal, central axis of a uniform thin circular plate of mass m and radius r makes an angle α with respect to the axis of a shaft to which the plate is welded at its center C. (See the figure for Problem 9.21.) The moment of inertia tensor of the plate is given in (9.33). The shaft has a constant angular velocity $\boldsymbol{\omega}$ and is supported in horizontal bearings symmetrically positioned at a distance d on each side of C. Ignore the mass of the shaft. Find the magnitude of the dynamic bearing reaction torque about C, and determine the magnitudes of the bearing reaction forces.

10.20. An inextensible cable of negligible mass wound around a stepped cylindrical pulley supports a weight W. The pulley is restrained by a spring of elasticity k, as shown in the figure. The system is released from rest when the spring is initially unstretched and performs small oscillations in the vertical plane. Determine the angular motion $\theta(t)$, and find the support bearing reaction force as a function of θ. Find the equilibrium angular placement θ_e.



Problem 10.20.

10.21. A homogeneous, plane rigid body has several holes drilled through it, all situated on a vertical line through the center of mass C. A small pin through any one of the holes provides a smooth horizontal axle O about which the body may oscillate in the vertical plane. (a) Find the period of small oscillations of the body in terms of the distance d of point O from C and the radius of gyration R_C about C. It may be seen from this equation that the period goes to infinity when $d = 0$ and when $d = \infty$, and hence there exists a value of d for which the period is least. (b) Show that this point is at a distance $d = R_C$ from C. (c) Show that there exists another point of suspension Q on the line OCQ at $b \neq d$ from C for which the period is the same as that about O. The point Q is called the *center of oscillation*. Find its distance b from C.

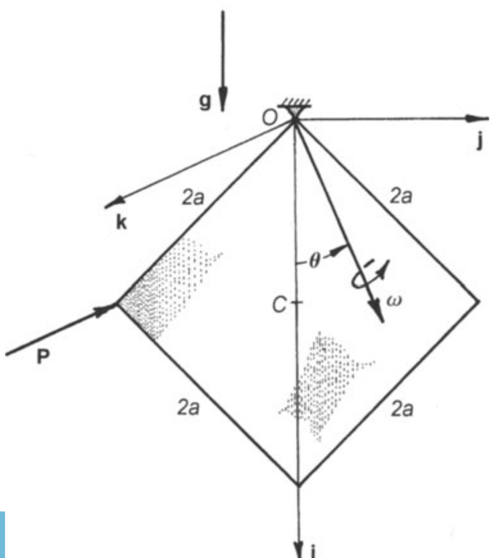
10.22. A plane rigid body of mass m similar to the physical pendulum shown in Fig. 10.15, page 467, can rotate about a smooth axle at Q. The body is suspended at rest in the vertical plane, and an instantaneous impulsive force $\mathbf{F}^*$ is applied in the plane of the body at its boundary and in a direction perpendicular to the vertical line joining Q and the center of mass G. Find the vertical distance d from the point of support Q to the line of action of $\mathbf{F}^*$ such that there is no impulsive reaction at Q in the direction of $\mathbf{F}^*$. The point at d on the vertical line QG is called the *center of percussion*.

10.23. A homogeneous square plate of mass m and side $2a$ rests on a smooth horizontal surface. A horizontal impulsive force $\mathbf{F}^*$ of specified intensity is applied at a corner point B in a direction perpendicular to the diagonal line at B. Find the subsequent angular velocity of the plate. What is the velocity of the center of mass?

10.24. A uniform slender rod of mass m and length $2a$ is supported in a bearing and rotates in the horizontal plane with angular velocity ω about a vertical axle at its center C. Let Q be an arbitrary point along the rod from C. (Note that the center of mass is fixed in the inertial frame.) Use (10.38) and thus demonstrate that $\mathbf{h}_Q = \mathbf{h}_C$. Suppose that a horizontal force $\mathbf{S}$ acts perpendicular to the rod at one end. Show that $\mathbf{M}_Q = \mathbf{M}_C$.

10.25. A homogeneous square plate $DEFG$ of mass m and side $2a$ rotates about a vertical diagonal axis DF with angular velocity ω, the point F being supported by a smooth horizontal plane. (a) Apply (10.38) to find $\mathbf{h}_E$, and thus show that $\mathbf{h}_E = \mathbf{h}_C$ for the center of mass C. (b) Now suppose that the corner point E is suddenly fixed so that the plate now turns with angular velocity $\boldsymbol{\Omega}$ about an axis through E and parallel to DF. Find the new angular velocity.

10.26. A homogeneous square plate of mass m and side $2a$ is suspended from a smooth ball joint at a corner point O. The plate is initially at rest in the vertical plane when an instantaneous impulse $\mathbf{P}$, acting in the horizontal plane and normal to the plane of the plate, is suddenly applied at an adjacent corner point, as shown. (a) Determine explicitly the axis $\mathbf{n}$ about which the plate begins to rotate; find the angle θ, and show the axis in a diagram. (b) Find the instantaneous impulse reaction $\mathbf{R}$ at O.



Problem 10.26.

10.27. An equilateral triangular frame formed from three homogeneous thin straight rods, each of mass m and length $2a$, is suspended from a smooth ball joint at a vertex O. Initially at rest in the vertical plane, the frame suffers an instantaneous impulse $\mathbf{P}$ in a direction perpendicular to the plane of the triangle at an adjacent vertex. The geometry setup is similar to that shown in the previous problem. (a) Establish that the axes $\mathbf{i}_k$ at O are principal axes. (b) Show that the magnitude of the instantaneous impulse reaction at O is $P/5$.

10.28. The ends of two homogeneous thin rods, each of mass m and length $2a$, are connected by a smooth hinge H and lie along a straight line AHB on a smooth horizontal plane, the free end points of the rods being at A and B. The rod HB is struck suddenly at B by a horizontal force impulse $\mathbf{P}$ perpendicular to the rod. (a) Find the velocity of the center of mass and the angular velocity of each rod immediately after the impact. (b) Determine the instantaneous impulse reaction at H. Find the velocity of H, and describe in a diagram the direction of its motion and that of the rods. Hint: First construct a free body diagram for each rod and apply the principles (10.30) and (10.56).

10.29. Two gears, A of radius a and B of radius b having respective axial moments of inertia I_A and I_B, can turn freely about their parallel axles. Initially, gear A has an angular speed ω, while B is at rest. The gears are suddenly engaged. Find the subsequent angular speed of each gear.

10.30. A homogeneous circular disk of radius a and mass M is pivoted at its center and set spinning with angular velocity $\boldsymbol{\omega}$ about its central axis normal to the horizontal plane. A particle of mass m falls vertically, strikes the disk at its edge with velocity $\mathbf{v}$, and adheres to it. Find the velocity $\mathbf{v}_f$ of the particle and the angular velocity $\boldsymbol{\omega}_f$ of the plate immediately after the impact. Show that $\mathbf{v}_f$ makes an angle α with the horizontal plane, the same angle that $\boldsymbol{\omega}_f$ makes with the vertical axis, given by

$$\tan\alpha = 4\frac{m(M+4m)}{M(M+8m)} \cdot \frac{v}{a\omega}. \tag{P10.30}$$

10.31. A uniform bicycle wheel of mass M is pivoted at its center and rotates with angular velocity $\boldsymbol{\omega}$ about its axle perpendicular to the horizontal plane. Ignore the mass of the spokes and model the wheel as a thin ring of radius a. A particle of mass m falls vertically, hits the wheel near its rim with velocity $\mathbf{v}$, and adheres to it. Show that the velocity of the particle immediately after impact makes an angle α with the horizontal plane given by (P10.30) in which m is replaced by $m/2$.

10.32. A uniform wheel of radius r and mass m, whose plane is vertical, spins and slips on a horizontal surface having a dynamic coefficient of friction ν. Initially, the center O of the wheel has a speed v_0^* and its angular speed is ω_0. The motion of O is along a straight line but $v_0^* \neq r\omega_0$, and hence the wheel at first skids or slips accordingly as $v_0^* > r\omega_0$ for skidding or $v_0^* < r\omega_0$ for slipping. (a) Consider both cases, and determine the time required to achieve pure rolling contact. (b) Find the subsequent speed v^* of the center of mass and the angular speed ω of the wheel, which are independent of ν.

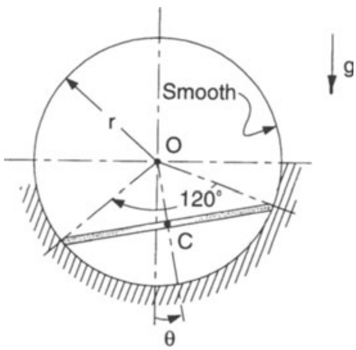
10.33. A thin rigid rod of length ℓ and mass m is suspended in the vertical XY-plane by a string attached to one end while the other end rests on a smooth horizontal surface. The rod makes an angle α with the vertical axis of the string. Find as a function of α the ratio N_D/N_S of the dynamic to the initial static surface reaction force on the rod immediately after the string is cut. Determine the physical range of values of N_D/N_S.

10.34. Describe the motion of the center of mass of the rod in the previous problem. Determine the angular speed $\dot{\theta}$ of the rod as a function of its angular placement $\theta(t)$, and thus derive an integral for the time t as a function of θ.

10.35. A slender rigid rod of length ℓ and mass m is hung vertically by two inextensible cords of equal length a attached to one end of the rod, each cord making an angle α with the horizontal ceiling. The left cord breaks. Find the tension in the remaining cord and the angular acceleration of the rod at that instant.

10.36. A thin rod AB of length ℓ and mass m is suspended by two vertical wires FA of length b and GB of length such that the rod makes an angle α with the horizontal. Find the initial angular accelerations of the rod and the wire FA, hinged at A, when the wire connector at B is released. What is the tension in the wire FA at that moment?

10.37. A uniform thin rigid rod of weight w slides in the vertical plane with its ends on a smooth circle of radius r, and subtending an angle of 120° at the center, as shown in the diagram. The rod is released from rest at a placement $\theta_0 = \tan^{-1} 3/4$. Find the contact forces acting on the rod in terms of θ and the assigned parameters alone.

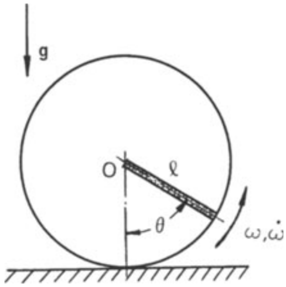


Problem 10.37.

10.38. A homogeneous cylinder of radius r and mass m rolls without slipping down a plane inclined at an angle α to the horizontal plane. (a) Determine the contact forces that act on the cylinder. (b) What is the critical inclined angle α_c, independent of m, at which slip is imminent? (c) How does the acceleration of the center of mass of the cylinder compare with that of a block of mass m sliding down a smooth plane, for the same translation coordinate?

10.39. A uniform slender rod of length ℓ and mass m, shown in the diagram, is fastened to a thin circular hoop of radius ℓ and negligible mass. The assembly is released from rest at the placement $\theta(0) = \theta_0$. (a) Derive the system of equations for the angular motion $\theta(t)$ and the surface reaction forces for rolling without slip.** (b) Find the initial values of the surface forces and the angular acceleration for $\theta_0 = 90°$. (c) Apply the results to derive the system of equations for small motions. (d) Find the small motion $\theta(t)$ starting from rest at a small angle θ_0, and describe its physical characteristics.

** Here is an example where we have rolling without slip, so $\mathbf{v}_Q = 0$ at the contact point, but the condition (ii) below (10.47) for use of Euler's law in (10.48) fails. Therefore, one must exercise caution in forming $\dot{\mathbf{h}}_Q$ and $\dot{\mathbf{h}}_{rQ}$ in use of (10.45) for a point of rolling contact without slip. You should try to solve Problem 10.39 by application of (10.42) and (10.46). Either equation will yield the same correct result consistent with that based on use of (10.48) for the center of mass, provided you use an arbitrary point on the rim of the hoop and exercise care in evaluating the derivatives, and afterwards in evaluating the result at the contact point itself. You will see that it is much easier to use Euler's principle for the center of mass.

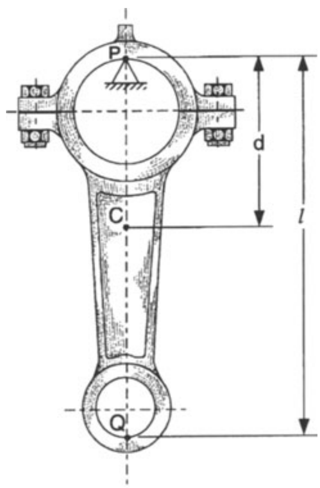


Problem 10.39.

10.40. Write the general equation of motion for $\theta(t)$ derived in the last problem, and solve it for the angular speed $\dot{\theta}(t)$ of the system when released from rest at an arbitrary angle $\theta(0) = \theta_0$. From this result, derive an integral formula for the motion time t as a function of θ.

10.41. Apply Euler's laws to derive the equations of motion for the spring and pulley system described in Problem 7.49. Model the pulley as a thick plate of radius a and mass m. The pulley rolls without slipping on an inextensible supporting belt. What are the static displacements of the pulley and the load? How many degrees of freedom does this system have?

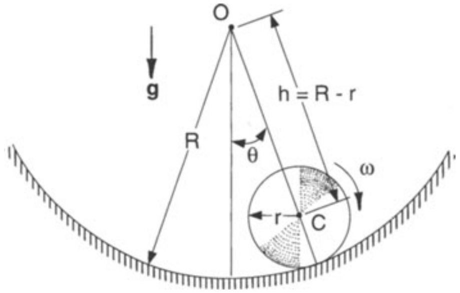
10.42. A connecting rod of mass m is suspended on a horizontal knife edge at P, given a small angular displacement ϕ, and released from rest. The measured frequency of its oscillation is f_P. When the connecting rod is similarly suspended at Q and the experiment is repeated, its measured oscillational frequency is f_Q. The distance between the points of support at P and Q is l. Find from these data the moment of inertia of the connecting rod about the normal axis at its center of mass C, and find the location d of C from P.



Problem 10.42.

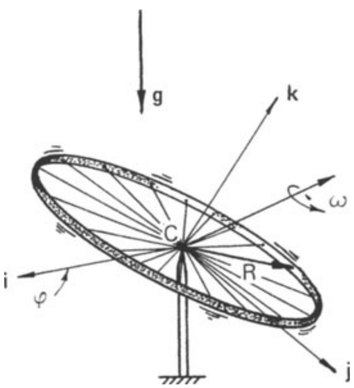
10.43. A homogeneous cylinder of mass m and radius r rolls without slipping on a fixed cylindrical surface of radius R, as shown in the figure. The cylinder is released from rest at the placement θ_0. (a) Apply the Newton–Euler laws to derive the equation for the rotational motion

$\theta(t)$. (b) Find the exact solution of this equation. What is the finite amplitude period of the motion? (c) Determine as functions of $\theta(t)$ the forces that act on the rolling cylinder.



Problem 10.43.

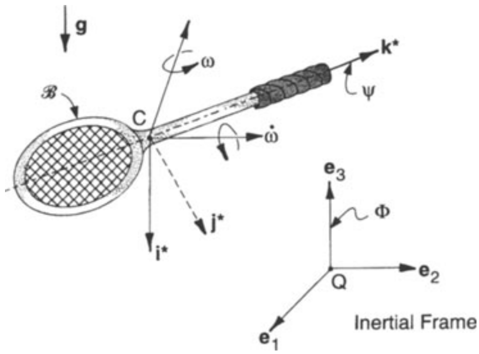
10.44. A bicycle wheel of mass m is modeled as a homogeneous, thin circular ring supported at its center C on a smooth needle point, as shown in the figure. The wheel is set spinning with angular velocity $\boldsymbol{\omega}_0 = \omega_0 \mathbf{n}$ in the inertial frame, where the *initial* axis of rotation is a unit vector $\mathbf{n}$ in the 23-body plane and makes an angle α with the normal vector $\mathbf{k}$. Ignore the mass of the spokes and the center support. Determine the angular velocity $\boldsymbol{\omega}(t)$ of the wheel referred to the principal body frame $\varphi = \{C; \mathbf{i}_k\}$.



Problem 10.44.

10.45. A rigid tennis racket $\mathscr{B}$ in an inertial frame Φ is tossed upward with an angular velocity $\boldsymbol{\omega}$, as suggested in the figure. (a) Show that three steady (constant) rotations with total angular velocity vectors $\boldsymbol{\omega} = (\alpha, 0, 0), (0, \beta, 0), (0, 0, \gamma)$ referred to principal axes $\psi = \{C; \mathbf{i}_k^*\}$ at the center of mass are exact solutions of the equations of motion. (b) The principal values of the inertia tensor referred to ψ are ordered so that $I_1 > I_2 > I_3$. Show that the steady rotations about the axes of greatest and least principal values of $\mathbf{I}_C$ are infinitesimally stable, while the rotation about the axis of the intermediate value of $\mathbf{I}_C$ is not.

10.46. Consider a homogeneous billiard ball shown in Fig. 10.14, page 457 and suppose that the horizontal impulsive action $\mathscr{I}^*$ occurs in the vertical plane through the center of mass, i.e. without "English." Assume that $\mathscr{I}^*$ is known. (a) Show that the height d above the mass center C at which the cue must hit the ball to produce pure rolling on the horizontal surface is given by $d = 2R/5$. How does this result compare with the solution of Exercise 10.9, page 460?



Problem 10.45.

(b) Consider a ball struck high at $h > d$ from C and struck low at $h < d$ from C, and account for slipping on a rough surface with coefficient of friction ν. Determine the initial velocity of C and the angular velocity of the ball immediately after the impact, and assess the direction of the frictional force for each case. (c) Find for each case the velocity of the mass center and the angular velocity of the ball for all time during the slipping phase, and determine the time for slipping to cease. (d) Determine the extent to which the velocity of C increases in a high shot and decreases in a low shot.

10.47. A homogeneous circular cylinder of radius r and mass m rotates horizontally about its central axis with angular velocity ω_0. The cylinder is carefully released, without bouncing, to commence sliding on a rough horizontal surface with coefficient of dynamic friction ν. Recall (9.31) for the moment of inertia tensor. (a) Determine the velocity of the center of mass and the angular velocity of the cylinder during and after the slipping phase. (b) What is the slip velocity at the point of contact with the surface? (c) Find the duration of the slipping phase, and determine the total angle through which the cylinder turned during this phase.

10.48. The inertia tensor of a rigid body referred to the body frame $\varphi = \{C; \mathbf{e}_k\}$ at its center of mass C is given by $\mathbf{I}_C = 10\mathbf{e}_{11} + 5\mathbf{e}_{22} + 7\mathbf{e}_{33} + \sqrt{3}(\mathbf{e}_{23} + \mathbf{e}_{32})$. The body has an angular velocity $\boldsymbol{\omega} = \omega_1\mathbf{i}_1 + \omega_2\mathbf{i}_2 + \omega_3\mathbf{i}_3$ referred to the principal body frame $\psi = \{C; \mathbf{i}_k\}$. Initially, $|\boldsymbol{\omega}(0)| = 2\sqrt{3}$ rad/sec about the line $x = y = z$ in ψ. (a) Derive Euler's equations referred to $\psi = \{C; \mathbf{i}_k\}$. (b) What is the rotational work done by the applied torques acting on the body?

10.49. Apply the energy method to derive the equation for the frequency of small plane oscillations of the rod described in Problem 10.12.

10.50. Apply the energy method to derive a formula for the period of the small amplitude oscillations of the plate described in Problem 10.13, expressed in terms independent of its mass.

10.51. Use the energy method to solve Problem 10.15 for the period of the small amplitude oscillation.

10.52. Solve Problem 10.18 by use of the energy principle.

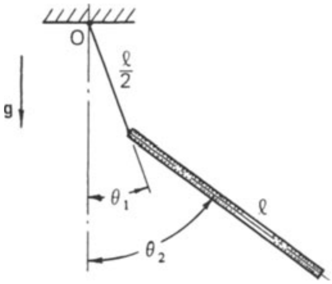
10.53. The sandbag body of the ballistic pendulum described in Fig.7.3, page 226, is replaced with a solid block B of mass M and moment of inertia I_O about point O. The block is supported by a rigid rod of negligible mass, and the length from O to the center of mass C of B is ℓ. A bullet of mass m is fired horizontally into the block in the direction of C, as shown previously, and the pendulum suffers a finite, maximum angular displacement θ_o. Find the muzzle speed of the bullet.

10.54. What is the small oscillation frequency of the mass m in Problem 6.57 when the mass M of the thin, rigid supporting rod, which has length $\ell = a + b$, is taken into account?

Formulate the problem using (i) Euler's equations and (ii) the energy method. What is the total force $\mathbf{R}(\theta)$ exerted by the hinge support bearing at O, as a function of θ?

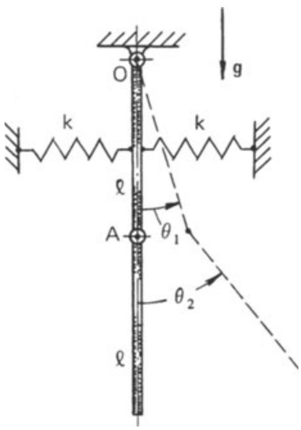
10.55. Apply the energy method to derive the first integral of the equation of motion for the system described in Problem 10.39. Derive an equation for the period of the finite amplitude motion.

10.56. The figure shows a slender rod of length ℓ and mass m suspended by an inextensible wire of negligible mass and length $\ell/2$, hinged at both ends. Derive exact expressions for the kinetic and potential energies of the system as functions of the angular placements θ_1 and θ_2 and their derivatives. Then determine their approximate quadratic forms when θ_k and $\dot{\theta}_k$ are small.



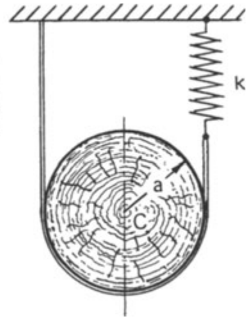
Problem 10.56.

10.57. Identical slender rods of length ℓ and mass m are hinged at O and A. Identical springs of stiffness k are attached at the midpoint of the upper rod, as shown. Determine the total kinetic and potential energies of the system for small angular placements θ_1 and θ_2 of the rods from their vertical positions.



Problem 10.57.

10.58. A homogeneous cylindrical log of radius a and mass m is supported by a thin inextensible belt of negligible mass, suspended by a linear spring, as shown in the figure. In the vertical displacement of the center of mass of the log from the natural state, the log rolls on the belt without slipping. (i) Apply Euler's laws to determine the frequency of the vertical oscillation about the equilibrium state. (ii) Apply the energy method to obtain the frequency.



Problem 10.58.

10.59. Apply the energy method to solve Problem 6.59 for small oscillations of the system. The thin rod has mass $3m$ and length $\ell = 4b$, very nearly. Express the results in terms of m and b.

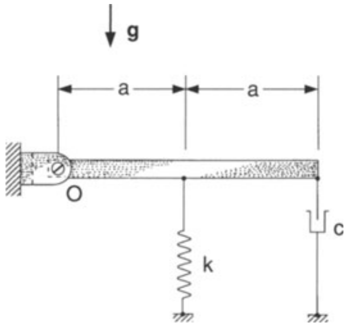
10.60. The punched plate described in Problems 9.36 is suspended in the vertical plane by a smooth hinge whose axis is perpendicular to the plane of the plate at a corner point Q. The plate is released from rest at an angle ϕ_0. (a) Apply the energy principle, and determine the support reaction force as a function of ϕ. (b) For small ϕ, determine the frequency f of the angular motion. What is the length of a simple pendulum having the same frequency?

10.61. A homogeneous, thick-walled circular tube of mass M and inner and outer radii r and R, respectively, rolls without slipping down a plane inclined at an angle β to the horizontal plane. (a) Find the angular velocity of the tube when its center has descended a vertical distance h from its initial place of rest. Express the result in terms of assigned parameters. (b) Determine the winner in a race down the plane between the tube and a solid cylinder of radius R and equal mass M, starting from the same line.

10.62. An inhomogeneous circular plate of radius R has a mass density σ, per unit area, that varies linearly with the distance from its center C where its value is σ_0. Its value at the rim is $2\sigma_0$. The plate is supported in the vertical plane by a smooth bearing at a point Q, at a distance $R/2$ from C. (a) Find as a function of R alone the small amplitude frequency of the oscillation of the plate about its vertical equilibrium position. (b) What is the length of an equivalent simple pendulum? (c) Find the exact finite amplitude solution expressed in terms of a Jacobian elliptic function?

10.63. A thin rigid rod of length $2a$ and mass m is statically supported horizontally by a smooth hinge at O, a linear spring of stiffness k, and a linear viscous damper with damping coefficient c, positioned as shown in the figure. (a) Apply the general energy principle (10.131) to derive the equation for the small angular motion $\theta(t)$ of the rod from its horizontal equilibrium state, in terms of the assigned parameters. (b) Characterize the lightly damped, free vibrational motion of the rod for the initial data $\theta(0) = 0, \dot{\theta}(0) = \omega_0$, and determine its frequency. (c) Describe the corresponding undamped motion and find its period.

10.64. A homogeneous, slender, curved rigid link of mass m and length l is hinged at its ends A and B to slider blocks that are constrained to move in smooth perpendicular slots in the horizontal plane similar to the device in Fig. 10.8, page 442. The center of mass is at the midpoint on the line of length $2d$ joining A and B. The link is moving in a viscous medium that exerts a drag force per unit length s along the link in accordance with Stokes's law. (a) Apply Euler's laws to derive the differential equation (10.151) for the motion $\theta(t)$ of the link. Find the angular motion $\theta(t)$ of a link whose initial angular velocity is ω_0 at θ_0. (b) What can be said about the physical nature of the slot reaction forces? Use the mechanical power rule (10.121) to derive the equation of motion.



Problem 10.63.

10.65. A uniform slender hoop of radius a and mass m rolls without slipping on a fixed horizontal surface. The motion occurs in a viscous medium that exerts a Stokes drag force per unit length s along the hoop. (a) Use Euler's laws to derive the differential equation for its angular motion $\theta(t)$. (b) Describe the nature of the nonviscous forces acting on the hoop. Apply the mechanical power rule (10.121) to derive a differential equation for the total kinetic energy, and thereby deduce the equation of motion for $\theta(t)$. Is this the same equation obtained in (a)? (c) Find the contact forces as functions of time, with $\omega(0) = \omega_0$ initially.

10.66. A homogeneous slender rod of length l and mass m turns about a smooth hinge at a point O distant $b > l/2$ from one end, and within a viscous medium that exerts a Stokes drag force per unit length s along the rod. Derive by two methods the equation motion for the rod (a) in the horizontal plane, and (b) in the vertical plane. (c) Compare the results when the rod is supported at its center of mass?

10.67. A homogeneous circular disk in the horizontal plane has radius R, mass m, and spins about a smooth fixed axle at its center C under a constant torque $\mathbf{T}_C = T\mathbf{k}$ in a Stokes medium. Initially, the plate has angular velocity $\boldsymbol{\omega}_0 = \boldsymbol{\omega}(0)$. (a) Determine the angular motion $\omega(t)$ of the disk. (b) Determine the motion when the torque is suddenly removed.

10.68. A homogeneous thin plate of mass m is suspended in the vertical plane by a smooth hinge at Q and released with initial speed ω_0 at the placement θ_0. The plate moves in a Stokes medium that exerts a light drag force over its entire surface. (a) Derive the equation of motion and provide its solution. (b) How are the results affected, if Q is at the center of mass of the plate?

11

Introduction to Advanced Dynamics

11.1. Introduction

At the age of 19, recognized for his extraordinary mathematical abilities, Joseph Louis de Lagrange (1736–1813) was appointed professor of geometry and mechanics at the Royal Artillery School at Turin, Italy, his birthplace. Here he developed his method of variations, invented earlier by Euler (in 1744) who later named it the calculus of variations. Lagrange left Turin in 1766 to become director of the Berlin Academy of Sciences until 1787 when, at the invitation of King Louis XVI of France, he was appointed to the Paris Academy of Sciences.[††] Shortly thereafter his most celebrated work, *Mécanique Analytique*, appeared in 1788, nearly a century after the appearance of Newton's *Principia*. Therein, Lagrange sets down an energy based approach for *dynamics—the analysis of motion*.

Inspired and strongly influenced by his senior contemporaries D'Alembert (1717–1785) and Euler (1707–1783), Lagrange linked the classical concepts and postulates of others in an invariant formulation of the equations of classical mechanics, now known as *Lagrange's equations*. The method begins with construction of a single scalar function of the total kinetic and potential energies, called the Lagrangian function, and for general dynamical systems it employs the method of virtual work to identify the nonconservative generalized forces. Although Lagrange's analytical mechanics embraces the theories of Newton and Euler, as it must, but in terms of work and energy, we shall see that it does not explicitly identify specific concepts of momentum, moment of momentum, center of mass, and rigidity. With these classical concepts in hand, Lagrange's method provides a systematic scheme for the formulation of the equations of motion and

[††] Lagrange's life and times are sketched in the translators' "Introduction" in Lagrange's *Analytical Mechanics* cited in the chapter References. See also Truesdell's *Essays*.

their first integrals for any multidegree of freedom dynamical system consisting of any number of particles and rigid bodies.

Although a detailed study of Lagrange's analytical dynamics is beyond the scope of this Introduction, still, we can accomplish a great deal. Our objective is to derive Lagrange's equations of motion for all sorts of classical (holonomic) dynamical systems, both conservative and nonconservative, consisting of a particle, a system of particles, a rigid body, several connected rigid bodies, in fact, any combination of these objects. First, various kinds of system constraints are discussed. Then, Lagrange's equations of motion for a particle are formulated and illustrated in some applications. Their straightforward extension for a system of particles follows. Hamilton's principle of stationary action, a method based upon the calculus of variations, is introduced, and Lagrange's equations are then derived from this principle without mention of any specific dynamical system. A number of examples are exhibited along the way.

11.2. Generalized Coordinates, Degrees of Freedom, and Constraints

We begin with a description of degrees of freedom and system constraints. Recall from Chapter 2 that the number of degrees of freedom of a dynamical system is the number of *independent* coordinates required to uniquely specify the location and orientation of all material points of the system relative to an assigned reference frame. A rigid disk free to move in the xy-plane, for example, has three degrees of freedom; two coordinates (x_p, y_p) specify the location of any disk point P and one coordinate θ provides the angle of rotation of the disk about its normal axis, say. If P is constrained to move on a specified path $y_p = f(x_p)$, only two coordinates x_p and θ are independent and hence the disk now has two degrees of freedom. In general, if there are c independent kinematical constraint equations relating the n coordinates, there remain $n - c = d$ independent coordinates, i.e. degrees of freedom.

The number of degrees of freedom is strictly a property of the system; it is independent of the particular coordinates used to uniquely specify the configuration of the system. Imagine, for example, that the dynamical system requires m Cartesian coordinates x_k, $k = 1, 2, \ldots, m$, to uniquely specify its configuration in a Cartesian frame ψ at an instant t, and these m coordinates are related by r kinematical equations of constraint. Then $d = m - r$. Suppose, on the other hand, that the x_k coordinates are related to another set of p *generalized coordinates* q_l, $l = 1, 2, \ldots, p$, that uniquely specify the system configuration in ψ at the instant t so that, in general, $x_k = x_k(q_1, q_2, \ldots, q_p, t) \equiv x_k(q_l, t)$, say. These equations describe the transformation from the set of ordinary coordinates x_k to the set of generalized coordinates q_l for a fixed t. If the new coordinates q_k are related by s kinematical equations of constraint, then, regardless of the particular set of coordinates used to specify the configuration in ψ at time t, $d = m - r = p - s$, the number of degrees of freedom of the system is the same. It may not be possible, however, to solve the constraint equations and thus eliminate the dependent variables. This is discussed next.

11.2.1. Holonomic Constraints

Kinematical constraints are classified as either holonomic or nonholonomic. First consider a system described by p generalized coordinates q_l related by s algebraic, kinematical constraint equations

$$f_j(q_l, t) \equiv f_j(q_1, q_2, \ldots, q_p, t) = 0, \qquad j = 1, 2, \ldots, s < p. \tag{11.1}$$

Kinematical constraints of this kind, or any that can be recast in this form as discussed later, are called *holonomic constraints.* In principle, these constraint equations for s coordinates can be solved in terms of the remaining $p - s = d$ coordinates and the time t, thus retaining only as many generalized coordinates as there are degrees of freedom. The elimination of s variables among the p generalized coordinates by use of (11.1), however, generally is quite awkward. In most cases, we try to find a set of generalized coordinates that describe the constrained system without our actually having to use the constraint equations.

For illustration, consider a simple pendulum whose bob is supported by a rigid rod of length ℓ and negligible mass, and pivoted at a support O. The general configuration of the bob is specified by the three Cartesian coordinates (x, y, z) in a reference frame $\psi = \{O; \mathbf{i}_k\}$. Now, suppose that the motion is confined to the vertical xy-plane in ψ. The two obvious constraint equations of the form (11.1) are $f_1(x, y, z, t) = z = 0$ and $f_2(x, y, z, t) = x^2(t) + y^2(t) - \ell^2 = 0$ relating x and y, and therefore this system has $3 - 2 = 1$ degree of freedom. On the other hand, for the plane motion of the pendulum, we also may write $x = \ell \cos q$, $y = \ell \sin q$, relating each of the Cartesian coordinates to the single generalized coordinate angle $q \equiv \theta(t)$ which completely describes the single degree of freedom motion of the pendulum without violating the constraints and without our having actually to use the constraint equations.

The aforementioned holonomic constraints do not depend *explicitly* on the time t, so these are further classified as *scleronomic constraints.* Holonomic constraints that depend explicitly on time are called *rheonomic constraints.* A holonomic dynamical system, therefore, is respectively classified as *rheonomic* if one or more constraints are time-dependent, or *scleronomic* when all constraints are time-independent. Suppose that a particle P at $\mathbf{x} = \hat{\mathbf{x}}(X, Y)$ in the frame $\Phi = \{O; \mathbf{I}_k\}$ is constrained by forces to move on an inclined plane of slope b that is moving toward the right with a speed $v(t)$, a prescribed function of t. Then the coordinates of P must satisfy the holonomic constraint $f(X, Y, t) = Y - b(X - \int_0^t v(t)dt) = 0$, and now the position vector of P may be written as an explicit function $\mathbf{x} = \mathbf{x}(X, t)$ of only one independent coordinate and the time. This is an example of a rheonomic constraint of the type (11.1) in which $q_1 = X$, $q_2 = Y$. The holonomic constraint on the motion of a pendulum suspended from a moving support is another example. Sometimes a holonomic equation of constraint may be an inequality $f(q_l, t) \leq 0$

restricting the values of the generalized coordinates. We shall not encounter these kinds of bounded constraints in our studies here.

11.2.2. Nonholonomic Constraints

Kinematical constraints that are not of the form (11.1), or cannot be recast in that form, are called *nonholonomic constraints.* These kinds of constraints are expressible only in terms of differentials of the generalized coordinates and time in the form

$$\sum_{k=1}^{p} a_{jk}(q_l, t)dq_k(t) + b_j(q_l, t)dt = 0, \qquad j = 1, 2, \ldots, s < p, \tag{11.2}$$

where a_{jk} and b_j are certain functions of the p generalized coordinates q_l and the time t. Clearly, nonholonomic constraints are characterized by their being nonintegrable; otherwise, upon integration they would reduce to holonomic constraints of the type (11.1). As a consequence of their nonintegrability, the nonholonomic constraint equations (11.2) cannot be used to reduce the number of generalized coordinates. Therefore, nonholonomic systems always require more coordinates to specify the configuration of the system than there are degrees of freedom.

A circular disk of radius a situated in the vertical plane and rolling without slipping along a curved path $\mathscr{C}$ in the horizontal plane is described by two nonholonomic constraint equations. Suppose the point of rolling contact has Cartesian coordinates (x, y) in the frame $\Phi = \{O; \mathbf{i}_k\}$. Let $d\sigma$ denote the elemental arc length along $\mathscr{C}$, and let ϕ denote the angle between the tangent vector to $\mathscr{C}$ and the x-axis. Then for rolling without slipping, we have $d\sigma = ad\theta$, the elemental arc length traced by the rim of the disk rotating through an angle $d\theta$. Thus, constraint equations of the kind (11.2) describing rolling contact without slip may be written as

$$dx - a\cos\phi d\theta = 0, \qquad dy - a\sin\phi d\theta = 0. \tag{11.3}$$

These are two independent constraint equations in four coordinates x, y, ϕ, and θ that cannot be used to reduce the number of generalized coordinates; nevertheless, the system has $4 - 2 = 2$ degrees of freedom. Because neither of these equations can be expressed as an exact differential of the form

$$df(x, y, \phi, \theta) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial \phi}d\phi + \frac{\partial f}{\partial \theta}d\theta = 0, \tag{11.4}$$

they are not integrable. If this were true, the constraints (11.3) could be recast as a holonomic constraint $f(x, y, \phi, \theta) = 0$ of the type (11.1). The difficulty arises because the value of the angle of rotation θ cannot be specified until the path or the length of the path along which the disk has rolled in reaching the point (x, y) is known. When the path is specified by the additional constraint that the disk roll parallel to the y-axis so that $\phi = \pi/2$ in (11.3), the system becomes holonomic with the integrable constraints $dx = 0$ and $dy = ad\theta = d\sigma$ that yield

$x = x_0,\ \ y = \sigma = y_0 + a\theta$, where $x_0,\ \ y_0$ are constants for which the origin O may be chosen so that these vanish.

A kinematical constraint given as a differential relation among the generalized coordinates may be holonomic or nonholonomic. This may prove quite difficult to decide. To assess its type, the relation must be tested for its integrability to determine the existence of integrals of the differential equations (11.2). The case $p = 2$ in (11.2) is exceptional, because a differential relation between two variables is always integrable, though not always exactly in closed form, and therein lies the difficulty. If relation (11.2) for any fixed $s < p$ is integrable, then that differential constraint is holonomic, otherwise not. If any one of the s constraints is not integrable, the system is nonholonomic. To illustrate the test procedure, let us consider a single differential, scleronomic constraint relation among $p = 3$ generalized coordinates (q_1, q_2, q_3); say, $a_{11}dq_1 + a_{12}dq_2 + a_{13}dq_3 = 0$, in which the coefficients $a_{jk} = a_{jk}(q_1, q_2, q_3)$ are certain nonzero, differentiable functions of $q_k(t)$. Then solving this relation for dq_3 and writing $C_k(q_1, q_2, q_3) \equiv -a_{1k}/a_{13}$, $k = 1, 2$, we form the differential relation

$$dq_3 = C_1(q_1, q_2, q_3)dq_1 + C_2(q_1, q_2, q_3)dq_2. \tag{11.5}$$

If there is a holonomic condition relating the three variables so that $q_3 = q_3(q_1, q_2)$, then in accordance with (11.5) we must have $C_1(q_1, q_2, q_3) = \partial q_3/\partial q_1$, $C_2(q_1, q_2, q_3) = \partial q_3/\partial q_2$, and hence the integrability condition $\partial^2 q_3/\partial q_2 \partial q_1 = \partial^2 q_3/\partial q_1 \partial q_2$, that is,

$$\frac{\partial C_1}{\partial q_2} + \frac{\partial C_1}{\partial q_3} C_2 = \frac{\partial C_2}{\partial q_1} + \frac{\partial C_2}{\partial q_3} C_1 \tag{11.6}$$

must be satisfied identically for all values of q_1 and q_2. If this result is an identity, then (11.5) is integrable and hence holonomic; but integration of the constraint to obtain $f(q_1, q_2, q_3)$ may not be apparent. We learn that an integral exists, but it is not revealed. On the other hand, if (11.6) yields a relation $q_3 = q_3(q_1, q_2)$, then we test this relation to see whether or not $\partial q_3/\partial q_1 = C_1(q_1, q_2, q_3)$ and $\partial q_3/\partial q_2 = C_2(q_1, q_2, q_3)$. If these hold, then the relation $q_3 = q_3(q_1, q_2)$ yields the desired holonomic constraint equation $f(q_1, q_2, q_3) = 0$. See Problem 11.1 for an equivalent alternative method.

Exercise 11.1. Show that the following differential, scleronomic constraint is holonomic, and determine its algebraic form (11.1):

$$(\sin^2 y - e^{2x} - z)e^x dx + (z - \sin^2 y - e^x \sin y)\cos y dy + e^x dz = 0. \qquad \square$$

This concludes the discussion of kinematical constraints. In this book, only holonomic constraints, mainly of the scleronomic type and usually so evident as to require no special attention, are encountered. The further study of nonholonomic constraints is left for advanced study. See the texts listed in the References.

11.3. Lagrange's Equations of Motion for a Particle

Our immediate objective is to derive the first fundamental form of Lagrange's equations of motion for a particle. Later, however, independent of the specific nature of the dynamical system, it is shown that the same energy based equations of motion hold for far more complex, multidegree of freedom dynamical systems consisting of several particles and rigid bodies. For simplicity, however, let us begin with a particle P with position vector $\mathbf{x}(P,t) = x_k\mathbf{i}_k$ in a Cartesian reference frame $\Phi = \{O; \mathbf{i}_k\}$. The use of rectangular Cartesian coordinates x_k, as we know, is not always convenient; so, a more suitable set of independent generalized coordinates q_k , say, that also may serve to specify uniquely and more naturally the configuration of P in Φ is introduced. Then the coordinates x_k are certain functions of these generalized coordinates q_k and perhaps time t. For example, if cylindrical coordinates $(r, \phi, \hat{z})$ are chosen as the generalized coordinates $q_1 = r,\ q_2 = \phi$, and $q_3 = \hat{z}$ to describe the motion of a particle in a frame ψ that has a specified motion $\zeta(t)$ along the z-axis in Φ, these are related to the regular Cartesian coordinates (x, y, z) in Φ by the coordinate transformation relations

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = \hat{z} + \zeta(t) = z(\hat{z}, t).$$

This typical sort of change of variables may be written as $x_k = x_k(r, \phi, \hat{z}, t) = x_k(q_1, q_2, q_3, t)$; or briefly, $x_k = x_k(q_j, t)$ in which $q_j = q_j(t)$. We thus introduce, more generally,

$$\mathbf{x} = \mathbf{x}(q_j, t), \tag{11.7}$$

in which the q_j are independent generalized coordinates, the actual number of which will depend on the number of degrees of freedom, hence the number of holonomic constraints, if any be imposed.

11.3.1. Two Useful Identities

We now derive two important identities relating partial derivatives that arise in the formulation of Lagrange's equations. Since $q_j = q_j(t)$ are functions of time t, differentiation of (11.7) with respect to time gives the velocity vector

$$\dot{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{x}}{\partial t}. \tag{11.8}$$

Here and throughout this chapter the summation rule (see Chapter 3) applies to twice repeated indices, unless explicitly noted otherwise. The quantities $\dot{q}_j$ are named the *generalized velocity components*, or briefly the *generalized velocities*. In view of (11.8), the particle's velocity vector is a function of the independent variables q_k, $\dot{q}_k$, and t; namely, $\dot{\mathbf{x}} = \dot{\mathbf{x}}(q_k, \dot{q}_k, t)$. Notice, however, that $\partial\mathbf{x}/\partial q_j$ and $\partial\mathbf{x}/\partial t$ are independent of the generalized velocities $\dot{q}_j$. Consequently, recalling the definition (3.2) of the Kronecker delta and noting that $\partial\dot{q}_j/\partial\dot{q}_k = \delta_{jk}$, we obtain from (11.8) our first identity, the *rule of cancellation of the dots*:

$$\frac{\partial \dot{\mathbf{x}}}{\partial \dot{q}_k} = \frac{\partial \mathbf{x}}{\partial q_k}. \tag{11.9}$$

Further, since $\partial \mathbf{x}/\partial q_k$ are functions of q_k and t alone, we have

$$\frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial q_k}\right) = \frac{\partial^2 \mathbf{x}}{\partial q_j \partial q_k}\dot{q}_j + \frac{\partial^2 \mathbf{x}}{\partial t \partial q_k}.$$

By (11.8), however,

$$\frac{\partial \dot{\mathbf{x}}}{\partial q_k} = \frac{\partial^2 \mathbf{x}}{\partial q_k \partial q_j}\dot{q}_j + \frac{\partial^2 \mathbf{x}}{\partial q_k \partial t}.$$

We shall require that $\mathbf{x}(q_k, t)$ has continuous second partial derivatives with respect to q_k and t in the domain considered. Then the last two expressions are identical, and hence follows our second identity, the *rule for interchange of derivatives*:

$$\frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial q_k}\right) = \frac{\partial \dot{\mathbf{x}}}{\partial q_k} = \frac{\partial}{\partial q_k}\left(\frac{d\mathbf{x}}{dt}\right). \tag{11.10}$$

11.3.2. First Fundamental Form of the Lagrange Equations of Motion

We are now prepared to derive Lagrange's equations for a particle, based on its kinetic energy. In the Lagrangian theory, however, it is customary to denote the kinetic energy by T so that in an arbitrary motion of the particle

$$T \equiv K = \frac{1}{2}m\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}. \tag{11.11}$$

Because $\dot{\mathbf{x}} = \dot{\mathbf{x}}(q_k, \dot{q}_k, t)$ in (11.8), $T = T(q_k, \dot{q}_k, t)$ as well. Hence, from (11.11),

$$\frac{\partial T}{\partial q_k} = m\dot{\mathbf{x}} \cdot \frac{\partial \dot{\mathbf{x}}}{\partial q_k}, \qquad \frac{\partial T}{\partial \dot{q}_k} = m\dot{\mathbf{x}} \cdot \frac{\partial \dot{\mathbf{x}}}{\partial \dot{q}_k} = m\dot{\mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial q_k}, \tag{11.12}$$

wherein the rule of cancellation of the dots (11.9) is introduced. Further, differentiation of the second equation in (11.12) with respect to t and use of rule (11.10) for the interchange of derivatives yields

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = m\ddot{\mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial q_k} + m\dot{\mathbf{x}} \cdot \frac{\partial \dot{\mathbf{x}}}{\partial q_k}.$$

Therefore, use of this relation and the first equation in (11.12) yields the result

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = m\ddot{\mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial q_k}. \tag{11.13}$$

Thus far, no laws of motion have been imposed; so (11.13), except for the presence of the mass m, is essentially a purely kinematical result. Now introduce on the right-hand side of (11.13) the Newton–Euler equation of motion $\mathbf{F} = m\ddot{\mathbf{x}}$ for a particle of mass m acted upon by a force $\mathbf{F} = \mathbf{F}(q_k, \dot{q}_k, t)$, where $\mathbf{x}$ is its position vector in an *inertial* reference frame $\Phi = \{O; \mathbf{i}_k\}$, and define the *generalized forces* $Q_k = Q_k(q_k, \dot{q}_k, t)$ by the relation

$$Q_k \equiv \mathbf{F} \cdot \frac{\partial \mathbf{x}}{\partial q_k}. \tag{11.14}$$

Then (11.13) yields the *first fundamental form of Lagrange's equations*,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k, \tag{11.15}$$

for the unconstrained ($k = 1, 2, 3$) *or at most holonomic constrained* ($1 \le k < 3$) *motion of a particle.* The force $\mathbf{F}(q_j, \dot{q}_j, t)$, hence the generalized forces $Q_k = Q_k(q_j, \dot{q}_j, t)$, includes all sorts of applied forces, such as conservative, nonconservative, time varying driving forces, and forces of constraint.

11.4. The Generalized Forces and Virtual Work

The generalized forces $Q_k(q_k, \dot{q}_k, t)$ may be found from the virtual work done by the total force $\mathbf{F}(q_k, \dot{q}_k, t)$ acting on a particle over a small *virtual* (*imaginary*) *displacement* $\delta\mathbf{x}(q_k)$ corresponding to arbitrary small virtual increments δq_k in the generalized coordinates during which *time is held fixed* and the *applied forces do not change.* The virtual increments δq_k, also called virtual displacements, must respect the kinematic constraints, any moving constraints being momentarily halted with time. By (11.7), the virtual displacement vector $\delta\mathbf{x}$ is given by

$$\delta\mathbf{x} = \frac{\partial \mathbf{x}}{\partial q_k}\delta q_k. \tag{11.16}$$

In the presence of any holonomic constraints (11.1), the virtual displacements must respect the corresponding constraint conditions $\delta f_j = (\partial f_j/\partial q_k)\delta q_k = 0$, valid for both scleronomic and rheonomic systems. This is not the same as the differential of f_j for "real" displacements dq_k for which $df_j = (\partial f_j/\partial q_k)dq_k + (\partial f_j/\partial t)dt = 0$. Here df_j is the infinitesimal change in $f_j(q_k, t)$ when both q_k and t are varied, whereas $\delta f_j(q_k, t)$ is the infinitesimal change in $f_j(q_k, t)$ when only q_k are varied. These are the same only for scleronomic constraints. The nonholonomic constraints given in (11.2) for real displacements are replaced by $\sum_{k=1}^{p} a_{jk}\delta q_k = 0$ for virtual displacements in both scleronomic and rheonomic systems; however, nonholonomic constraints are not encountered in this text.

The *virtual work* $\delta\mathscr{W}$ done by the total force acting on the particle over its virtual displacement $\delta\mathbf{x}$ is defined by

$$\delta\mathscr{W} = \mathbf{F} \cdot \delta\mathbf{x}. \tag{11.17}$$

Substitution of (11.16) and the use of (11.14) yields, equivalently,

$$\delta\mathscr{W} = Q_k \delta q_k. \tag{11.18}$$

This is the virtual work done by the generalized forces frozen in time, i.e. treated as constants, and acting over the generalized virtual displacements that satisfy the constraints. The generalized forces may be found from these results.

If the virtual work done by the total force **F** in a virtual displacement compatible with the constraints vanishes in (11.17), by (11.18), the generalized total forces Q_k also are workless. Moreover, for holonomic systems, the δq_k may be independently chosen, and hence, by (11.18), $Q_k = 0$, the corresponding generalized total forces must vanish. Consider the part **P** of the total force **F** with corresponding generalized forces $Q_k^P = \mathbf{P} \cdot \partial \mathbf{x}/\partial q_k$, defined in accordance with (11.14), which does no virtual work. Then $Q_k^P = 0$. Hence, nontrivial generalized forces arise only from those applied forces that do virtual work.* Consequently, workless forces of constraint contribute nothing to Lagrange's equations (11.15) for the motion of a particle.

Example 11.1. Apply Lagrange's equations (11.15) to derive the equations of unconstrained motion of a particle in cylindrical coordinates.

Solution. The three independent generalized coordinates and generalized velocity components for the unconstrained motion of a particle in terms of cylindrical coordinates are defined by

$$(q_1, q_2, q_3) = (r, \phi, z), \qquad (\dot{q}_1, \dot{q}_2, \dot{q}_3) = (\dot{r}, \dot{\phi}, \dot{z}). \tag{11.19a}$$

It should be noted that the generalized coordinates and velocities are not the respective physical scalar components of either the actual position vector $\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z$ or the velocity vector $\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z$ in cylindrical coordinates. The kinetic energy function in cylindrical coordinates is given by

$$T = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2). \tag{11.19b}$$

Hence, with (11.19a) and (11.15) in mind, we use (11.19b) to first derive

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} &= \frac{d}{dt}(m\dot{r}) - mr\dot{\phi}^2 = m(\ddot{r} - r\dot{\phi}^2),\\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} &= \frac{d}{dt}(mr^2\dot{\phi}) = m\frac{d}{dt}(r^2\dot{\phi}),\\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} &= \frac{d}{dt}(m\dot{z}) = m\ddot{z}.
\end{aligned} \tag{11.19c}$$

To complete the formulation of the equations of motion from the Lagrange equations (11.15), we next consider the generalized forces in (11.15). The virtual work

* We may think of the result (11.18) in the space of the q_ks as a generalized scalar product $\mathbf{Q}^P \cdot \delta\mathbf{q}$ that vanishes if and only if $\mathbf{Q}^P$ is "perpendicular" to $\delta\mathbf{q}$. Because the only vector perpendicular to every vector is the zero vector, $\mathbf{Q}^P \cdot \delta\mathbf{q} = 0$ for all $\delta\mathbf{q} \neq \mathbf{0}$ implies that $\mathbf{Q}^P = \mathbf{0}$ for all workless holonomic constraints.

(11.17) done by the total force $\mathbf{F} = F_r\mathbf{e}_r + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z$ in the virtual displacement $\delta\mathbf{x} = \delta r\mathbf{e}_r + r\delta\phi\mathbf{e}_\phi + \delta z\mathbf{e}_z$ is given by

$$\delta\mathscr{W} = F_r\delta r + rF_\phi\delta\phi + F_z\delta z. \tag{11.19d}$$

By (11.18), the virtual work done by the generalized forces $(Q_1, Q_2, Q_3) = (Q_r, Q_\phi, Q_z)$ acting over the generalized virtual displacements $(\delta q_1, \delta q_2, \delta q_3) = (\delta r, \delta\phi, \delta z)$ is

$$\delta\mathscr{W} = Q_r\delta r + Q_\phi\delta\phi + Q_z\delta z. \tag{11.19e}$$

Since there are no constraints, the virtual work relations in (11.19d) and (11.19e) must be equal *for all* arbitrary virtual displacements $\delta r, \delta\phi, \delta z$, and hence the generalized force components are given by

$$Q_r = F_r, \qquad Q_\phi = rF_\phi, \qquad Q_z = F_z. \tag{11.19f}$$

Notice that $[Q_\phi] = [FL]$ has dimensional units of torque. We thus see that the generalized forces need not have dimensional units of force, as do Q_r and Q_z.

Use of the two sets of results (11.19c) and (11.19f) in the Lagrange equations (11.15) now yields the familiar equations of unconstrained motion for a particle in terms of its cylindrical coordinates:

$$m(\ddot{r} - r\dot{\phi}^2) = F_r, \qquad \frac{m}{r}\frac{d}{dt}(r^2\dot{\phi}) = F_\phi, \qquad m\ddot{z} = F_z. \tag{11.19g}$$

These agree with equations (6.4) based on Newton's second law. □

Exercise 11.2. (a) Recall the position vector in cylindrical coordinates and note that $\mathbf{e}_r$ and $\mathbf{e}_\phi$ are known functions of ϕ alone. Apply the definition (11.14) to derive the generalized force components (11.19f). (b) In view of (11.9), the generalized forces (11.14) also may be written as

$$Q_k = \mathbf{F} \cdot \frac{\partial \mathbf{v}}{\partial \dot{q}_k}. \tag{11.20}$$

Use this result to derive the generalized forces in (11.19f). This is the simpler of these two methods, because the basis vectors depend on the generalized coordinates q_k, not the generalized velocities $\dot{q}_k$. □

Exercise 11.3. Apply Lagrange's equations (11.15) to derive the equations of unconstrained motion of a particle in spherical coordinates (r, θ, ϕ). Find the generalized forces (a) by use of the method of virtual work, and (b) by application of both (11.14) and (11.20). Compare the results with (6.5). □

We have seen that the generalized force components corresponding to a workless applied force $\mathbf{P}$, including any workless force of holonomic constraint, must vanish. The same result follows readily from (11.20). To see this, consider a force

P that does no work in the real motion. This force is perpendicular to the particle's path, and hence to its velocity vector $\mathbf{v} = \mathbf{v}(\dot{q}_k, q_k, t) = v(\dot{q}_k, q_k, t)\mathbf{t}(q_k, t)$, where $\mathbf{t}(q_k, t) = d\mathbf{x}(q_k, t)/ds$ is the unit tangent vector. By (11.20), since $\mathbf{P} \cdot \mathbf{t}(q_k, t) = 0$, the corresponding generalized forces $Q_k^P = \partial v(\dot{q}_j, q_j, t)/\partial \dot{q}_k \mathbf{P} \cdot \mathbf{t}(q_j, t) \equiv 0$. For a specific illustration, let the reader consider the workless force of constraint $\mathbf{P} = T(\theta)\mathbf{n}(\theta)$ that acts on the bob of a simple pendulum having a single degree of freedom with $q_1 = \theta$, the usual angular placement. Note that $\mathbf{v} = l\dot{\theta}\mathbf{t}(\theta)$, and thus confirm that the corresponding generalized force $Q_1^P = Q_\theta^P = 0$. The gravitational part **W** of the total force, however, does virtual work $\mathbf{W} \cdot \delta\mathbf{x} = \mathbf{W} \cdot l\delta\theta\mathbf{t} = Q_\theta^W \delta\theta$ from which $Q_\theta^W = -mgl \sin\theta$.

11.5. The Work–Energy Principle for Scleronomic Systems

For holonomic systems of scleronomic type, the first integral of Lagrange's equations with respect to the generalized displacements is the familiar work–energy principle (7.36). To demonstrate this result, let us consider only those dynamical systems for which (11.7) is not an explicit function of t so that $\partial\mathbf{x}/\partial t \equiv \mathbf{0}$ in (11.8); then,

$$\mathbf{x} = \mathbf{x}(q_r(t)), \qquad \dot{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial q_k}\dot{q}_k. \tag{11.21}$$

This is a special case for which all of the previous results apply.

First, recall (7.21) and introduce the second relation in (11.21) to obtain

$$\mathscr{W} = \int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{x} = \int_{\mathscr{C}} \mathbf{F} \cdot \frac{\partial \mathbf{x}}{\partial q_k} dq_k, \tag{11.22}$$

in which $\mathbf{F} = \mathbf{F}(\mathbf{x}(q_r))$ and $\mathscr{C}$ is the particle's path. Then use of (11.14) in which $Q_k = Q_k(q_r)$ delivers *the work done by the generalized forces acting over the generalized displacements*:

$$\mathscr{W} = \int_{\mathscr{C}} Q_k dq_k. \tag{11.23}$$

The kinetic energy (11.11) may be cast in terms of the generalized velocities by use of the second expression in (11.21) to obtain

$$T = \frac{1}{2} M_{jk}(q_r)\dot{q}_j\dot{q}_k. \tag{11.24}$$

Remember that repeated indices are summed over their range, and note that $T(q_r, \dot{q}_r)$ is not an explicit function of time, which is generally the case when there are no moving constraints. Hence, (11.24) is a homogeneous function of degree 2 wherein the symmetric coefficient matrix $M_{jk} = M_{kj}$, which need not have physical dimensions of mass, is a function of the generalized coordinates only:

$$M_{jk}(q_r) \equiv m \frac{\partial \mathbf{x}}{\partial q_j} \cdot \frac{\partial \mathbf{x}}{\partial q_k}. \tag{11.25}$$

It thus follows from (11.24) that

$$\frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T. \tag{11.26}$$

Finally, we consider

$$\frac{dT(\dot{q}_r, q_r)}{dt} = \frac{\partial T}{\partial \dot{q}_k} \ddot{q}_k + \frac{\partial T}{\partial q_k} \dot{q}_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k} \dot{q}_k\right) - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) \dot{q}_k + \frac{\partial T}{\partial q_k} \dot{q}_k.$$

Use of (11.26) in the first time derivative on the far right-hand side yields

$$\frac{dT(\dot{q}_r, q_r)}{dt} = \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right) \dot{q}_k. \tag{11.27}$$

This is equivalent to our multiplying the k^{th} Lagrange equation (11.15) by $\dot{q}_k$ and adding the results, as indicated by the right-hand sum of (11.27), to obtain

$$\frac{dT(\dot{q}_r, q_r)}{dt} = Q_k \dot{q}_k. \tag{11.28}$$

Integration of this equation in accord with (11.23) yields *the work–energy principle for scleronomic systems*:

$$\mathscr{W} = \Delta T, \tag{11.29}$$

as a first integral of the Lagrange equations (11.15). Moreover, based on (7.38), this result also provides an expression for the *mechanical power*: $\mathscr{P} = dT/dt = d\mathscr{W}/dt$. Notice that if T were to depend explicitly on time, additional terms would arise in (11.24) and (11.27), and the work–energy equation (11.29) would not hold.

Exercise 11.4. Begin with (11.11) for the kinetic energy of a particle and derive (11.26). □

Exercise 11.5. Begin with (11.7) and (11.8) and show that the kinetic energy for a particle has the general explicit time dependent form

$$T(\dot{q}_r, q_r, t) = A_{ij}(q_r, t)\dot{q}_i \dot{q}_j + B_j(q_r, t)\dot{q}_j + C(q_r, t), \tag{11.30}$$

in which $A_{ij} = A_{ji}$. Identify the coefficient functions and thus establish that for scleronomic systems, $B_j(q_r, t) = 0$, $C(q_r, t) = 0$, and $2A_{ij}(q_r, t) = M_{ij}(q_r)$ defined by (11.25). Clearly, the general form of the time dependent kinetic energy function in generalized coordinates is far more complex than its scleronomic form. □

11.6. Conservative Scleronomic Dynamical Systems

When the total force is conservative, there exists a potential energy function $\hat{V}(\mathbf{x}(q_k)) \equiv V(q_k)$ that depends only on the generalized coordinates such that $\mathbf{F} = -\boldsymbol{\nabla}\hat{V}$. Then, in accordance with (11.14),

$$Q_k = -\boldsymbol{\nabla}\hat{V} \cdot \frac{\partial \mathbf{x}}{\partial q_k} = -\frac{\partial \hat{V}}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial q_k},$$

and hence the *conservative generalized forces* are given by

$$Q_k = -\frac{\partial V}{\partial q_k}. \tag{11.31}$$

The same relation holds if a part of the total force is workless and the remaining part is conservative. The generalized forces for the workless part vanish and the conservative part yields (11.31). In all such cases, the dynamical system is called conservative. If the system also is scleronomic, it follows from (11.23) that *the work done by conservative generalized forces of a scleronomic dynamical system is equal to the decrease in the total potential energy*:

$$\mathscr{W} = -\Delta V(q_r). \tag{11.32}$$

This is equivalent to (7.45). Moreover, by (11.29), *the total energy for a conservative, scleronomic dynamical system is constant throughout the motion*:

$$T(\dot{q}_k, q_k) + V(q_k) = E, \ \text{ a constant.} \tag{11.33}$$

Exercise 11.6. For conservative forces, the virtual work (11.18) may be expressed as $\delta\mathscr{W} = -\delta V$. Begin with this relation and show conversely that for unconstrained or holonomic systems (11.31) follows. □

11.7. Lagrange's Equations for General Conservative Systems

Let us return to the first fundamental form of Lagrange's equations (11.15) in which $T = T(\dot{q}_r, q_r, t)$ and all the Q_ks are derivable from a potential energy function in accord with (11.31). Now, introduce the *Lagrangian function* $L(\dot{q}_r, q_r, t)$ defined by

$$L(\dot{q}_r, q_r, t) \equiv T(\dot{q}_r, q_r, t) - V(q_r), \tag{11.34}$$

called briefly the *Lagrangian*. The potential energy does not depend on the generalized velocities, so $\partial V/\partial \dot{q}_k \equiv 0$. Therefore, upon substitution of (11.31) in (11.15) and introduction of the Lagrangian (11.34), we obtain the classical form of *Lagrange's equations of motion for a general conservative dynamical system*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \tag{11.35}$$

with $k = 1, 2, 3$ for a single, unconstrained particle.

This result also holds more generally for holonomic systems in which the potential function $V = V(q_r, t)$ may depend on time. The virtual work $\delta\mathscr{W} = -\delta V(q_r, t) = Q_k \delta q_k$ yields $Q_k(q_r, t) = -\partial V(q_r, t)/\partial q_k$. Therefore, with the Lagrangian defined by $L(\dot{q}_r, q_r, t) \equiv T(\dot{q}_r, q_r, t) - V(q_r, t)$, (11.15) again transforms to (11.35). The principle of conservation of energy, however, does not hold for a time dependent potential energy function. See the discussion regarding (7.79).

Thus, to obtain the equations of motion for a general conservative dynamical system, we need only determine the Lagrangian function (11.34) and apply (11.35). Lagrange's method, like the work–energy method, provides no information about the inessential workless forces of constraint.

Example 11.2. Derive Lagrange's equations of motion for the simple pendulum shown in Fig. 6.15, page 138.

Solution. The motion of the pendulum is restricted to the vertical plane and a rigid wire of negligible mass constrains the motion to a circle, so the scleronomic constraints are $z = 0$ and $r = \ell$. The tension in the string is workless, and the weight of the bob is a conservative force for which $V = mg\ell(1 - \cos\theta)$. Hence, this scleronomic system is conservative with one degree of freedom described by $q_1 = \theta$. The kinetic energy of the bob is $T = \frac{1}{2}m\ell^2\dot{\theta}^2$. Therefore, the Lagrangian (11.34) is

$$L = T - V = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1 - \cos\theta), \tag{11.36a}$$

and hence with

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}(m\ell^2\dot{\theta}) = m\ell^2\ddot{\theta}, \qquad \frac{\partial L}{\partial \theta} = -mg\ell\sin\theta, \tag{11.36b}$$

the Lagrange equations (11.35) yield the differential equation of motion for the pendulum bob:

$$m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0. \tag{11.36c}$$

The inessential, workless string tension of constraint, an incidental result of the Newton–Euler law in (6.67b), does not explicitly enter the argument. Of course, since only derivatives of V with respect to the generalized coordinates appear in Lagrange's equations (11.35), any constant in the potential energy function also is unimportant. Thus, in the present problem we could have written, for example, $V = -mg\ell\cos\theta$. □

Example 11.3. Derive the equations of motion for a particle of mass m moving in a plane under a central force $\mathbf{F} = -(\mu m/r^2)\mathbf{e}_r$, where μ is a constant and cylindrical coordinates are used.

Solution. The holonomic constraint $z = 0$ is evident, and the velocity vector is $\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi$. Hence, the kinetic energy is given by

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2). \tag{11.37a}$$

The central force is conservative with potential energy

$$V = -\mu m/r, \tag{11.37b}$$

as shown in (7.62) with $\mu \equiv MG$. Thus, the Lagrangian (11.34) for this conservative scleronomic dynamical system is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{\mu m}{r}. \tag{11.37c}$$

The generalized coordinates are $(q_1, q_2) = (r, \phi)$, the generalized velocity components are $(\dot{q}_1, \dot{q}_2) = (\dot{r}, \dot{\phi})$, and with (11.35) in mind we use (11.37c) to obtain

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \qquad \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}, \qquad \frac{\partial L}{\partial r} = mr\dot{\phi}^2 - \frac{\mu m}{r^2}, \qquad \frac{\partial L}{\partial \phi} = 0. \tag{11.37d}$$

Lagrange's equations (11.35) thus yield the equations of motion for this conservative system with two degrees of freedom:

$$\ddot{r} - r\dot{\phi}^2 + \frac{\mu}{r^2} = 0, \qquad \frac{d}{dt}(r^2\dot{\phi}) = 0. \tag{11.37e}$$

□

11.8. Second Fundamental Form of Lagrange's Equations

Suppose that some of the generalized forces are conservative with a total potential energy $V(q_k)$. Let the remaining generalized forces that are not workless be denoted by $Q_k^N = Q_k^N(\dot{q}_r, q_r, t)$; these are called *nonconservative generalized forces*. The dynamical system, in this case, is called *nonconservative*. In the absence of any constraints, with $Q_k = -\partial V/\partial q_k + Q_k^N$, and use of (11.34), (11.15) may be rewritten to obtain *the second fundamental form of Lagrange's equations*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k^N. \tag{11.38}$$

If it happens that the nature of a force is uncertain, the ambiguous force may be treated as a "nonconservative force" in the formulation of (11.38). Moreover, generalized forces of rheonomic constraints that are specified functions of time alone do no virtual work; for, with time held fixed, the corresponding virtual displacements are zero. Hence, these nonconservative forces do not enter the Lagrange equations of motion. On the other hand, these same nonconservative forces generally do work in the real motion of the system. This is illustrated in a subsequent example.

Example 11.4. Derive the equation of motion for a particle P that falls from rest in a Stokes medium.

Solution. The kinetic energy of P is $T = \frac{1}{2}m\dot{y}^2$, its gravitational potential energy is $V = -mgy$, and the nonconservative Stokes force is $\mathbf{F} = -c\mathbf{v} = -c\dot{y}\mathbf{j}$, where $\mathbf{j}$ is the downward direction of the motion. Let the reader apply the method of virtual work to find Q_1^N. Here we consider (11.20). Accordingly, with $q_1 = y$, the nonconservative generalized force in (11.38) is $Q_1^N = \mathbf{F} \cdot \partial\mathbf{v}/\partial\dot{q}_1 = -c\dot{y}\mathbf{j} \cdot \mathbf{j} = -c\dot{y}$. Now form the Lagrangian $L = T - V = \frac{1}{2}m\dot{y}^2 + mgy$, and apply (11.38) to obtain the equation of motion $m\ddot{y} - mg = -c\dot{y}$. That is, with $v = \dot{y}$ and $\nu = c/m$, we recover (6.34a): $dv/dt = g - \nu v$. □

Example 11.5. (a) Derive the Lagrange equations of motion for a heavy bead of mass m that slides freely in a smooth circular tube of radius a, as the tube spins with constant angular speed $\dot{\phi} = \omega$ about its fixed vertical axis, as shown in the diagram for Problem 6.66. Obtain the first integral of the equation of motion, and find the tangential constraint force normal to the plane of the tube. (b) Relax the rheonomic constraint, treat $\phi(t)$ as an independent generalized coordinate, and derive the Lagrange equations of motion for the bead.

Solution of (a). Introduce the spherical coordinates (a, ϕ, θ) of the bead in its constrained motion referred to a moving, spherical reference frame $\psi = \{O; \mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_\theta\}$ in which $\mathbf{e}_k$ are in the directions of their increasing coordinates, and note that the angle between $\mathbf{k}$ and $\mathbf{e}_r$ in the problem diagram is $\pi - \theta$. The workless scleronomic constraint is $r = a$, constant, and the working rheonomic constraint is $\dot{\phi} = \omega$, a constant, that is, $\phi = \phi_0 + \omega t$. Notice, alternatively, that a rheonomic constraint $\dot{\phi} = \omega(t)$, a specified function of t, also is holonomic with $\phi = \phi_0 + \int_{t_0}^{t} \omega(t)dt$. In either instance, ϕ is a specified function of time; it is not a generalized coordinate. The radial component of the nonconservative force $\mathbf{F} = -N\mathbf{e}_r + R\mathbf{e}_\phi$ exerted by the tube on the mass m is workless, but due to the moving tube constraint the component R perpendicular to the plane of the tube is not, so the system is nonconservative. Nevertheless, the *virtual work* done by this force in the virtual displacement $\delta\mathbf{x} = a\delta\theta\mathbf{e}_\theta + a\sin\theta\delta\phi\mathbf{e}_\phi$ compatible with the constraint $\delta\phi = \omega\delta t \equiv 0$ vanishes, $\delta\mathcal{W} = \mathbf{F} \cdot \delta\mathbf{x} = Ra\sin\theta\delta\phi = 0$, because the moving constraint in the virtual displacement is frozen. The bead has one degree of freedom described by $q_1 = \theta$ with the corresponding generalized force $Q_\theta^N = 0$.

The gravitational potential energy of m is given by

$$V = mga(1 - \cos\theta). \tag{11.39a}$$

The absolute velocity of the bead is $\mathbf{v} = a\dot{\phi}\sin\theta\mathbf{e}_\phi + a\dot{\theta}\mathbf{e}_\theta$, and hence its total kinetic energy is

$$T = \frac{1}{2}ma^2(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2). \tag{11.39b}$$

Now introduce $\dot{\phi} = \omega$, a constant, form the Lagrangian function (11.34),

$$L = \frac{1}{2}ma^2(\omega^2\sin^2\theta + \dot{\theta}^2) - mga(1 - \cos\theta), \tag{11.39c}$$

and thus derive

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = ma^2\ddot{\theta}, \qquad \frac{\partial L}{\partial\theta} = ma^2\omega^2\sin\theta\cos\theta - mga\sin\theta. \tag{11.39d}$$

Collecting the results in (11.38), we reach the Lagrange equation of motion for the bead:

$$\ddot{\theta} + \sin\theta\left(\frac{g}{a} - \omega^2\cos\theta\right) = 0. \tag{11.39e}$$

Notice that the result is independent of the mass m.

Equation (11.39e) written as $\ddot{\theta} = \dot{\theta}d\dot{\theta}/d\theta = f(\theta)$ has an easy first integral given by

$$\dot{\theta}^2 = \dot{\theta}_0^2 + \omega^2(\sin^2\theta - \sin^2\theta_0) + \frac{2g}{a}(\cos\theta - \cos\theta_0), \tag{11.39f}$$

in which $\dot{\theta}_0 = \dot{\theta}(0)$ and $\theta_0 = \theta(0)$ denote the initial values. For small angular hoop speeds the term in ω^2 may be neglected to obtain from (11.39e) and (11.39f) the equation of motion and its first integral for the finite amplitude oscillations of the bead as an equivalent simple pendulum.

The constraint force R is not determined; it is inconsequential to the determination of the general motion of the bead by Lagrange's method. Nevertheless, we can find R easily by writing the moment of momentum equation about the z-axis. The bead is at the instantaneous distance $a\sin\theta$ from this axis, and its momentum in the direction perpendicular to the plane of the tube is $ma\omega\sin\theta$. Therefore, the moment of momentum of the bead about the axis of rotation is $h_z = ma^2\omega\sin^2\theta$. The constraint driving torque relation about the spin axis is given by $\dot{h}_z = Ra\sin\theta$; and hence

$$R = 2ma\omega\dot{\theta}\cos\theta. \tag{11.39g}$$

Solution of (b). Now consider a different situation in which we ignore the rheonomic constraint and treat $\phi(t)$ as an independent variable. The bead now has two degrees of freedom described by $(q_1, q_2) = (\theta, \phi)$ with the corresponding nonconservative generalized forces (Q_θ^N, Q_ϕ^N). The virtual work done by these forces is given by $\delta \mathscr{W} = \mathbf{F} \cdot \delta\mathbf{x} = Ra \sin\theta \delta\phi = Q_\theta^N \delta\theta + Q_\phi^N \delta\phi$ for all $\delta\theta$ and $\delta\phi$, and hence

$$Q_\theta^N = 0, \qquad Q_\phi^N = Ra \sin\theta. \tag{11.39h}$$

The Lagrangian $L = \frac{1}{2} ma^2(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) - mga(1 - \cos\theta)$ is the same as (11.39c) in which $\omega = \dot{\phi}$. In addition to (11.39d), we now have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = ma^2\ddot{\phi} \sin^2\theta + 2ma^2\dot{\phi}\dot{\theta} \sin\theta \cos\theta, \qquad \frac{\partial L}{\partial \phi} = 0. \tag{11.39i}$$

Assembling the results (11.39d), (11.39h), and (11.39i) in (11.38), we obtain the θ-equation (11.39e), as before, and the new ϕ-equation:

$$ma\ddot{\phi} \sin\theta + 2ma\dot{\phi}\dot{\theta} \cos\theta = R. \tag{11.39j}$$

If $R(\dot{\theta}, \dot{\phi}, \theta, \phi, t)$ is some specified function, (11.39j) is a nonlinear differential equation for $\phi(t)$ coupled with (11.39e) in which $\omega = \dot{\phi}(t)$. On the other hand, for a specified function $\phi(t)$, and with $\theta(t)$ determined by (11.39e), (11.39j) is an equation that determines $R(t)$. In particular, for $\dot{\phi} = \omega$, a constant, (11.39j) gives $R = 2ma\omega\dot{\theta} \cos\theta$, which is the same nonconservative, rheonomic constraint force obtained in (11.39g). The original Lagrange method eliminates the need to determine the inconsequential working rheonomic constraint force, which may be found by other methods, if needed. □

Exercise 11.7. Apply Lagrange's equations (11.38) to derive the equation of motion for the forced vibration of the system shown in Fig. 6.20, page 152, for a linear viscous damper and a linear elastic spring. □

11.9. Lagrange's Equations for a System of Particles

Lagrange's equations for a single particle are equivalent to the Newton–Euler equations of motion; they contain no new principles. Unlike the Newton–Euler approach, however, the Lagrangian method never involves workless forces of holonomic constraint, the sometimes laborious calculation of accelerations is avoided, and for conservative systems the equations of motion are readily derived from a single scalar Lagrangian function. A bit more effort is required to determine nonconservative generalized forces, but the procedure is straightforward. In all, Lagrange's method often is easier to apply because only generalized coordinates and velocities are involved. On the other hand, the physics of the analysis is not always apparent. Subsequent developments, however, will shed light on the physical interpretation of certain terms that arise in the Lagrange formulation.

The aforementioned advantages are significantly magnified in applications of Lagrange's method to more complex systems. Recall that the equation of motion (5.41) for a system of particles "determines" only the motion of the center of mass. Finding the motions of the individual particles requires formulation of a vector equation of motion (5.39) for each particle, so it is necessary to include all of the forces that act on each particle separately, including mutual forces exerted by all of the other particles, and to introduce all of their accelerations. This procedure, even for simple problems, is cumbersome. The Lagrangian approach avoids these difficulties.

So far, the Lagrangian theory is strictly valid only for a single particle. We shall see that precisely the same equations and analytical structure emerge for a system of N particles. Without constraints, this system has $3N$ degrees of freedom, and hence $3N$ generalized coordinates are required to describe its configuration. When the constraints are *holonomic*, some of the generalized coordinates may be eliminated. In this case the number of degrees of freedom n is fewer than $3N$, and the remaining generalized coordinates are independent variables. In the following description, all functions of the n independent generalized variables $q_1, q_2, \ldots, q_n$ and their time derivatives are abbreviated by use of q_r and $\dot{q}_r$ alone, as done previously for a single particle for which $n \leq 3$. For example, $f(\dot{q}_r, q_r, t) \equiv f(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, q_1, q_2, \ldots, q_n, t)$.

Consider a system of N particles. The position vector (11.7) for the i^{th} particle is written as $\mathbf{x}_i = \mathbf{x}_i(q_r, t)$ in frame $\Phi = \{O; \mathbf{i}_k\}$, $\dot{\mathbf{x}}_i = (\partial \mathbf{x}_i / \partial q_k)\dot{q}_k + \partial \mathbf{x}_i / \partial t$ coincides with (11.8), and corresponding rules of the form (11.9) and (11.10) hold as well. Here we suppose that the functions $\mathbf{x}_i(q_r, t)$ have continuous second partial derivatives in the domain of interest so that $\partial^2 \mathbf{x}_i / \partial q_j \partial q_k = \partial^2 \mathbf{x}_i / \partial q_k \partial q_j$ and $\partial^2 \mathbf{x}_i / \partial q_j \partial t = \partial^2 \mathbf{x}_i / \partial t \partial q_j$. The total kinetic energy is defined by $T(\dot{q}_r, q_r, t) \equiv \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i$, in accordance with (8.50). Therefore, upon retracing the steps starting from (11.11), introducing the equation of motion $\mathbf{F}_i = m_i \ddot{\mathbf{x}}_i$ (no sum) for the total force on the i^{th} particle, and noting that $Q_k(\dot{q}_r, q_r, t) \equiv \sum_{i=1}^{N} \mathbf{F}_i(\dot{\mathbf{x}}_j, \mathbf{x}_j, t) \cdot \partial \mathbf{x}_i(q_r, t)/\partial q_k$, the reader will find that Lagrange's first fundamental form (11.15) holds for a general holonomic system of N particles having $k = n$ degrees of freedom. Clearly, collinear, mutual internal forces between pairs of particles of the system contribute nothing to the total force and they are workless. And workless forces of holonomic constraint of any sort contribute nothing to the generalized forces, so these forces of constraint do not enter Lagrange's equations for a system of particles.

For scleronomic systems, we can say more. In this case, the total kinetic energy is given by (11.24) in which the symmetric matrix $M_{jk}(q_r) \equiv \sum_{i=1}^{N} m_i \partial \mathbf{x}_i / \partial q_j \cdot \partial \mathbf{x}_i / \partial q_k$, with $j, k = 1, 2, \ldots, n \leq 3N$ for a system having n degrees of freedom, and the work–energy principle (11.29) thus holds for a system of particles, as shown differently in (8.60). For conservative forces $\mathbf{F}_i(\mathbf{x}_j(q_k)) = -\partial V_i / \partial \mathbf{x}_i$ (no sum), we recall (11.31) in which $V(q_r) = \sum_{i=1}^{N} V_i(\mathbf{x}_j(q_k))$ is the total potential

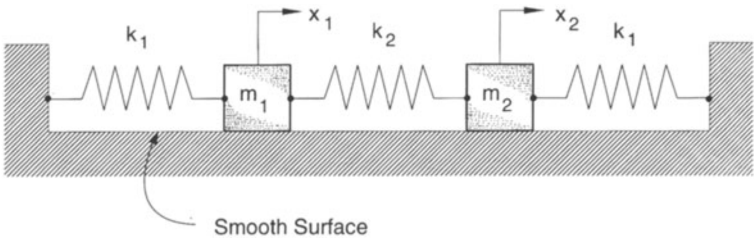


Figure 11.1. A conservative holonomic system of two particles.

energy derived from all of the conservative forces that act on the separate particles, both external and internal as described in (8.84). It is not necessary, however, to elaborate on these details in the construction of the total potential energy function. It thus follows from (11.29) and (11.32) that the principle of conservation of energy (11.33) holds for a system of particles under scleronomic constraints, which is a more precise description of (8.86).

The Lagrangian (11.34) for a general holonomic conservative system is now defined as the difference of the total kinetic and potential energies. It is readily seen that the second fundamental form of Lagrange's equations (11.38) holds for general holonomic constraints. We now explore an application of Lagrange's equations for a conservative system of two mass points and study the problem solution in a special case.

Example 11.6. (i) Derive the equations for the uniaxial motion of the spring-mass system shown in Fig. 11.1. The supporting surface is smooth and all springs are linearly elastic and unstretched initially. (ii) Determine the motion of the system for the special symmetric case when $m_1 = m_2 = m$ and $k_1 = k_2 = k$.

Solution of (i). The holonomic constraints for the uniaxial motion are evident. Let each mass m_1 and m_2 be displaced uniaxially an amount x_1 and x_2, respectively. The system has two degrees of freedom with independent generalized coordinates $(q_1, q_2) = (x_1, x_2)$. The total kinetic energy for the system of particles is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 . \tag{11.40a}$$

Notice that this has the form (11.24) in which $\left[M_{jk}\right]$ =diag$[m_1, m_2]$ is a diagonal matrix . The spring forces are conservative, and all other forces are workless. In consequence, the system is conservative with total potential energy

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2\left(x_2 - x_1\right)^2 + \frac{1}{2}k_1x_2^2. \tag{11.40b}$$

Therefore, the Lagrangian (11.34) is given by

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1\left(x_1^2 + x_2^2\right) - \frac{1}{2}k_2(x_2 - x_1)^2, \tag{11.40c}$$

and hence

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2\,\ddot{x}_2, \tag{11.40d}$$

$$\frac{\partial L}{\partial x_1} = -k_1x_1 + k_2(x_2 - x_1), \qquad \frac{\partial L}{\partial x_2} = -k_1x_2 - k_2(x_2 - x_1). \tag{11.40e}$$

Use of (11.40d) and (11.40e) in (11.35) for $k = 1, 2$ delivers two coupled, ordinary, linear differential equations of motion for the conservative system:

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0, \qquad m_2\ddot{x}_2 + (k_1 + k_2)x_2 - k_2x_1 = 0. \tag{11.40f}$$

Solution of (ii). The general solution of this coupled system of equations may be readily obtained; however, to simplify the analysis and preserve the essential methodology, the motion will be determined for the special symmetrical case when $m_1 = m_2 = m$ and $k_1 = k_2 = k$. Then with $p^2 = k/m$, the equations in (11.40f) simplify to

$$\ddot{x}_1 + p^2\,(2x_1 - x_2) = 0, \qquad \ddot{x}_2 + p^2\,(2x_2 - x_1) = 0. \tag{11.40g}$$

The typical procedure for solving this class of problems is to assume a trial solution for x_1 and x_2 having the *same circular frequency* α *and initial phase* ϕ. We thus consider a trial solution of the form

$$x_1^T = C_1\sin(\alpha t + \phi), \qquad x_2^T = C_2\sin(\alpha t + \phi), \tag{11.40h}$$

in which C_1 and C_2 are constant amplitudes. Substitution of (11.40h) into (11.40g) yields a system of two homogeneous algebraic equations that determine α and the amplitude ratio C_1/C_2:

$$\begin{aligned}(2p^2 - \alpha^2)C_1 - p^2C_2 &= 0,\\ -p^2C_1 + (2p^2 - \alpha^2)C_2 &= 0.\end{aligned} \tag{11.40i}$$

For nontrivial amplitudes C_k, we must have

$$\det\begin{bmatrix} 2p^2 - \alpha^2 & -p^2 \\ -p^2 & 2p^2 - \alpha^2 \end{bmatrix} = 0. \tag{11.40j}$$

This leads to the quadratic equation $(2p^2 - \alpha^2)^2 - p^4 = 0$ with the following two solutions for the circular frequency in (11.40h):

$$\alpha_1 = p, \qquad \alpha_2 = \sqrt{3}p. \tag{11.40k}$$

Equation (11.40j) is called the *characteristic equation* for the system, and its positive roots (11.40k) are called *characteristic frequencies*. The latter also are known as *eigenfrequencies*, *normal mode*, or *natural frequencies*.

For each of these frequencies, the system (11.40i) determines a corresponding amplitude ratio and separate trial solutions of the type (11.40h). It is useful, therefore, to denote by C_{jk} the different amplitudes associated with each generalized coordinate x_j, frequency α_k, and phase angle ϕ_k. In particular, $x_1^T = C_{11}\sin(\alpha_1 t + \phi_1)$ and $x_1^T = C_{12}\sin(\alpha_2 t + \phi_2)$, and hence the general solution for x_1 is provided by their sum. Similarly for x_2. Hence, with the aid of (11.40h) and (11.40k), the general solution of the linear system (11.40g) is given by

$$\begin{aligned} x_1 &= C_{11}\sin(pt + \phi_1) + C_{12}\sin(\sqrt{3}pt + \phi_2), \\ x_2 &= C_{21}\sin(pt + \phi_1) + C_{22}\sin(\sqrt{3}pt + \phi_2). \end{aligned} \tag{11.40l}$$

When α_k is used in (11.40i), the former amplitudes C_1, C_2 are replaced by the respective amplitudes C_{1k}, C_{2k}. Successive use of (11.40k) in (11.40i) yields the following amplitude ratios corresponding to α_1 and α_2:

$$\frac{C_{11}}{C_{21}} = 1, \qquad \frac{C_{22}}{C_{12}} = -1. \tag{11.40m}$$

In consequence, the general solution (11.40l) may be written as

$$\begin{aligned} x_1 &= C_{11}\sin(pt + \phi_1) - C_{22}\sin(\sqrt{3}pt + \phi_2), \\ x_2 &= C_{11}\sin(pt + \phi_1) + C_{22}\sin(\sqrt{3}pt + \phi_2). \end{aligned} \tag{11.40n}$$

The four constants C_{11}, C_{22}, ϕ_1, and ϕ_2 are determined by assigned initial data. Suppose we specify the initial data so that $C_{22} = 0$, then the solution (11.40n) has the form

$$x_1 = x_2 = C_{11}\sin(pt + \phi_1). \tag{11.40o}$$

On the other hand, if we specify initial data so that $C_{11} = 0$, we obtain

$$-x_1 = x_2 = C_{22}\sin(\sqrt{3}pt + \phi_2). \tag{11.40p}$$

Each of these motions is described by a single amplitude, frequency, and phase. In general, a motion of a multidegree of freedom vibrating system that can be described by a single frequency is called a *mode*. The solutions (11.40o) and (11.40p) correspond to modes having the distinct natural frequencies p and $\sqrt{3}p$. In the case (11.40o), the masses move in the same direction with the same circular frequency p, initial phase ϕ_1, and amplitude C_{11}. In the case (11.40p), the masses move symmetrically, in opposite directions with the same circular frequency $\sqrt{3}p$, initial phase ϕ_2, and amplitude C_{22}.

Indeed, when the masses are equally displaced in the same direction an amount B and released from rest, the initial data $x_1(0) = x_2(0) = B$, $\dot{x}_1(0) = \dot{x}_2(0) = 0$

applied to the system (11.40n) provides four algebraic equations that are satisfied with $C_{22} = 0$, $\phi_1 = \pi/2$, and $B = C_{11}$. Thus, (11.40o) has the explicit form

$$x_1 = x_2 = C_{11} \cos pt. \tag{11.40q}$$

Similarly, when initially the masses are equally but oppositely displaced an amount D and released from rest so that $-x_1(0) = x_2(0) = D$ and $\dot{x}_1(0) = \dot{x}_2(0) = 0$, we find $C_{11} = 0$, $\phi_2 = \pi/2$, and $D = C_{22}$. So, the explicit solution is

$$-x_1 = x_2 = C_{22} \cos \sqrt{3} pt. \tag{11.40r}$$

The two natural frequencies of the system are the characteristic frequencies given by (11.40k). By introduction of new coordinates ξ_k, the most general motion of the system (11.40n) for arbitrary initial data may be viewed as the superposition of normal mode motions corresponding to these natural frequencies. Indeed, we see from (11.40n) that each of the coordinates ξ_k, which may be defined in terms of the physical coordinates x_k by

$$\begin{aligned}\xi_1 &\equiv \frac{1}{2}(x_1 + x_2) = C_{11} \sin(pt + \phi_1),\\ \xi_2 &\equiv \frac{1}{2}(x_1 - x_2) = -C_{22} \sin(\sqrt{3} pt + \phi_2),\end{aligned} \tag{11.40s}$$

has only one frequency. These are the natural modes of vibration of the system. The coordinates ξ_k are called *normal coordinates* and their corresponding modes given on the right-hand side of equations (11.40s) are known as the *normal modes of vibration*. It is now evident that the general motion (11.40n) is a superposition of normal mode motions described by $x_1 = \xi_1 + \xi_2$ and $x_2 = \xi_1 - \xi_2$. These relations uncouple the original equations of motion. Let the reader use these results to show that (11.40g) may be written as

$$\ddot{\xi}_1 + p^2 \xi_1 = 0, \qquad \ddot{\xi}_2 + 3p^2 \xi_2 = 0. \tag{11.40t}$$

These are the normal equations of motion for which the normal mode frequencies, now evident, are given in (11.40k). □

11.10. First Integrals of the Lagrange Equations

Two first integrals of the Lagrange equations of motion (11.35) for a conservative system that are equivalent to the three classical principles of conservation of momentum, moment of momentum, either of which may hold also for nonconservative systems with appropriate forces as described later, and energy are derived next. Presentation of the general energy principle for arbitrary generalized forces follows. Finally, a first integral of Lagrange's equations (11.15) with respect to time leads to the generalized impulse–momentum principle.

11.10.1. Ignorable Coordinates: An Easy First Integral of Lagrange's Equations

Let us consider a conservative dynamical system for which the Lagrangian function L is independent of a generalized coordinate q_e, say. Then $\partial L/\partial q_e \equiv 0$, and the corresponding Lagrange equation for q_e, in accord with (11.35) for a general conservative system, yields a first integral

$$\frac{\partial L}{\partial \dot{q}_e} = \gamma_e, \tag{11.41}$$

where γ_e is a constant of integration fixed by the initial conditions. The absentee coordinate q_e is called an *ignorable coordinate.* Since q_e is absent from L, equation (11.41) can be used to obtain $\dot{q}_e$ as a function of all of the other generalized coordinates and their velocities; and consequently $\dot{q}_e$ can then be removed from the remaining differential equations. In this way, a conservative dynamical system having n degrees of freedom and k ignorable coordinates can be reduced to a problem having $n - k$ degrees of freedom. The general reduction procedure for conservative systems is discussed in advanced texts; see Whittaker (Section 38), for example.

Notice in Example 11.3, page 509, that ϕ is absent from the Lagrangian in (11.37c); and, therefore, from the second and fourth equations in (11.37d), $mr^2\dot{\phi} = \gamma_\phi$, a constant, in accordance with (11.41), that is,

$$\dot{\phi} = \frac{\gamma}{r^2}, \tag{11.42a}$$

where $\gamma \equiv \gamma_\phi/m$, a constant. Now, use of this relation in the first equation in (11.37e) yields a differential equation in r alone:

$$\ddot{r} + \frac{\mu}{r^2} - \frac{\gamma^2}{r^3} = 0. \tag{11.42b}$$

In principle, with the solution $r = r(t)$ of this equation in hand, (11.42a) may be used to find $\phi(t)$. The result (11.42a) is the same as (7.72b) leading to Kepler's equal area rule—it is a reflection of the principle of conservation of moment of momentum, as shown in (7.72a). A generalized principle of momentum that includes this case is described later on.

It is seen in Example 11.5(b), page 512, that ϕ is an absentee coordinate for the *nonconservative problem* described by the additional equations in (11.39i), the first of which does not vanish; rather, it leads to (11.39j). Hence, (11.41) based on Lagrange's equations (11.35) for a general conservative system does not hold for the absentee coordinate ϕ in (11.39i), because the nonconservative part of the generalized force $Q_\phi^N \neq 0$ in (11.39h). On the other hand, depending on the nature of the generalized forces, it is quite possible that momenta may be conserved in a nonconservative system. Let us look more closely at the role of ignorable coordinates in relation to the familiar classical conservation principles.

11.10.2. Principle of Conservation of Generalized Momentum

The fundamental principles of conservation of momentum and moment of momentum arise as first integrals of the Newton–Euler equations of motion when a specific component $F_e \equiv \mathbf{F} \cdot \mathbf{e}$ of the total force $\mathbf{F}$, or $M_e^Q \equiv \mathbf{M}_Q \cdot \mathbf{e}$ of the total torque $\mathbf{M}_Q$ about an appropriate point Q, for a fixed direction $\mathbf{e}$ in an inertial frame Φ, vanishes. Consequently, the specific corresponding component $p_e \equiv \mathbf{p} \cdot \mathbf{e}$ of the momentum , or $h_e^Q \equiv \mathbf{h}_Q \cdot \mathbf{e}$ of the moment of momentum, is a constant throughout the motion in Φ. The total force $\mathbf{F}$, however, need not be conservative. We now show that these important fundamental principles are imbedded within the Lagrangian theory.

To motivate the principal idea involved, let us rewrite the second equation in (11.12) as $\partial T(\dot{q}_r, q_r, t)/\partial \dot{q}_k = \mathbf{p}(\dot{q}_r, q_r, t) \cdot \partial \mathbf{x}(q_r, t)/\partial q_k$; and, similarly, by (11.24), obtain $\partial T(\dot{q}_r, q_r)/\partial \dot{q}_k = M_{kj}(q_r)\dot{q}_j$ (sum on j). Notice that both relations have the form of a kind of general momentum component. Consequently, we are led to introduce the *generalized momenta* $p_k(\dot{q}_r, q_r, t)$ defined by

$$p_k(\dot{q}_r, q_r, t) \equiv \frac{\partial T(\dot{q}_r, q_r, t)}{\partial \dot{q}_k} = \frac{\partial L(\dot{q}_r, q_r, t)}{\partial \dot{q}_k}. \tag{11.43}$$

Use of (11.43) in (11.15) casts the Lagrange equations of motion in a somewhat familiar form:

$$\dot{p}_k(\dot{q}_r, q_r, t) = Q_k(\dot{q}_r, q_r, t) + \frac{\partial T(\dot{q}_r, q_r, t)}{\partial q_k} \equiv \mathscr{F}_k(\dot{q}_r, q_r, t), \tag{11.44}$$

in which the quantities $\partial T/\partial q_k$ are called *pseudoforces* (See Problem 11.12.) and $\mathscr{F}_k$, the totals of the corresponding generalized and pseudoforces, are named the *Lagrange forces.* The pseudoforces include the familiar inertial forces. Because the generalized force Q_k has the physical interpretation as either a force or a torque, by dimensional homogeneity, the pseudoforces and the Lagrange forces share a corresponding physical interpretation. With $Q_k = Q_k^N(\dot{q}_r, q_r, t) - \partial V(q_r)/\partial q_k$, (11.44) may be rewritten as

$$\dot{p}_k(\dot{q}_r, q_r, t) = Q_k^N(\dot{q}_r, q_r, t) + \frac{\partial L(\dot{q}_r, q_r, t)}{\partial q_k} \equiv \mathscr{F}_k(\dot{q}_r, q_r, t), \tag{11.45}$$

which also follows directly from (11.38) and (11.43). For a general conservative dynamical system, $Q_k^N = 0$ and (11.45) reduces to

$$\dot{p}_k(\dot{q}_r, q_r, t) = \frac{\partial L(\dot{q}_r, q_r, t)}{\partial q_k} = \mathscr{F}_k(\dot{q}_r, q_r, t). \tag{11.46}$$

Equation (11.44), or equivalently (11.45), is the Lagrange form of the Newton–Euler laws: *The time rate of change of a generalized momentum is equal to the corresponding Lagrange force*: $\dot{p}_k = \mathscr{F}_k$. *Hence, the generalized momentum* p_e *corresponding to a generalized coordinate–velocity pair* $(\dot{q}_e, q_e)$ *is constant throughout the motion if and only if the corresponding Lagrange force* $\mathscr{F}_e = 0$.

If a generalized coordinate q_e is ignorable, then $\partial L(\dot{q}_r, q_r, t)/\partial q_e \equiv 0$ and (11.45) yields the corresponding generalized momentum equation $\dot{p}_e = Q_e^N$. If the generalized force Q_e^N also vanishes, as it does when the system is conservative, then $\dot{p}_e = 0$, and hence

$$p_e(\dot{q}_r, q_r, t) = \frac{\partial L(\dot{q}_r, q_r, t)}{\partial \dot{q}_e} = \gamma_e, \text{ a constant.} \tag{11.47}$$

This is the *principle of conservation of generalized momentum*: *The generalized momentum p_e corresponding to an ignorable coordinate q_e for which $Q_e^N = 0$ is a constant throughout the motion.* In this case, (11.41) for a conservative system is the same as the more general momentum rule (11.47). If the system is not conservative, however, in order that (11.47) may hold for an ignorable coordinate q_e, the nonconservative part of the corresponding generalized force Q_e^N also must vanish. In particular, recall again equations (11.39h) and (11.39i), for which $\partial L/\partial \phi = 0$, but $Q_\phi^N \neq 0$; so, in this instance (11.47) does not hold.

Example 11.7. A nonconservative holonomic system having two degrees of freedom with generalized coordinates (q_1, q_2) and corresponding generalized forces $Q_1^N = -mb^2\nu\dot{q}_1$, $Q_2^N = 0$, has a Lagrangian function

$$L = \frac{1}{2}ma^2\sin^2 q_1 + mb^2\left(\dot{q}_2 + \frac{a}{b}\cos q_1\right)^2 + \frac{1}{2}mb^2(\dot{q}_1 + c)^2, \tag{11.48a}$$

in which $a,\ b,\ c$, and m are constants. Derive the Lagrange equations.

Solution. Notice that q_2 is ignorable *and* $Q_2^N = 0$. Therefore, we have immediately by (11.47) the corresponding momentum integral

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = 2mb^2\left(\dot{q}_2 + \frac{a}{b}\cos q_1\right) = \gamma_2, \text{ a constant.} \tag{11.48b}$$

Caution: We must continue to apply the Lagrange equations to (11.48a) in which all of the variables are considered independent. Equation (11.48b) is a partial solution of one of these equations that necessarily relates these variables; but it is not to be substituted into the Lagrangian, it is to be used in connection with the companion equation for q_1. The second of Lagrange's equations (11.38) yields

$$mb^2\ddot{q}_1 + 2mab\left(\dot{q}_2 + \frac{a}{b}\cos q_1\right)\sin q_1 - ma^2\sin q_1\cos q_1 = Q_1^N = -mb^2\nu\dot{q}_1. \tag{11.48c}$$

We now use (11.48b) to eliminate $\dot{q}_2$ from (11.48c) to obtain

$$\ddot{q}_1 + \nu\dot{q}_1 + \frac{a}{b}\sin q_1\left(\frac{\gamma_2}{mb^2} - \frac{a}{b}\cos q_1\right) = 0. \tag{11.48d}$$

The two equations (11.48b) and (11.48d), in principle, determine $q_1(t)$ and $q_2(t)$. □

Consider two additional easy examples. The Lagrangian for the conservative particle motion on the constraint-free path from B to C in Example 7.10, page 249, is given by $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$, in which x is an ignorable generalized coordinate. Hence, by (11.47), we have $p_x = m\dot{x} = \gamma_x$, a constant, which is equivalent to the result obtained earlier from the principle of conservation of momentum. The Lagrangian $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) + mgz$ for the conservative spherical pendulum problem in Example 7.15, page 260, shows that ϕ is an ignorable coordinate, and hence by (11.47), $p_\phi = mr^2\dot{\phi} = \gamma_\phi$, a constant, as shown earlier in (7.83d) based on conservation of moment of momentum. In the Lagrange formulation, however, it is not necessary to recognize the specific application of the principles of conservation of momentum and moment of momentum; these emerge naturally and easily from the Lagrangian structure with ignorable coordinates.

11.10.3. The Principle of Conservation of Energy for Scleronomic Systems Revisited

The familiar principle of conservation of energy for a particle in (11.33) holds only when the kinetic energy is not an explicit function of time, always the case when the constraints are scleronomic. Here we show that this conservation law for a conservative, scleronomic system may be derived differently to reveal a more general analytical structure that leads to a first integral applicable to dynamical systems other than merely a single particle.

We begin with $L = L(\dot{q}_r, q_r)$, which does not involve t explicitly, observe that

$$\frac{d}{dt}L(\dot{q}_r, q_r) = \frac{\partial L}{\partial \dot{q}_r}\ddot{q}_r + \frac{\partial L}{\partial q_r}\dot{q}_r,$$

introduce (11.35) for a conservative system to write

$$\frac{d}{dt}L(\dot{q}_r, q_r) = \frac{\partial L}{\partial \dot{q}_r}\ddot{q}_r + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right)\dot{q}_r = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\dot{q}_r\right),$$

and thus obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\dot{q}_r - L\right) = 0. \tag{11.49}$$

This yields the *Lagrangian form of the law of conservation of energy for a conservative, scleronomic dynamical system*:

$$\frac{\partial L}{\partial \dot{q}_r}\dot{q}_r - L = E, \text{ a constant.} \tag{11.50}$$

Finally, introduce the Lagrangian function $L(\dot{q}_r, q_r) = T(\dot{q}_r, q_r) - V(q_r)$ and recall (11.26), in which $T(\dot{q}_r, q_r)$ is given by (11.24) for a scleronomic system, to deduce from (11.50) the *principle of conservation of energy for a conservative, scleronomic dynamical system*:

$$T(\dot{q}_r, q_r) + V(q_r) = E, \text{ constant.} \tag{11.51}$$

The derivation of (11.51) involved only the time independence of the Lagrangian function $L(\dot{q}_r, q_r)$, the Lagrange equations (11.35) for a conservative system, and the general representation (11.24) for the kinetic energy $T(\dot{q}_r, q_r)$ in terms of a certain symmetric coefficient matrix $M_{jk}(q_r)$ to be determined by the system, and which does not involve t explicitly. Consequently, the result (11.51) may be applied to any conservative, scleronomic dynamical system. In particular, the two degree of freedom spring-mass system described by (11.40a) and (11.40b) is conservative with constant total energy given by

$$\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}k_1(x_1^2 + x_2^2) + \frac{1}{2}k_2\,(x_2 - x_1)^2 = E,$$

which certainly is not evident from the equations of motion in (11.40f).

11.10.4. The General Energy Principle

Let us write each generalized force as the sum $Q_k = Q_k^C + Q_k^N$ of a conservative part $Q_k^C = -\partial V(q_r)/\partial q_k$ with total potential energy $V(q_r)$ and a nonconservative part $Q_k^N = Q_k^N(\dot{q}_r, q_r, t)$. Then (11.28) for a scleronomic system may be written as

$$\frac{d}{dt}(T(\dot{q}_r, q_r) + V(q_r)) = Q_k^N \dot{q}_k,$$

whose integration yields as a first integral of (11.15), hence also (11.38), *the general energy principle for a nonconservative scleronomic system*:

$$\Delta\mathscr{E} = \int_{\mathscr{C}} Q_k^N dq_k \equiv \mathscr{W}_N, \tag{11.52}$$

where $\mathscr{E}(\dot{q}_r, q_r) \equiv T(\dot{q}_r, q_r) + V(q_r)$ is the *total energy of the system. Accordingly, the change in the total energy of a scleronomic system is equal to the total work done by all of the nonconservative generalized forces. Thus, the total energy is conserved and* (11.51) *holds if and only if the nonconservative parts of all generalized forces are workless, i.e. trivially, when the scleronomic system is conservative*. This result also follows directly from the work–energy principle (11.29). The rule (11.52) is applicable to all scleronomic, multidegree of freedom dynamical systems. It subsumes all previous special results (7.80) for a particle, (8.93) for a system of particles, and (10.131) for a rigid body—in fact, it holds for any combination of these dynamical systems, as shown more precisely later on.

Exercise 11.8. Begin with (11.38) and show that

$$\Delta\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L\right) = \int_{\mathscr{C}} Q_k^N dq_k \equiv \mathscr{W}_N, \tag{11.53}$$

and thus derive the general energy principle (11.52). □

11.10.5. The Generalized Impulse–Momentum Principle

The impulse–momentum and torque–impulse principles for particles and rigid bodies were obtained in earlier chapters as first integrals of the Newton–Euler equations with respect to time. We are now able to derive an inclusive, generalized impulse–momentum principle as a first integral of Lagrange's equations for an arbitrary holonomic dynamical system having n degrees of freedom. First, however, it is important to recognize that an impulsive force acting on a single particle (or a separate body) has no *instantaneous effect* on the system as a whole unless that particle (or body) is connected to or has contact (impact) with another particle (or body). An impulsive force acting on one of two particles attached to the ends of a massless rigid rod, on the other hand, clearly affects the instantaneous motion of both particles of the system. Similarly, the motion of a system of two rigid rods situated in a plane and connected by a smooth hinge at one end is affected by a force suddenly applied at any point along either one. The configuration of an entire assembly of contacting billiard balls, initially at rest and impacted by another ball, is instantaneously affected. A glass jar full of marbles or pebbles that explodes upon striking a hard surface is another example where the motion of an entire system is affected by impulsive forces that act on all of the "particles" at the same moment. There are lots of other examples that may be built upon these. Therefore, in subsequent developments for a system having n degrees of freedom, it is important to bear in mind the limitations imposed by the nature of the system at the impulsive instant. Further, by extension of our earlier model of an instantaneous impulse, during an impulsive action, the generalized impulsive forces vary only with time; all other applied forces, like those due to gravity or attached springs, remain finite during the impulsive instant and contribute nothing to the effect of the impulse. The affected particles (or bodies) suffer an instantaneous change in their velocities, but no instantaneous change in their positions; therefore, the impulsive forces, without accounting in some fashion for deformation of the objects, do no work in the real motion. On the other hand, the instantaneous impulsive forces can do work in an admissible virtual motion during which time is frozen.

With these observations in mind, let us consider the integral of Lagrange's equations (11.15) with respect to time over the interval $[t_0, t]$ to obtain

$$\left.\frac{\partial T}{\partial \dot{q}_k}\right|_{t_0}^{t} - \int_{t_0}^{t} \frac{\partial T}{\partial q_k} dt = \mathscr{Q}_k(t_0; t), \tag{11.54}$$

wherein, by definition, the *impulse of the generalized force* is

$$\mathscr{Q}_k(t_0;t) \equiv \int_{t_0}^{t} Q_k dt. \tag{11.55}$$

In accordance with (11.43) the first term in (11.54) is the change $\Delta p_k \equiv p_k(t) - p_k(t_0)$ in the generalized momenta during the impulsive interval $[t_0, t]$:

$$\left.\frac{\partial T}{\partial \dot{q}_k}\right|_{t_0}^{t} = \Delta p_k. \tag{11.56}$$

During the impulsive instant the generalized coordinates q_k do not change and the generalized velocities $\dot{q}_k$ remain finite. Therefore, the quantities Δp_k and $\partial T(\dot{q}_r, q_r)/\partial q_k$ in (11.54) remain finite. In the limit as $\Delta t = t - t_0 \to 0$, $\Delta p_k \to \Delta p_k^*$, the instantaneous change in the generalized momenta, and the integral of the pseudoforces vanishes: $\text{limit}_{\Delta t \to 0}(\partial T/\partial q_k)\Delta t = 0$. (This is trivially satisfied for all t when all of the generalized coordinates are ignorable.) The instantaneous impulse of any finite generalized forces that act on the system likewise will vanish as $\Delta t \to 0$. The impulsive forces in an instantaneous impulse tend to infinity in such a way that the limit of (11.55) exists and is finite; its value is *the instantaneous impulse of the generalized forces* defined by

$$\mathscr{Q}_k^* \equiv \lim_{\Delta t \to 0} \mathscr{Q}_k(t_0;t) = \lim_{\Delta t \to 0} \int_{t_0}^{t} Q_k dt. \tag{11.57}$$

Thus, in the limit of (11.54) as $\Delta t \to 0$, we obtain the *generalized impulse–momentum principle:*

$$\mathscr{Q}_k^* = \Delta p_k^*; \tag{11.58}$$

the instantaneous impulse of the generalized force is equal to the instantaneous change in the corresponding generalized momentum of the system.

Notice from (11.44) that $\Delta p_k = \int_{t_0}^{t} \mathscr{F}_k dt$; that is, the impulse of the Lagrange force is equal to the change in the corresponding generalized momentum. It is easily seen that bounded quantities vanish in the limit as $\Delta t \to 0$ and this reduces to the generalized impulse–momentum principle (11.58). The generalized impulse–momentum principle includes both the impulse–momentum and torque-impulse principles of earlier studies, depending on the physical interpretation of the generalized momenta.

The generalized impulsive forces may be calculated from their virtual work $\delta \mathscr{W}^* = \mathscr{Q}_k^* \delta q_k$ done in an admissible general virtual displacement consistent with any holonomic constraints, as though they were ordinary fixed forces. Given the initial values of q_k and $\dot{q}_k$ prior to the impulsive action, the system of n algebraic equations (11.58) then relates the instantaneous impulsive forces $\mathscr{Q}_k^*$ and instantaneous generalized velocities $\dot{q}_k$ following the impulsive action. Therefore, these equations may be used to provide the initial conditions for the subsequent motion of the system. Since no specific form of the kinetic energy was introduced in the construction, the principles hold for all dynamical systems. This observation is reinforced later.

Example 11.8. Two particles of equal mass m are attached to the ends of a massless rigid rod of length ℓ initially oriented parallel to the y-axis of a frame $\psi = \{O; \mathbf{i}_k\}$ and at rest on a smooth horizontal surface. An instantaneous impulsive normal force $\mathbf{P} = P\mathbf{i}$ acts on the particle closer to O. Determine the subsequent instantaneous generalized velocities of the system, find the instantaneous increase of the total energy of the system due to the impulse, and describe the subsequent motion for all time.

Solution. Following the impulse, the center of mass of the system is at (x^*, y^*) in ψ and the rod makes an angle θ with the y-axis. The three generalized coordinates, therefore, are $(q_1, q_2, q_3) = (x^*, y^*, \theta)$. The total kinetic energy of the system in its subsequent general motion is given by (8.53); we find

$$T = \frac{1}{2}(2m)(\dot{x}^{*2} + \dot{y}^{*2}) + \frac{1}{2}\left(\frac{m\ell^2}{2}\right)\dot{\theta}^2. \tag{11.59a}$$

All of the generalized variables are ignorable, and hence $\partial T/\partial q_r \equiv 0$. Since the system is at rest initially, the instantaneous changes in the generalized momenta (11.56) are given by

$$\Delta p_1^* = 2m\dot{x}^*, \qquad \Delta p_2^* = 2m\dot{y}^*, \qquad \Delta p_3^* = \frac{m\ell^2}{2}\dot{\theta}. \tag{11.59b}$$

Notice that Δp_1^* and Δp_2^* are instantaneous changes in linear momenta due to an impulsive force, and Δp_3^* is the instantaneous change in the moment of momentum due to a torque–impulse.

Let $(\mathcal{Q}_1^*, \mathcal{Q}_2^*, \mathcal{Q}_3^*)$ denote the corresponding generalized impulsive forces. The position vector $\mathbf{x}_P$ of the point of application of the impulsive force $\mathbf{P} = P\mathbf{i}$ is given by $\mathbf{x}_P = (x^* + \frac{1}{2}\ell\sin\theta)\mathbf{i} + (y^* - \frac{1}{2}\ell\cos\theta)\mathbf{j}$ so its virtual displacement is $\delta\mathbf{x}_P = (\delta x^* + \frac{1}{2}\ell\delta\theta\cos\theta)\mathbf{i} + (\delta y^* + \frac{1}{2}\ell\delta\theta\sin\theta)\mathbf{j}$. Hence, at the impulsive instant at which $\theta = 0$, the virtual work of the applied forces and the generalized impulsive forces is given by

$$\delta\mathcal{W}^* = \mathcal{Q}_1^*\delta x^* + \mathcal{Q}_2^*\delta y^* + \mathcal{Q}_3^*\delta\theta = P\left(\delta x^* + \frac{1}{2}\ell\delta\theta\right), \tag{11.59c}$$

for all arbitrary virtual displacements. Therefore,

$$\mathcal{Q}_1^* = P, \qquad \mathcal{Q}_2^* = 0, \qquad \mathcal{Q}_3^* = \frac{P\ell}{2}. \tag{11.59d}$$

We thus see that $\mathcal{Q}_1^*$ is the instantaneous impulsive force while $\mathcal{Q}_3^*$ is its instantaneous moment about the center of mass. Use of (11.59b) and (11.59d) in (11.58) yields the instantaneous values of the generalized velocities $(\dot{x}_i^*, \dot{y}_i^*, \dot{\theta}_i)$:

$$\dot{x}_i^* = \frac{P}{2m}, \qquad \dot{y}_i^* = 0, \qquad \dot{\theta}_i = \frac{P}{m\ell}. \tag{11.59e}$$

The instantaneous increase in the total energy due to the impulse on the system, initially at rest, follows by substitution of (11.59e) into (11.59a): $T_i = P^2/2m$.

The values (11.59e) of the instantaneous translational velocity $\dot{\mathbf{x}}^* = \dot{x}_i^* \mathbf{i}$ of the center of mass and the instantaneous angular velocity $\boldsymbol{\omega} = \dot{\theta}_i \mathbf{k}$ of the system about the center of mass are the initial conditions for the subsequent motion of the system under no forces. It follows that the motion of the center of mass is uniform with velocity $\mathbf{v}^* = P/2m\mathbf{i}$, its initial value. Moreover, there are no applied torques, so the moment of momentum for the system about the center of mass is constant: $\mathbf{h}_C = m\ell^2\dot{\theta}/2\mathbf{k}$, and hence the angular velocity is constant, $\boldsymbol{\omega} = P/m\ell\mathbf{k}$. Alternatively, this being a conservative system with total kinetic energy (11.59a), and all of whose generalized coordinates are ignorable and whose generalized forces are zero, (11.47) yields the principles of conservation of generalized momenta:

$$p_1 = 2m\dot{x}^* = \gamma_1, \qquad p_2 = 2m\dot{y}^* = \gamma_2, \qquad p_3 = \frac{m\ell^2}{2}\dot{\theta} = \gamma_3, \tag{11.59f}$$

where the constants γ_k are determined by the initial values (11.59e). We thus find $\gamma_1 = P$, $\gamma_2 = 0$, $\gamma_3 = P\ell/2$, and hence, $\mathbf{v}^* = P/2m\mathbf{i}$ and $\boldsymbol{\omega} = P/m\ell\mathbf{k}$ for all t. □

11.11. Hamilton's Principle

Let us consider a holonomic system having n degrees of freedom. To compare two neighboring motions of the same system with n independent generalized coordinates $q_r^*(t)$ and $q_r(t)$ described in the *same time* between the *same end states* so that $q_r^*(t_1) = q_r(t_1)$ and $q_r^*(t_2) = q_r(t_2)$ at the respective arbitrary times t_1 and t_2, as shown in Fig. 11.2, suppose that the motions differ by only a small amount described by the generalized coordinate *variations* $\delta q_r(t) \equiv q_r^*(t) - q_r(t) = \varepsilon\eta_r(t)$, where ε is an arbitrary small parameter and $\eta_r(t)$ are n arbitrary, continuously differentiable functions. Then $\delta q_r(t_1) = \varepsilon\eta_r(t_1) = 0$ and $\delta q_r(t_2) = \varepsilon\eta_r(t_2) = 0$ must hold at the end states. The process of variation requires that only the independent generalized coordinates $q_r(t)$ are varied, not the time t; hence $\delta t = 0$. The parameter ε allows us to modify the functions $q_r(t)$ by arbitrarily small amounts, and because ε may decrease to zero, the variation of a function $q_r(t)$ thus constitutes an infinitesimal virtual change of that function.

Now consider a given function $f(q_r(t), t) \equiv f(q_1, q_2, \ldots, q_n, t)$ whose variation is caused by the variation $\delta q_r(t)$ of the independent variables $q_r(t)$, noting that the functional dependence is not altered by the variation. Then $\delta f(q_r, t) \equiv f(q_r^*, t) - f(q_r, t) = f(q_r + \varepsilon\eta_r, t) - f(q_r, t)$; and by the Taylor series

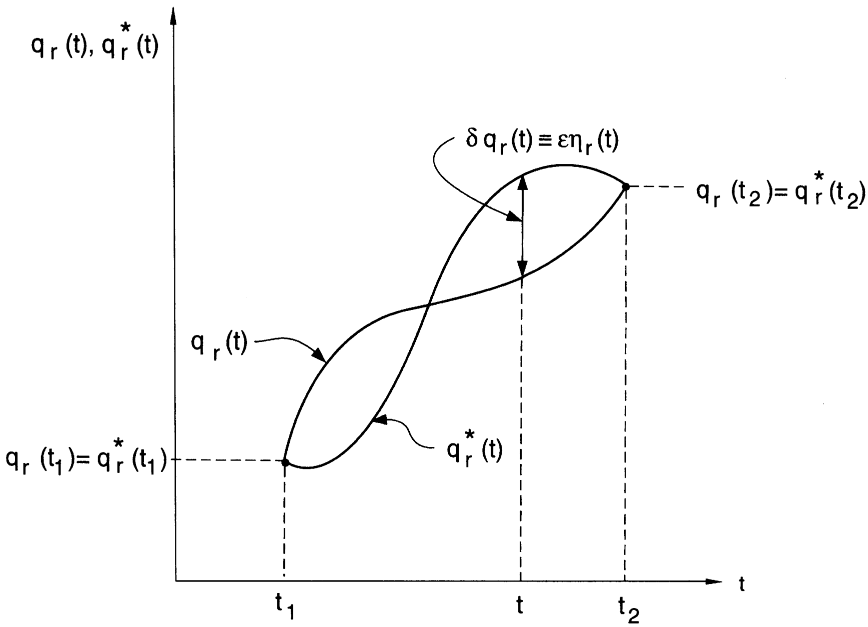


Figure 11.2. Comparison of the generalized coordinates for two motions having the same end states, for a system with *n* degrees of freedom.

approximation about $\varepsilon = 0$, we find $f(q_r + \varepsilon\eta_r, t) = f(q_r, t) + (\partial f/\partial q_r)\varepsilon\eta_r(t)$ to the first order in ε. The variation of the function is thus given by $\delta f(q_r, t) = (\partial f/\partial q_r)\delta q_r(t) = (\partial f/\partial q_r)\varepsilon\eta_r(t)$, sum on r. Therefore, the variation of a function with $\delta t = 0$ behaves like a differential in terms of the variation $\delta q_r(t)$ of its independent variables, the explicit time dependence in $f(q_r, t)$ playing no role.

11.11.1. Hamilton's Principle for a Conservative, Holonomic System

The *action* $\mathscr{A}$ for a general conservative, holonomic system is defined by the definite integral

$$\mathscr{A} \equiv \int_{t_1}^{t_2} L(\dot{q}_r(t), q_r(t), t)dt, \tag{11.60}$$

in which $L(\dot{q}_r, q_r, t) \equiv T(\dot{q}_r, q_r, t) - V(q_r)$ is the Lagrangian function for an otherwise unspecified dynamical system. Now suppose that the curves $q_r(t)$ in Fig. 11.2 represent the actual or natural motion of the system between arbitrarily assigned end states, and that δq_r defines an arbitrary infinitesimal variation in $q_r(t)$ in passing from the natural motion to a neighboring motion defined by the variables $q_r^*(t)$ between the same end states. This results in an infinitesimal variation $\delta\mathscr{A}$ in the action (11.60) defined by

$$\begin{aligned}\delta\mathscr{A} &= \delta\int_{t_1}^{t_2} L(\dot{q}_r, q_r, t)dt \equiv \int_{t_1}^{t_2} L(\dot{q}_r^*, q_r^*, t)dt - \int_{t_1}^{t_2} L(\dot{q}_r, q_r, t)dt \\ &= \int_{t_1}^{t_2} (L(\dot{q}_r^*, q_r^*, t) - L(\dot{q}_r, q_r, t))dt = \int_{t_1}^{t_2} \delta L(\dot{q}_r, q_r, t)dt;\end{aligned} \tag{11.61}$$

that is, the variation of the definite integral is equal to the integral of the variation of its integrand. Of course, the actual motion is not yet known, because the equations of motion have not been introduced and solved. To deduce Lagrange's equations for a general conservative, holonomic system without mention of the specific nature of the Lagrangian energy function, we introduce a variational principle due to Sir William Rowan Hamilton (1805–1865) applicable to both conservative and nonconservative, holonomic dynamical systems.

Hamilton's principle: *Among all possible motions between assigned end states of any holonomic dynamical system, the actual motion is the one for which the action is stationary; that is,*

$$\delta\mathscr{A} = 0. \tag{11.62}$$

First, let us consider a conservative dynamical system for which $\mathscr{A}$ is defined by (11.60), and apply the previous description of the variation of a function $f(q_r, t)$ to the Lagrangian function $L(\dot{q}_r, q_r, t)$ in the action integral (11.61) to obtain

$$\delta\mathscr{A} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \dot{q}_r}\delta\dot{q}_r + \frac{\partial L}{\partial q_r}\delta q_r\right) dt, \tag{11.63}$$

repeated indices being summed. Since $\delta\dot{q}_r \equiv \dot{q}_r^* - \dot{q}_r$, we have $\delta\dot{q}_r = \dfrac{d}{dt}\delta q_r$, and hence the first term in (11.63) may be integrated by parts to obtain

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_r}\delta\dot{q}_r dt = \left.\frac{\partial L}{\partial \dot{q}_r}\delta q_r\right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right)\delta q_r dt. \tag{11.64}$$

In view of the null end conditions, however, the first of the right-hand terms vanishes, and hence use of this result in (11.63) and application of Hamilton's principle (11.62) yield the stationary action condition

$$\delta\mathscr{A} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right)\right)\delta q_r dt = 0. \tag{11.65}$$

We shall assume that the integrand is a continuous function of t. For holonomic systems, $\delta q_r = \varepsilon\eta_r(t)$ are arbitrary infinitesimal quantities in which $\eta_r(t)$ are any continuously differentiable functions. Since (11.65) must vanish for all arbitrary variations δq_r that satisfy the end conditions $\delta q_r(t_1) = \delta q_r(t_2) = 0$, it follows that in the integrand sum all of the functions in the parentheses must vanish. To prove this, let us write the integral (11.65) briefly as $I \equiv \varepsilon \int_{t_1}^{t_2} \mathscr{L}_r(t)\eta_r(t)dt$. Now take all $\eta_r(t) = 0$ except any one you wish, $\eta_a(t)$, say. Now choose this function to vanish everywhere except on an arbitrary small interval $\tau - \alpha \leq t \leq \tau + \alpha$ around the point $t = \tau$ and on which, without loss of generality, we may choose $\eta_a(t) > 0$, say. For example, the function $\eta_a(t) = (t - \tau + \alpha)^2(t - \tau - \alpha)^2$ for $t \in [\tau - \alpha, \tau + \alpha]$ and $\eta_a(t) = 0$ elsewhere satisfies the conditions and is continuously differentiable. Within this interval the continuous function $\mathscr{L}_a(t) = \mathscr{L}_a(\tau)$, very nearly, and hence $I = \varepsilon \mathscr{L}_a(\tau) \int_{\tau-\alpha}^{\tau+\alpha} \eta_a(t)dt$, approximately, the error tending to zero as α tends to zero. Since the integral $\int_{\tau-\alpha}^{\tau+\alpha} \eta_a(t)dt > 0$, it follows that (11.65) holds only if $\mathscr{L}_a(\tau) = 0$. The point $t = \tau$, however, may be chosen as any point of the interval (t_1, t_2), and hence the continuous function $\mathscr{L}_a(t)$ must vanish for all $t \in [t_1, t_2]$. Because our choice of the a^{th} member of the sum in (11.65) was arbitrary, (11.65) holds only if every term $\mathscr{L}_r(t)$ of its integrand sum vanishes. Conversely, if all $\mathscr{L}_r(t) = 0$ holds, (11.65) is satisfied. Hence, $\delta\mathscr{A} = 0$ holds for arbitrary variations $\delta q_r(t)$ that satisfy the end conditions, if and only if the Lagrangian function $L(\dot{q}_r, q_r, t)$ for a general conservative, holonomic dynamical system satisfies Lagrange's equations,[†]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0, \qquad r = 1, 2, \ldots, n. \tag{11.66}$$

These equations are the same as (11.35). The difference, however, is that they now apply to every conservative, holonomic dynamical system; it is only the Lagrangian function for a specific conservative, holonomic dynamical system that must be defined.

[†] For holonomic systems for which elimination of the constraints by direct substitution may be inconvenient or cumbersome, the Lagrange multiplier method may be used to derive a modified form of Lagrange's equations. For a system of $p < m$ holonomic constraints,

$$f_j(q_r, t) \equiv f_j(q_1, q_2, \ldots, q_m, t) = 0, \qquad j = 1, 2, \ldots, p, \tag{a}$$

(hence $n = m - p$ degrees of freedom), p Lagrange multipliers λ_j are introduced, and Hamilton's principle is then applied to a modified Lagrangian function $\hat{L}(\dot{q}_r, q_r, t) = L(\dot{q}_r, q_r, t) + \sum_{j=1}^{p} \lambda_j f_j(q_r, t)$ to obtain the modified Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} - \sum_{j=1}^{p} \lambda_j \frac{\partial f_j}{\partial q_r} = 0, \qquad r = 1, 2, \ldots, m \tag{b}$$

subject to the constraints (a). The system of equations (a) and (b) determine the $m + p$ generalized variables q_r and multipliers λ_j. We shall find no need to apply this method in our studies here. Further discussion of these matters and extension of Hamilton's principle and Lagrange's equations to nonholonomic dynamical systems may be found in advanced works on analytical dynamics. See, for example, the books by Lanczos, Pars, Rosenberg, and Whittaker, among others listed in the chapter references.

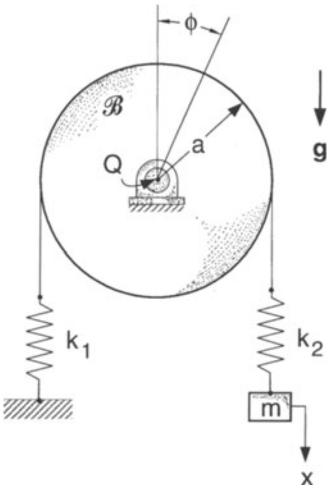


Figure 11.3. A conservative holonomic dynamical system of two bodies.

Example 11.9. A system of rigid bodies in its static equilibrium position is shown in Fig. 11.3. The homogeneous wheel $\mathscr{B}$ of radius a is free to rotate in a smooth bearing about Q, and the block of mass m is supported by a linear spring of stiffness k_2 attached to an inextensible cable that wraps around the wheel. The other end of the cable is fastened to a linear spring of modulus k_1 fixed to the machine foundation. There is no slip between the cable and the wheel when the block is displaced vertically and released. Derive the equations of motion for the system.

Solution. Because the elastic spring response is linear, we may consider the motion about the prestretched static equilibrium position of the system. In consequence, gravity has no further influence on the motion. This system has two degrees of freedom characterized by the independent generalized coordinates $(q_1, q_2) = (x, \phi)$ measured from the equilibrium state shown in Fig. 11.3. Due to the no slip and inextensibility constraints, the extension $\tilde{x}$ of the foundation spring is $\tilde{x} = a\phi$. The smooth bearing reaction force is workless, and the remaining forces that act on the system are conservative. Therefore, relative to the static equilibrium state, the Lagrangian function for the system is given by

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 - \left[\frac{1}{2}k_1 a^2\phi^2 + \frac{1}{2}k_2(x - a\phi)^2\right], \qquad (11.67a)$$

where I is the moment of inertia of the wheel about its principal axis at Q. Notice in passing that the total kinetic energy in (11.67a) has the form (11.24) in which

$[M_{jk}]$ =diag$[m, I]$. With (11.66) in mind, we first determine

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial L}{\partial x} = -k_2(x - a\phi), \tag{11.67b}$$

$$\frac{\partial L}{\partial \dot{\phi}} = I\dot{\phi}, \qquad \frac{\partial L}{\partial \phi} = -k_1 a^2\phi + ak_2(x - a\phi), \tag{11.67c}$$

and thereby obtain the following coupled pair of linear differential equations of motion for the system:

$$m\ddot{x} + k_2 x - k_2 a\phi = 0, \qquad I\ddot{\phi} + (k_1 + k_2)a^2\phi - k_2 ax = 0. \tag{11.67d}$$

The structure of these equations is similar to (11.40f) studied earlier. □

11.11.2. Hamilton's Principle for a Nonconservative, Holonomic System

Let us recall that the total work $\mathscr{W}$ done by a conservative system of forces is equal to the decrease in the total potential energy, and hence, for a conservative dynamical system, the integrand in (11.60) may be rewritten as $L = T + \mathscr{W}$. This suggests that for an arbitrary nonconservative, holonomic dynamical system having n degrees of freedom the action is appropriately defined by

$$\mathscr{A} \equiv \int_{t_1}^{t_2} \left[T(\dot{q}_r, q_r, t) + \mathscr{W}(\dot{q}_r, q_r, t)\right] dt, \tag{11.68}$$

where the total work $\mathscr{W}$ is defined by the sum of integrals in (11.23) in which $\mathscr{C} = \cup_{k=1}^{n} \mathscr{C}_k$ and each integral is over a path $\mathscr{C}_k$ for the generalized coordinate q_k corresponding to the generalized force Q_k, in the same time interval $[t_1, t]$. Hence, Hamilton's principle (11.62) applied to (11.68) yields the stationary action condition

$$\delta\mathscr{A} = \int_{t_1}^{t_2} \left[\delta T(\dot{q}_r, q_r, t) + \delta\mathscr{W}(\dot{q}_r, q_r, t)\right] dt = 0. \tag{11.69}$$

First, consider the variation $\delta\mathscr{W}$. In accordance with (11.23),

$$\delta\mathscr{W} = \sum_{k=1}^{n} \int_{q_k^*(t_1)}^{q_k^*(t)} Q_k(\dot{q}_r, q_r, t) dq_k - \sum_{k=1}^{n} \int_{q_k(t_1)}^{q_k(t)} Q_k(\dot{q}_r, q_r, t) dq_k, \tag{11.70}$$

in which $q_k^*(t_1) = q_k(t_1)$ for all k at the end point and $q_k^*(t) = q_k(t) + \delta q_k(t)$. Then, because $\delta q_k(t) = \varepsilon\eta_k(t)$ are infinitesimal quantities, (11.70) may be rewritten as

$$\delta\mathscr{W} = \sum_{k=1}^{n} \int_{q_k(t)}^{q_k(t)+\delta q_k(t)} Q_k(\dot{q}_r, q_r, t) dq_k = \sum_{k=1}^{n} \hat{Q}_k(\dot{q}_r, q_r, t)\delta q_k(t), \tag{11.71}$$

where $\hat{Q}_k(\dot{q}_r, q_r, t) = Q_k(\dot{q}_r, q_r, t) + \Delta Q_k(\varepsilon)$, the last term being an infinitesimal quantity of order ε. Therefore, to the first order in ε, (11.71) yields the variation $\delta \mathscr{W} = \sum_{k=1}^{n} Q_k \delta q_k \equiv Q_k \delta q_k$, the virtual work done by all of the generalized forces at the fixed time t.

Retracing the procedure used earlier to obtain the variation δL leading to (11.65) and now applied to $\delta T(\dot{q}_r, q_r, t)$ with vanishing end conditions, we find that Hamilton's principle (11.69), to the first order in ε, requires

$$\delta \mathscr{A} = \int_{t_1}^{t_2} \left(\frac{\partial T}{\partial q_r} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r} \right) + Q_r \right) \delta q_r dt = 0, \tag{11.72}$$

for all $\delta q_r = \varepsilon \eta_r(t)$ such that $\delta q_r(t_1) = \delta q_r(t_2) = 0$, repeated indices being summed over $r = 1, 2, \ldots, n$. We shall assume that each integrand term in parentheses is a continuous function $\mathscr{L}_r(t)$ of time t—all being independent of ε. It then follows by our previous argument that each integrand function $\mathscr{L}_r(t)$ must vanish for all t. Consequently,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = Q_r, \qquad r = 1, 2, \ldots, n, \tag{11.73}$$

for every nonconservative, holonomic dynamical system of n degrees of freedom. These equations, while the same as (11.15), are now applicable to every nonconservative, holonomic dynamical system. Writing $Q_r(\dot{q}_r, q_r, t) = -\partial V(q_r)/\partial q_r + Q_r^N(\dot{q}_r, q_r, t)$ in terms of its conservative and nonconservative parts, we deduce from (11.73) the generalized form of Lagrange's equations (11.38) for nonconservative, holonomic dynamical systems.

Example 11.10. A rigid body shown in Fig. 11.4 is driven by a torque $\mu(t)$ about a fixed, principal body axis $\mathbf{k}$ in a smooth bearing at H. (i) Apply (11.73) to

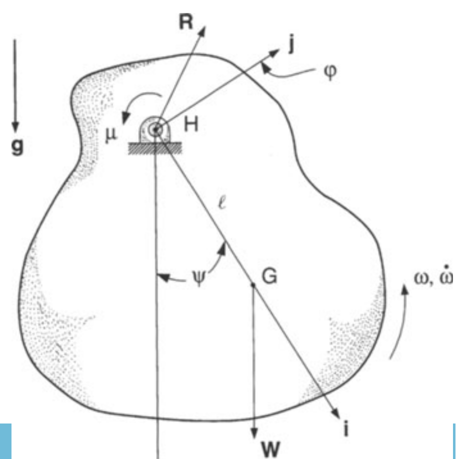


Figure 11.4. Torque driven rotation of a rigid body—a nonconservative holonomic dynamical system.

derive the equation of motion for the body. (ii) Repeat the derivation from (11.38). Show that the result has the familiar form of the equation of motion of a driven pendulum. (iii) Apply Euler's law to obtain the equation of motion.

Solution of (i). The system is holonomic with one degree of freedom described by $q_1 = \psi$; hence, (11.73) yields

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = Q_\psi. \tag{11.74a}$$

With the total kinetic energy of the body $T = \frac{1}{2}I\dot{\psi}^2$, where I is the principal moment of inertia about the body axis at H, (11.74a) becomes

$$I\ddot{\psi} = Q_\psi. \tag{11.74b}$$

We next determine the generalized force Q_ψ. The bearing reaction force $\mathbf{R}$ is workless, and the total external torque $\mathbf{M}_H$ about H is the sum of the gravitational torque $-W\ell \sin\psi\mathbf{k}$ and the applied driving torque $\boldsymbol{\mu} = \mu\mathbf{k}$. The virtual work $\delta\mathscr{W}$ done by the total torque in the virtual displacement $\delta\boldsymbol{\psi} \equiv \delta\psi\mathbf{k}$ is thus given by

$$\delta\mathscr{W} = \mathbf{M}_H \cdot \delta\boldsymbol{\psi} = (-W\ell \sin\psi + \mu)\delta\psi \equiv Q_\psi \delta\psi. \tag{11.74c}$$

Hence, $Q_\psi = -W\ell \sin\psi + \mu$, and (11.74b) yields the equation of motion:

$$I\ddot{\psi} + mg\ell \sin\psi = \mu(t). \tag{11.74d}$$

Solution of (ii). Application of the Lagrange equations (11.38) yields

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = Q_\psi^N. \tag{11.74e}$$

Again, the workless constraint force $\mathbf{R}$ need not be considered, and the gravitational force is conservative with total potential energy $V = mg\ell(1 - \cos\psi)$. Therefore, the Lagrangian is

$$L = T - V = \frac{1}{2}I\dot{\psi}^2 - mg\ell(1 - \cos\psi). \tag{11.74f}$$

The virtual work done by the nonconservative generalized force is $\delta\mathscr{W}_N = \boldsymbol{\mu} \cdot \delta\boldsymbol{\psi} = \mu\delta\psi = Q_\psi^N \delta\psi$. Hence, $Q_\psi^N = \mu$, and (11.74e) leads to (11.74d).

With $I = mR^2$ in terms of the radius of gyration R, (11.74d) may be written in the form of the equation of motion of a driven pendulum for which $p^2 \equiv g\ell/R^2$ and $\hat{\mu}(t) \equiv \mu(t)/I$; namely,

$$\ddot{\psi} + p^2 \sin\psi = \hat{\mu}(t). \tag{11.74g}$$

Solution of (iii). Euler's law for the rotation about a fixed principal axis at H requires $\mathbf{M}_H = I_H \dot{\boldsymbol{\omega}}$, wherein $\mathbf{M}_H = (\mu - W\ell \sin\psi)\mathbf{k}$ and $I_H \dot{\boldsymbol{\omega}} = I\ddot{\psi}\,\mathbf{k}$. This yields the equation of motion $I\ddot{\psi} = \mu - W\ell \sin\psi$, which is the same as (11.74d). □

Exercise 11.9. Begin with the action (11.68) and introduce from the start the decomposition of $\mathscr{W}$ into its conservative and nonconservative parts: $\mathscr{W}(\dot{q}_r, q_r, t) = \mathscr{W}_C(q_r) + \mathscr{W}_N(\dot{q}_r, q_r, t)$. Show that the action integral for the nonconservative system may be written as

$$\mathscr{A} \equiv \int_{t_1}^{t_2} (L(\dot{q}_r, q_r, t) + \mathscr{W}_N(\dot{q}_r, q_r, t))\, dt. \tag{11.75}$$

Then work out the details for $\delta\mathscr{A} = 0$ and thus derive (11.38). □

11.12. Additional Applications of the Lagrange Equations

Several additional applications of Lagrange's equations are investigated, starting with analysis of an experimental technique useful in engineering design for evaluation of the moment of inertia of a complex structured body. Then the finite amplitude oscillation of a rotating simple pendulum, for which generalized forces arise from the moving constraint, is studied. Next, we revisit the problem of the gyrocompass with torsional damping and conclude with analysis of the general motion of a spinning top about a fixed point.

11.12.1. A Problem in Engineering Design Analysis

The moment of inertia of a table assembly $\mathscr{T}$ shown in Fig. 11.5 in the horizontal plane is calibrated experimentally to have a principal value $I_O(\mathscr{T})$ about

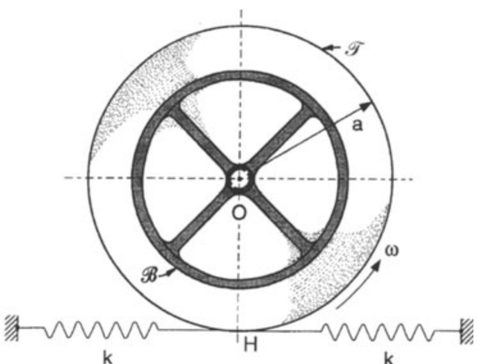


Figure 11.5. Experimental apparatus to determine the moment of inertia of a complex structured body.

its normal axis of rotation in a smooth bearing at O. Identical springs of stiffness k, initially unstretched, are attached symmetrically to the table in a tangential line at point H. The moment of inertia of another complex structured body $\mathscr{B}$ placed on the table in a specified orientation can be found experimentally by measuring the frequency $f_{\mathscr{S}}$ of small oscillations of the system $\mathscr{S} = \mathscr{T} \cup \mathscr{B}$ consisting of the table and the body. We thus derive an equation for the frequency of the system and thereby determine the moment of inertia $I_O(\mathscr{B})$ of a flywheel $\mathscr{B}$ placed centrally at O.

Let us consider a small angular placement $q_1 = \theta(t)$ of the system $\mathscr{S} = \mathscr{T} \cup \mathscr{B}$ from its initial, natural equilibrium state in the horizontal plane in Fig. 11.5. No work is done by the smooth bearing reaction forces; so, the system is conservative with total kinetic and potential energies given by

$$T = \frac{1}{2} I_O(\mathscr{S})\dot{\theta}^2(t), \qquad V = 2\left(\frac{1}{2}ka^2\theta^2\right) = ka^2\theta^2, \tag{11.76a}$$

where $I_O(\mathscr{S})$ is the moment of inertia of the system about the normal axis at O. Hence, $L = \frac{1}{2} I_O(\mathscr{S})\dot{\theta}^2(t) - ka^2\theta^2$, and Lagrange's equations (11.35) yield

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = I_O(\mathscr{S})\ddot{\theta} + 2ka^2\theta = 0. \tag{11.76b}$$

This is the equation for a simple harmonic oscillator whose small amplitude circular frequency $p_{\mathscr{S}}$ is defined by

$$p_{\mathscr{S}} = 2\pi f_{\mathscr{S}} = \sqrt{\frac{2ka^2}{I_O(\mathscr{S})}}, \tag{11.76c}$$

in which the measured frequency is $f_{\mathscr{S}}$ and the values for k and a are known. With $I_O(\mathscr{S}) = I_O(\mathscr{T}) + I_O(\mathscr{B})$ in (11.76c), we obtain from this data the moment of inertia $I_O(\mathscr{B})$ of the flywheel about its central axis:

$$I_O(\mathscr{B}) = \frac{ka^2}{2\pi^2 f_{\mathscr{S}}^2} - I_O(\mathscr{T}). \tag{11.76d}$$

In the event that $I_O(\mathscr{T})$ is not known or the system may need to be recalibrated, the frequency $f_{\mathscr{T}}$ of the table assembly alone may be measured to obtain by the same process $I_O(\mathscr{T}) = ka^2/2\pi^2 f_{\mathscr{T}}^2$. Thus, in terms of measurable data alone,

$$I_O(\mathscr{B}) = \frac{ka^2}{2\pi^2}\left(\frac{1}{f_{\mathscr{S}}^2} - \frac{1}{f_{\mathscr{T}}^2}\right). \tag{11.76e}$$

This example is typical of useful applications of dynamics in engineering design analysis.

11.12.2. Rotating Simple Pendulum

A rotating simple pendulum shown in the figure for Problem 6.47 consists of a bob of mass m constrained by a rigid wire of length l and negligible mass fastened to a smooth hinge O at r from the center C of a smooth table that rotates in the horizontal plane with a constant angular speed ω, as shown. Relative to an observer in the table frame, the pendulum oscillates with a finite amplitude angle β_0. First, we apply Lagrange's equations subject to the rheonomic constraint to derive the equation of motion of the bob relative to the table. We then relax the constraint, use Lagrange's equations to derive the equations of motion, and find exactly the nonconservative constraint tension in the wire as a function of the finite angular placement β alone. Finally, the angular placement and period of the finite amplitude motion of the bob are determined, thus solving the problem exactly and entirely.

11.12.2.1. Application of the Rheonomic Constraint

Introduce generalized coordinates $(q_1, q_2) = (\beta, \theta)$, where β is the angular placement of the pendulum relative to the table, and θ is the angular placement of the table in the ground frame. The rheonomic constraint yields $\theta = \omega t$, so the system has only one degree of freedom. The absolute velocity $\mathbf{v}_m$ of the pendulum bob referred to the moving frame $\varphi = \{O; \mathbf{i}_k\}$ is given by

$$\mathbf{v}_m = r\omega \sin\beta \mathbf{i} + (r\omega\cos\beta + l(\omega + \dot{\beta}))\mathbf{j}, \tag{11.77a}$$

and hence its total kinetic energy is

$$T = \frac{1}{2}m[r^2\omega^2 + 2rl\omega(\omega + \dot{\beta})\cos\beta + l^2(\omega + \dot{\beta})^2], \tag{11.77b}$$

where $\omega = \dot{\theta}$, a constant. The total potential energy $V = 0$; so $L = T$, which is independent of θ. The planar wire force $\mathbf{F} = -P\mathbf{i}$ on the bob does no work in the motion relative to the table, so it is evident that the generalized force $Q_\beta = 0$. The system is conservative; hence use of (11.77b) in (11.66) delivers the equation for the finite amplitude motion of the pendulum relative to the table:

$$\ddot{\beta} + \frac{r}{l}\omega^2 \sin\beta = 0. \tag{11.77c}$$

The wire constraining force, inconsequential to the bob's motion, is not determined by this analysis.

11.12.2.2. Equations of Motion with Relaxed Constraint

Now let us consider a new problem in which θ is treated as an independent variable in consideration of Lagrange's equations (11.73) to obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}}\right) - \frac{\partial T}{\partial \beta} = Q_\beta, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = Q_\theta. \tag{11.77d}$$

The generalized forces Q_k are determined from the virtual work done by the wire constraining force $\mathbf{F} = -P\mathbf{i}$, due to the motion of the table frame. The virtual displacement may be read from (11.77a) by replacing ω with $\delta\theta$ and $\dot{\beta}$ with $\delta\beta$ to obtain $\delta\mathbf{x} = r\sin\beta\delta\theta\mathbf{i} + [(l + r\cos\beta)\delta\theta + l\delta\beta]\mathbf{j}$. Then

$$\delta\mathscr{W} = \mathbf{F}\cdot\delta\mathbf{x} = -rP\sin\beta\delta\theta = Q_\theta\delta\theta + Q_\beta\delta\beta, \tag{11.77e}$$

yields the generalized forces

$$Q_\theta = -rP\sin\beta, \qquad Q_\beta = 0. \tag{11.77f}$$

Since $Q_\theta \neq 0$, the absentee coordinate θ in (11.77b) is not ignorable. The wire tension does no work in the β-motion of the bob relative to the table, so it is evident that the generalized force Q_β should vanish, which it does. Moreover, notice that use of the constraint $\delta\theta = \omega\delta t = 0$ in (11.77e) shows only that $Q_\beta = 0$, but says nothing about Q_θ in the constrained case studied above. A neat alternative derivation of (11.77f) uses (11.20) for a particle, in which $\mathbf{F} = -P\mathbf{i}$ and $\dot{\mathbf{x}} = \mathbf{v}_m$ in (11.77a); namely,

$$Q_\beta = -P\mathbf{i}\cdot\frac{\partial\mathbf{v}_m}{\partial\dot{\beta}} = -P\mathbf{i}\cdot l\mathbf{j} = 0, \tag{11.77g}$$

$$Q_\theta = -P\mathbf{i}\cdot\frac{\partial\mathbf{v}_m}{\partial\dot{\theta}} = -P\mathbf{i}\cdot(r\sin\beta\mathbf{i} + (l + r\cos\beta)\mathbf{j}) = -Pr\sin\beta. \tag{11.77h}$$

Substituting (11.77b) and (11.77f) into (11.77d) and following some simplifications, we obtain the two equations

$$\ddot{\beta} + \frac{r}{l}\omega^2\sin\beta + \left(1 + \frac{r}{l}\cos\beta\right)\dot{\omega} = 0, \tag{11.77i}$$

$$ml(\ddot{\beta}(l + r\cos\beta) - r(2\omega + \dot{\beta})\dot{\beta}\sin\beta) + m(r^2 + l^2 + 2rl\cos\beta)\dot{\omega} = -Pr\sin\beta. \tag{11.77j}$$

These are two equations in three unknown quantities: P, β, and ω. Thus, suppose that ω is constant. Then (11.77i) reduces to (11.77c) which determines the oscillatory motion $\beta(t)$ of the bob relative to the table, and (11.77j) determines the wire tension force P acting on the bob in the moving table frame. The former is the primary equation of interest, readily derived without our having to find the inconsequential wire constraining force. On the other hand, it is important in engineering analysis that the nature of the forces that act on a dynamical system be known. A variety of methods are available to evaluate these. Here we continue with the case when ω is constant.

11.12.2.3. General and Exact Solution of the Equations of Motion

Integration of (11.77c) for the initial data $\dot{\beta}(0) = 0$ at $\beta(0) = \beta_0$ yields

$$\dot{\beta}^2 = 2\frac{r}{l}\omega^2(\cos\beta - \cos\beta_0). \tag{11.77k}$$

Now use of (11.77c) and (11.77k) in (11.77j) delivers the equation for the wire tension as an exact function of its placement β alone:

$$P(\beta) = ml\omega^2 \left[1 + \frac{r}{l}(3\cos\beta - 2\cos\beta_0) \pm 2\sqrt{2\frac{r}{l}(\cos\beta - \cos\beta_0)} \right], \quad (11.77\text{l})$$

in which the $+$ sign corresponds to the case when $\beta(t)$ is increasing with time.

Further, integration of (11.77k) for increasing values of $\beta(t)$ determines the travel time as a function of the finite angular placement of the pendulum:

$$pt = \int_0^{\beta} \frac{d\beta}{\sqrt{2(\cos\beta - \cos\beta_0)}}, \qquad p \equiv \omega\sqrt{\frac{r}{l}}, \quad (11.77\text{m})$$

where p, which depends on ω, characterizes the circular frequency of the pendulum in its small amplitude motion. The exact solution, therefore, is given by an elliptic integral of the first kind defined by (7.87d) with $k = \sin(\beta_0/2)$, $\sin(\beta/2) = k\sin\phi$, in accord with (7.87b). In consequence, the exact period of the oscillation is provided by (7.87f) and the motion may be read from (7.89b). Thus,

$$\beta(t) = 2\sin^{-1}[k\,\mathrm{sn}(pt)], \qquad \tau^* = \frac{4}{p}K(k), \quad (11.77\text{n})$$

in which $K(k)$ is the complete elliptic integral of the first kind in (7.87e) and $\mathrm{sn}(pt)$ is the Jacobian elliptic sine function with properties (7.88h) and (7.88i). This concludes the fully exact solution for the motion (11.77n) and the wire tension (11.77l) of the rotating simple pendulum.

11.12.3. The Gyrocompass with Torsional Damping Revisited

Lagrange's equations are applied here to derive the equations for the small motion of the torsionally damped gyrocompass shown schematically in Fig. 10.12, page 450. The rotor has two degrees of freedom with generalized coordinates $(q_1, q_2) = (\theta, \alpha)$, where θ is the angular placement of the rotor about the **j**-axis and α is the angular placement of the gimbal frame about the **k**-axis of the gimbal reference frame $\varphi = \{C; \mathbf{i}_k\}$. With the aid of (10.81b) and (10.81d), the total kinetic energy of the rotor relative to its center of mass C is provided by (10.100):

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{h}_C = \frac{1}{2}I_{11}\Omega^2\cos^2\lambda\sin^2\alpha + \frac{1}{2}I_{22}(\dot{\theta} + \Omega\cos\lambda\cos\alpha)^2 + \frac{1}{2}I_{11}(\dot{\alpha} + \Omega\sin\lambda)^2, \quad (11.78\text{a})$$

where λ is the latitude angle and Ω is the Earth's very small angular rate of rotation, a rheonomic constraint. Here and below all infinitesimal terms of order Ω^2 are neglected. Equation (11.78a) thus simplifies to

$$T = \frac{1}{2}I_{22}(\dot{\theta}^2 + 2\Omega\dot{\theta}\cos\lambda\cos\alpha) + \frac{1}{2}I_{11}(\dot{\alpha}^2 + 2\Omega\dot{\alpha}\sin\lambda). \quad (11.78\text{b})$$

The potential energy $V = 0$ and the supporting constraint forces are workless. In view of (10.81a) there is no torque about $\mathbf{j}$, so $\mathbf{M}_C \cdot \mathbf{j} = 0 = Q_\theta$, and hence θ is an ignorable coordinate. Therefore, the Lagrange equations (11.73) yield

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) - \frac{\partial T}{\partial \alpha} = Q_\alpha, \qquad \frac{\partial T}{\partial \dot{\theta}} = \gamma_\theta, \text{ a constant,} \tag{11.78c}$$

the last reflecting conservation of the generalized momentum $p_\theta = \gamma_\theta$, that is, the principle of conservation of moment of momentum about $\mathbf{j}$. This fact was not so apparent in our earlier discussion of this problem.

The damping torque about the $\mathbf{k}$-axis is defined by $\mathbf{M}_C \cdot \mathbf{k} = -2I_{11}\nu\dot{\alpha} = Q_\alpha$. Therefore, it follows from (11.78b) for a small compass drift angle α that the Lagrange equations (11.78c) for the damped gyrocompass, to the first order in α, yield

$$I_{22}(\dot{\theta} + \Omega \cos\lambda) = \gamma_\theta, \tag{11.78d}$$

$$\ddot{\alpha} + 2\nu\dot{\alpha} + p^2\alpha = 0, \tag{11.78e}$$

where p^2 is defined by (10.81j). By (11.78d), $\dot{\theta}$ is a constant: $\dot{\theta} = \dot{\theta}(0) \equiv \omega_0$ given by the initial data. Hence, $\theta(t) = \omega_0 t$. It is seen that (11.78e) is the same as (10.81l), whose general solution is provided in (10.81m).

11.12.4. Motion of a Symmetrical Top about a Fixed Point

A homogeneous rigid top (a body of revolution) of mass m rotates about a point O fixed on a rough horizontal surface in the ground frame $\Phi = \{O; \mathbf{I}_k\}$ in a gravitational field. The three independent Euler angles (ϕ, θ, ψ) introduced in Chapter 3, Fig. 3.14, page 209, characterize the general rotational orientation of the top, as shown in Fig. 11.6. The equations of motion will be derived by Lagrange's method. Afterwards, the physical nature of the top's motion is described.

11.12.4.1. Equations of Motion for the Top

To find the total kinetic energy of the top, we first determine its total angular velocity referred to the principal body frame $\varphi = \{O; \mathbf{i}_k\}$ in Fig. 11.6, wherein several reference frames are identified. The top turns about the $\mathbf{K}$-axis of the ground frame $0 = \Phi$ with angular velocity $\boldsymbol{\omega}_{10} = \dot{\phi}\mathbf{K}$, followed by a rotation about the $\mathbf{i}'$-axis of frame $1 = \{O; \mathbf{i}'_k\}$ with angular velocity $\boldsymbol{\omega}_{21} = \dot{\theta}\mathbf{i}'$, and finally, it spins about the $\mathbf{k}$-axis of the body frame $3 = \varphi$, with angular spin $\boldsymbol{\omega}_{32} = \dot{\psi}\mathbf{k}$. Hence, the total angular velocity of the top in Φ is $\boldsymbol{\omega} \equiv \boldsymbol{\omega}_{30} = \dot{\psi}\mathbf{k} + \dot{\theta}\mathbf{i}' + \dot{\phi}\mathbf{K}$. With $\mathbf{i}' = \cos\psi\mathbf{i} - \sin\psi\mathbf{j}$, $\mathbf{K} = \cos\theta\mathbf{k} + \sin\theta\mathbf{j}'$, $\mathbf{j}' = \sin\psi\mathbf{i} + \cos\psi\mathbf{j}$, the total angular velocity of the top referred to φ is given by

$$\boldsymbol{\omega} = (\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)\mathbf{i} + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\mathbf{j} + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{k}. \tag{11.79a}$$

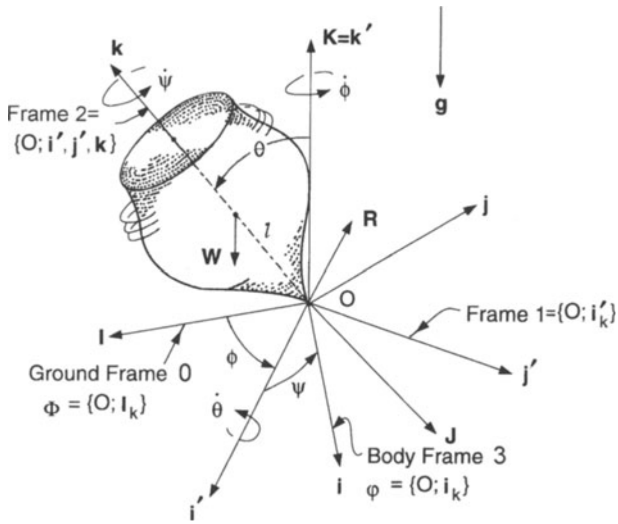


Figure 11.6. Symmetrical top rotating about a fixed point.

The homogeneous top is symmetrical about its **k**-axis with principal moments of inertia $I_{11} = I_{22}$. For future convenience, let $I_1 \equiv I_{11},\ I_3 \equiv I_{33}$, and assume that $I_1 \neq I_3$. The rough surface assures that point O of the top does not slide on the surface. Then, by (10.102), the total kinetic energy relative to the fixed point O of the principal body frame φ is given by $T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}_O\boldsymbol{\omega} = \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2$, which, by (11.79a), yields

$$T = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2. \tag{11.79b}$$

The total potential energy is

$$V = mgl\cos\theta. \tag{11.79c}$$

The supporting force **R** at O is workless, and hence the holonomic system is conservative with Lagrangian function $L = T - V$:

$$L = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mgl\cos\theta, \tag{11.79d}$$

in which both ψ and ϕ are ignorable coordinates. Hence, in accordance with (11.41),

$$\frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\psi} + \dot{\phi}\cos\theta) = \alpha, \tag{11.79e}$$

$$\frac{\partial L}{\partial \dot{\phi}} = I_1\dot{\phi}\sin^2\theta + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta = \beta, \tag{11.79f}$$

are constants of the motion provided by initial conditions. These equations determine $\dot{\phi}$ and $\dot{\psi}$ as functions of θ; we find

$$\dot{\phi} = \frac{\beta - \alpha\cos\theta}{I_1 \sin^2\theta}, \qquad \dot{\psi} = \frac{\alpha}{I_3} - \cos\theta\left(\frac{\beta - \alpha\cos\theta}{I_1\sin^2\theta}\right). \tag{11.79g}$$

Finally, by (11.66), the third of Lagrange's equations $d(\partial L/\partial\dot{\theta})/dt - \partial L/\partial\theta = 0$ yields

$$I_1\ddot{\theta} + (I_3 - I_1)\dot{\phi}^2 \sin\theta\cos\theta + I_3\dot{\psi}\dot{\phi}\sin\theta - mgl\sin\theta = 0. \tag{11.79h}$$

This equation, upon integration with specified initial data, determines the motion $\theta(t)$ of the top in a vertical plane through the **K**-axis, and which rotates about this fixed spatial axis with variable angular speed $\dot{\phi}(t)$ given by (11.79g), as shown in Fig. 11.6.

11.12.4.2. General Solution of the Equations of Motion

The main problem now is to solve the nonlinear equation (11.79h) for $\theta(t)$, and then use the result in (11.79g) to obtain $\phi(t)$ and $\psi(t)$. This is possible, in principle, but not entirely in terms of elementary functions. To begin, recall (11.79e) and the component ω_3 in (11.79a) to obtain

$$I_3(\dot{\psi} + \dot{\phi}\cos\theta) = I_3\omega_3 = \alpha, \tag{11.79i}$$

i.e., the total angular spin of the top about its **k**-axis is constant: $\omega_3 = \alpha/I_3$.

Because the system is conservative, the total energy $T + V = E$, a constant. Therefore, with (11.79b) and (11.79c), the first integral of (11.79h) is thus given by

$$\frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 + mgl\cos\theta = E, \tag{11.79j}$$

certainly not evident. With the aid of (11.79i), this reduces to

$$\frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mgl\cos\theta = E_0, \tag{11.79k}$$

in which the constant $E_0 \equiv E - \frac{1}{2}I_3\omega_3^2$. Use of the first expression in (11.79g) yields

$$\frac{1}{2}I_1\dot{\theta}^2 + V(\theta) = E_0, \tag{11.79l}$$

where

$$V(\theta) \equiv \frac{(\beta - \alpha\cos\theta)^2}{2I_1\sin^2\theta} + mgl\cos\theta. \tag{11.79m}$$

Equation (11.79l) has the appearance of an energy equation for a single degree of freedom system for which $V(\theta)$ is the *apparent potential energy*. Its integration delivers the travel time t in the motion $\theta(t)$:

$$t = \pm\sqrt{\frac{I_1}{2}} \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{E_0 - V(\theta)}}. \tag{11.79n}$$

The appropriate sign is to be fixed in accordance with the initial conditions for which $\theta_0 = \theta(0)$. The inverse of this integral determines $\theta(t)$, and integration of (11.79g) yields $\phi(t)$ and $\psi(t)$. The three Euler angles (ϕ, θ, ψ) thus determine the orientation of the top at each instant and provide the complete formal solution of the problem. For brevity, however, we omit these details‡ and focus on some interesting qualitative aspects of the top's rotational motion.

11.12.4.3. Physical Characterization of the Motion

Let us return to the apparent energy equation (11.79l) and recall that the phase plane diagram $\dot{\theta}$ versus θ characterizes the curves of constant energy. These curves are closed and the motion is periodic if and only if there are exactly two values $\theta^* = \theta_k^*,\ k = 1, 2$, called the *turning points*, for which $\dot{\theta}(\theta^*) = 0$ in (11.79l). The turning points of the motion $\theta(t)$ are determined by

$$V(\theta^*) = \frac{(\beta - \alpha\cos\theta^*)^2}{2I_1 \sin^2\theta^*} + mgl\cos\theta^* = E_0, \tag{11.79o}$$

wherein the energy constant E_0 is fixed by the initial data.

‡ By the introduction of a change of variable $x = \cos\theta$ for $-1 \le x \le 1$ it can be shown that the energy equation (11.79l) may be written as $\dot{x}^2 = f(x)$, where

$$f(x) \equiv \frac{2}{I_1}(E_0 - mglx)(1 - x^2) - \frac{1}{I_1^2}(\beta - \alpha x)^2. \tag{A}$$

This is a real cubic polynomial having three real roots x_k such that $-1 \le x_1 \le x_2 \le 1 \le x_3$, the root x_3 having no physical relevance. Since $f(x) = \dot{x}^2$ cannot be negative, x oscillates between the physically realizable values x_1 and x_2; that is, θ oscillates between the turning points θ_1 and θ_2 at which $\dot{\theta}(\theta) = 0$. (Note that in the text, we write $\theta_1^* = \theta_2 \le \theta_1 = \theta_2^*$.) With

$$\dot{x}^2 = \frac{2mgl}{I_1}(x - x_1)(x - x_2)(x - x_3), \tag{B}$$

we are led to an elliptic integral of the first kind. This eventually follows from (11.79n). Then it turns out that the inverse function for the nutational motion $\theta(t) \in [\theta_2, \theta_1]$ is determined precisely in terms of a Jacobian elliptic sine function of t; namely,

$$\cos\theta = x = x_1 + (x_2 - x_1)\mathrm{sn}^2[p(t - t_0)], \tag{C}$$

where $p = \sqrt{mgl(x_3 - x_1)/2I_3}$. Use of (C) in (11.79g) yields exact formal solutions for $\phi(t)$ and $\psi(t)$ in terms of elliptic integrals of the third kind with modulus $k = \sqrt{(x_2 - x_1)/(x_3 - x_1)}$. These and other analytical details and further discussion of the nutational–precessional paths of the motion traced by the unit vector $\mathbf{k}$ on the surface of a unit sphere centered at the fixed point O, as shown in Fig. 11.7, may be found in the texts by Greenwood, Marion, Rosenberg, Synge and Griffith, and Whittaker.

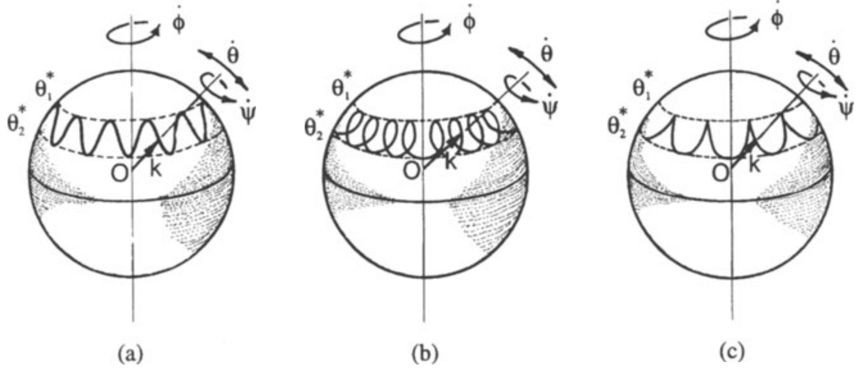


Figure 11.7. Geometrical description of a top's nutational–precessional motion.

In all real motions, (11.79l) and (11.79o) show that $E_0 = V(\theta^*) \geq V(\theta)$. Hence, the portion of the plane curve $y = V(\theta)$ of interest is situated below the horizontal line $y = E_0$. It is seen from (11.79m) that $V(\theta) \to +\infty$ at the extremes§ $\theta = 0, \pi$; and hence the apparent potential energy function $y = V(\theta)$ must have a minimum value at some intermediate point θ_0^*, the point at which, by (11.79l), $\dot{\theta}$ has its greatest value. Consequently, the graph $y = V(\theta)$ is concave upward, somewhat like a skewed parabola; so the horizontal line $y = E_0$ must intersect this graph at precisely two points θ_1^* and θ_2^* for which (11.79o) holds. Therefore, the motion $\theta(t)$ is periodic; the top oscillates between the extreme angular positions θ_1^* and θ_2^* from the vertical spatial **K**-axis in Fig. 11.6, points at which $\dot{\theta}(\theta_k^*) = 0$. This oscillation phenomenon is called *nutation*. At the same time, by the first relation in (11.79g), the axis of the top in Fig. 11.6 turns as a function of θ about the vertical **K**-axis. This variable rotational motion $\phi(\theta(t))$, induced by the gravitational torque about O, is called *precession*. In the special case for which the line $y = E_0 = V(\theta_0^*)$, the minimum value of $V(\theta)$, $\theta(t)$ is restricted to the single fixed value θ_0^*; and the top, inclined at this fixed angle, turns about the **K**-axis with a constant angular speed $\dot{\phi}(\theta_0^*)$, so this motion is called a *steady precession*.

The essential features of the simultaneous nutational–precessional motion of the top may be visualized in Fig. 11.7 by tracking the end point of the unit vector **k** (the spin axis) on the surface of a unit sphere centered at O. The precessional rotation of the axis of the top as a function of the nutation angle $\theta(t)$ is described by $\dot{\phi}$ in (11.79g). The specific geometry depends on how the top is started, that is, it depends on the initial values of θ, $\dot{\theta}$, $\dot{\phi}$, and $\dot{\psi}$ (these determine α, β, E_0). The simplest undulatory case described above in which the top traces a sinusoidal-like

§ The actual physical model is of no concern in the geometrical description of the function $V(\theta)$. The motion of a top (any body of revolution) as we commonly think of it, however, is restricted to values of $\theta < \pi/2$. For values of $\theta > \pi/2$, any body of revolution supported by a smooth ball joint at O on the axis of symmetry is known as a *gyroscopic pendulum*.

trajectory between two horizontal colatitude limit circles θ_1^* and θ_2^* is shown in Fig. 11.7a. In this case, $\dot{\phi}(t) > 0$ for all t during the motion; the **k**-axis of the top moves up and down (nutation) at the rate $\dot{\theta}$ as the top rotates (precession) about the fixed vertical **K**-axis at the rate $\dot{\phi}$. The looping motion shown in Fig. 11.7b is characterized by $\dot{\phi}$ increasing, then decreasing, over and over again. Therefore, there must be a value θ_l at which $\dot{\phi}(\theta_l) = 0$. The criterion for loops to occur, therefore, by (11.79g), is that $\theta_l = \cos^{-1}(\beta/\alpha)$ must lie in the interval between θ_1^* and θ_2^*. Finally, when initially the axis of the spinning top is fixed at an angle $\theta_0 = \theta_1^*$ and released with $\dot{\phi}(\theta_1^*) = 0$, the top at first falls to θ_2^*, but recovers and rises again to $\theta_1^* = \cos^{-1}(\beta/\alpha)$, and this nodding motion is repeated over and over. As the top falls, the gravitational torque induces a precession in the direction of the torque, so the axis of spin turns about **K** at the rate $\dot{\phi}$. This phenomenon leads to the cuspidal motion in Fig. 11.7(c). We shall not pause to explore the analytical details characterizing these geometrical properties; rather, we turn to the general problem of a steady precession for which $\theta(t) = \theta_0^*$, $\dot{\theta} = 0$, and $\dot{\phi}$ and $\dot{\psi}$ are constants.

11.12.4.4. Steady Precession and Stability of the Motion of a Top

Differentiation of (11.79l) with respect to θ yields the modified equation of motion for $\theta(t)$:

$$I_1\ddot{\theta} + \frac{dV(\theta)}{d\theta} = 0. \tag{11.79p}$$

The geometrical description of the apparent potential function, however, showed that $V(\theta)$ has a minimum at a point $\theta_0^* \in [\theta_1^*, \theta_2^*]$; hence, $dV(\theta)/d\theta = 0$ at $\theta = \theta_0^*$. (It is shown later that in fact $V(\theta)$ has a minimum at θ_0^*.) Hence, from (11.79p), the position θ_0^* is a relative equilibrium position in θ at which the top maintains a tilted position at an angle θ_0^* from the vertical axis, and the motion is a *steady precession* of the top about the vertical **K**-axis at a constant angular rate $\dot{\phi}_0 \equiv \dot{\phi}(\theta_0^*)$, that is,

$$\dot{\phi}_0 = \frac{\beta - I_3\omega_3\cos\theta_0^*}{I_1\sin^2\theta_0^*}, \tag{11.79q}$$

the top now having a constant spin $\dot{\psi}_0 \equiv \dot{\psi}(\theta_0^*)$ about its body axis **k**, in accordance with (11.79g). In addition to θ_0^*, both rates depend on the constants of the motion, β, ω_3, whose values must be appropriately chosen to support the steady precession in accordance with assigned initial data.

The point θ_0^* of steady precession, from (11.79m), is thus determined by

$$\left.\frac{dV}{d\theta}\right|_{\theta_0^*} = (I_3\omega_3\dot{\phi}_0 - I_1\dot{\phi}_0^2\cos\theta_0^* - mgl)\sin\theta_0^* = 0, \tag{11.79r}$$

where $\theta_1^* < \theta_0^* < \theta_2^*$. The same relation follows from (11.79h) in which $\theta(t) = \theta_0^*$.

The solutions $\theta_0^* = 0, \pi$ are considered separately later; otherwise, (11.79r) yields a quadratic equation for $\dot{\phi}_0(\theta_0^*)$:

$$(I_1 \cos\theta_0^*)\dot{\phi}_0^2 - I_3\omega_3\dot{\phi}_0 + mgl = 0, \tag{11.79s}$$

with solution

$$\dot{\phi}_0 = \frac{I_3\omega_3}{2I_1\cos\theta_0^*}\left(1 \pm \sqrt{1 - \frac{4mglI_1\cos\theta_0^*}{I_3^2\omega_3^2}}\right). \tag{11.79t}$$

By (11.79q), this is a transcendental equation for $\beta - I_3\omega_3\cos\theta_0^*$, and hence θ_0^*, whose solution depends on the constants β, ω_3. This is solvable for θ_0^* only by trial and error, when all constants are assigned. Of course, (11.79t) also is an equation for $\dot{\phi}_0$ that now depends only on ω_3 and θ_0^*; and for real values of $\dot{\phi}_0$, it is necessary that

$$I_3^2\omega_3^2 \geq 4mglI_1\cos\theta_0^*, \tag{11.79u}$$

in which case there are two speeds of steady precession. It follows that a steady precession at a fixed angle $\theta_0^* \in (\theta_1^*, \theta_2^*)$ is possible only when the total angular spin ω_3 is not less than its critical limit $\omega_3^c \equiv (2/I_3)(mglI_1\cos\theta_0^*)^{1/2}$ for which the corresponding critical steady precessional rate is

$$\dot{\phi}_0^c \equiv \frac{I_3\omega_3^c}{2I_1\cos\theta_0^*} = \frac{2mgl}{I_3\omega_3^c}. \tag{11.79v}$$

Therefore, consider a fast spinning top for which $\omega_3 \gg \omega_3^c$. Then use of the binomial theorem in the radicand of (11.79t) leads to the following two approximate speeds of steady precession for the fast top:

$$\dot{\phi}_0^+ = \frac{I_3\omega_3}{I_1\cos\theta_0^*}\left(1 - \frac{mglI_1\cos\theta_0^*}{I_3^2\omega_3^2}\right), \qquad \dot{\phi}_0^- = \frac{mgl}{I_3\omega_3}, \tag{11.79w}$$

where the signs correspond to those in (11.79t). The first, $\dot{\phi}_0^+$, is called the *fast precession*; it grows increasingly larger with ω_3. For sufficiently great values of ω_3 this further simplifies to $\dot{\phi}_0^+ = I_3\omega_3/(I_1\cos\theta_0^*)$, which is independent of the weight of the top. The second speed, $\dot{\phi}_0^-$, is called the *slow precession*; this is independent of θ_0^* and it goes toward zero as ω_3 grows increasingly great. The latter is the precession rate commonly observed in a fast spinning top (or gyroscope). Returning to (11.79s), we see that for $\theta_0^* = \pi/2$, $\dot{\phi}_0 = \dot{\phi}_0^-$ is an exact slow precession result. The foregoing results hold for $0 < \theta_0^* < \pi/2$, certainly the physical case for a top. More generally, however, for a gyroscopic pendulum $\pi/2 < \theta_0^* < \pi$ is possible. In this case, the radicand of (11.79t) is always positive, so there is no critical value of ω_3. The slow precession of the gyroscopic pendulum is the same as before, but its fast precession is greater and has the opposite direction.

Either of the speeds (11.79w) is possible provided that the top is started so that the initial data precisely satisfies (11.79s) and (11.79u). But it is not yet known whether these steady motions are stable. To investigate the infinitesimal stability of a steady precession, let $\theta(t) = \theta_0^* + \delta(t)$ and $\dot{\phi}(t) = \dot{\phi}_0 + \dot{\varepsilon}(t)$, where $\delta(t)$ and $\dot{\varepsilon}(t)$ are infinitesimal disturbances from the top's steady motion at θ_0^*, and assume that the constants β and ω_3 of the steady motion are unchanged by the disturbance. Only the change in $\theta(t)$ need be considered in the perturbation of (11.79p); and, by the first of (11.79g), $\dot{\varepsilon}$ is proportional to δ. Thus, with $dV(\theta)/d\theta = dV(\theta)/d\theta|_{\theta_0^*} + d^2V(\theta)/d\theta^2|_{\theta_0^*}\,\delta$ to the first order in δ, and use of (11.79r), we obtain from (11.79p) the equation for the perturbed motion:

$$I_1\ddot{\delta} + d^2V(\theta)/d\theta^2\big|_{\theta_0^*}\,\delta = 0. \tag{11.79x}$$

Therefore, for infinitesimal stability it is necessary that $d^2V(\theta)/d\theta^2\big|_{\theta_0^*} > 0$. This is a condition necessary in order that $V(\theta)$ shall have a minimum at θ_0^*, as required earlier. This is now confirmed analytically. With the aid of the first relation in (11.79g) and noting (11.79s), we find

$$\left.\frac{d^2V(\theta)}{d\theta^2}\right|_{\theta_0^*} = \frac{1}{I_1\dot{\phi}_0^2}\left[\left(mgl - I_1\dot{\phi}_0^2\cos\theta_0^*\right)^2 + I_1^2\dot{\phi}_0^4\sin^2\theta_0^*\right] > 0. \tag{11.79y}$$

Hence, the function $V(\theta)$ has a minimum at θ_0^*; and the arbitrarily small disturbance of the steady precession is oscillatory, and hence stable¶. The proportionality of $\dot{\varepsilon}$ and δ implies that ε also is periodic. Therefore, both values of $\dot{\phi}_0$ given by (11.79t) and subject to (11.79u) correspond to stable motions of steady precession.

Finally, let us consider a so-called *sleeping top* that merely spins about the vertical axis so that $\theta(t) = \theta_0^* = 0$ and $\dot{\theta}(t) = 0$. This steady motion is a solution of the equation of motion (11.79h) and for which, from (11.79e) and (11.79f), the constants of the motion satisfy $\beta = \alpha = I_3\omega_3$; and, by (11.79g), $\dot{\phi}(\theta_0^*) = \dot{\phi}_0 = \alpha/2I_1$. It is evident that if the total spin rate ω_3 of the top in its vertical position is not sufficiently great, any slight disturbance of the top will cause it to wobble, fall down, and roll to rest. So, our intuition suggests that there exists a critical total spin rate below which the motion of the top is unstable. To explore this, we put $\theta(t) = \theta_0^* + \delta(t) = \delta(t)$ and $\dot{\phi} = \dot{\phi}_0 + \dot{\varepsilon}$ into the equation of motion (11.79h),

¶ Alternatively, introducing both $\theta(t) = \theta_0^* + \delta(t)$ and $\dot{\phi}(t) = \dot{\phi}_0 + \dot{\varepsilon}(t)$ in (11.79g) and (11.79h), assuming that the moment of momentum $I_3\omega_3$ is unchanged and ultimately removing it by use of (11.79s), we eventually derive the two equations

$$\begin{aligned} &I_1\dot{\phi}_0\sin\theta_0^*\dot{\varepsilon} + \left(I_1\dot{\phi}_0^2\cos\theta_0^* - mgl\right)\delta = 0,\\ &I_1\dot{\phi}_0\ddot{\delta} + \sin\theta_0^*\left(mgl - I_1\dot{\phi}_0^2\cos\theta_0^*\right)\dot{\varepsilon} + I_1\dot{\phi}_0^3\sin^2\theta_0^*\delta = 0. \end{aligned} \tag{D}$$

Substitution of the first relation for $\dot{\varepsilon}$ into the second equation yields the incremental equation of motion for δ:

$$I_1^2\dot{\phi}_0^2\ddot{\delta} + \left(\left(mgl - I_1\dot{\phi}_0^2\cos\theta_0^*\right)^2 + I_1^2\dot{\phi}_0^4\sin^2\theta_0^*\right)\delta = 0, \tag{E}$$

in which the coefficient of δ is plainly positive and has the same form as (11.79y). The solution $\delta(t)$ is periodic. It follows from the first equation in (D) that the disturbance ε has the same period. Therefore, the motion of steady precession is stable, as shown more directly by (11.79x).

note that $\dot{\psi}(\theta_0^*) = (\alpha/2I_1)(2I_1/I_3 - 1)$, and thus obtain to the first order,

$$4I_1^2\ddot{\delta} + \left(I_3^2\omega_3^2 - 4I_1 mgl\right)\delta = 0. \tag{11.79z}$$

Consequently, the disturbance of the sleeping top is oscillatory, hence stable, provided that its total spin $\omega_3^s > \dfrac{2}{I_3}\sqrt{I_1 mgl}$. This value is just a bit greater than the critical rate $\omega_3^c = (2/I_3)(I_1 mgl \cos\theta_0^*)^{1/2}$ for a steady precession at an angle $\theta_0^* \neq 0$, which reduces to the former when $\theta_0^* = 0$. Hence, the sleeping top is stable provided its spin $\omega_3^s > \omega_3^c = (2/I_3)(I_1 mgl)^{1/2}$.

Exercise 11.10. Show that the vertical configuration of a "sleeping" gyroscopic pendulum for which $\theta(t) = \theta_0^* = \pi$ and $\dot{\theta}(t) = 0$ always is a stable configuration. □

This concludes‖ our study of the motion of a top about a fixed point. Note, however, that the Earth behaves like a top whose center revolves around the Sun, and similar top phenomena are characteristic of the motions of gyroscopes, spinning projectiles, bicycles, motorcycles, engine flywheels, propellers, jet engines, and more. So, with the beginnings sketched above, we have accomplished more than simply analyzing the motion of a child's toy. The reader is now equipped to explore by a variety of methods more advanced areas of gyrodynamics.

11.13. Introduction to the Theory of Vibrations

The equations of motion of many dynamical systems are nonlinear differential equations for which exact solutions are beyond reach of analysis, so either computational or analytical perturbation methods are applied to effect useful approximate solutions that shed light on important and interesting nonlinear phenomena. Discussion of various perturbation techniques may be found in books dedicated to this field. Here we focus on a fundamental perturbation method of linearization to study the theory of small vibrations of multidegree of freedom dynamical systems, a general method of approximate analysis based on a second order power series expansion of the motion about a stable, static equilibrium configuration of a nonlinear, holonomic system. The smallness approximation leads to a system of linearized differential equations for which the general theory of simultaneous linear equations is directly applicable. While certainly some interesting nonlinear phenomena and potentially useful information may be lost in our adopting only

‖ We have barely scratched the surface of a fascinating but difficult class of problems analyzed systematically and extensively by F. Klein and A. Sommerfeld in their treatise *Über die Theorie des Kreisels*, B.G. Teubner, Leipzig, 1910. See also the text by A. Gray, *A Treatise on Gyrostats and Rotational Motion*, Macmillan, London, 1918; Dover, New York, 1959.

a lowest order approximation in the final equations of motion, the linearization procedure offers useful insight into the first order physical nature of an otherwise complicated multidegree of freedom nonlinear system. Moreover, the theory provides a general framework within which many difficult problems may be solved.

11.13.1. Small Oscillations of a Simple Pendulum Revisited

In our earlier studies, linearization of the equations of motion for small vibrations of a specific dynamical system was applied after the general system of nonlinear equations was derived. This procedure, however, can be greatly simplified for conservative holonomic systems with scleronomic constraints. For these dynamical systems, the terms in the Lagrangian function can be expanded in power series at the outset of the problem formulation to retain all terms up to those quadratic in the variables. Then the equations of motion automatically will be linear in these variables. To illustrate the idea, consider the small motion of a pendulum about its stable equilibrium position. Recall that quadratic terms in θ and $\dot{\theta}$ are neglected in linearization of the equation of motion (11.36c). Since this equation is obtained by differentiation of the Lagrangian function, the linearization process may begin by our writing the potential and kinetic energies as quadratic functions of θ and $\dot{\theta}$, respectively. The Lagrangian $L(\dot{\theta}, \theta)$ for the pendulum problem is given exactly in (11.36a), where the kinetic energy $T(\dot{\theta}, \theta) = \frac{1}{2}M\dot{\theta}^2$ is independent of θ and already quadratic in $\dot{\theta}$, and $M \equiv m\ell^2$. The potential energy, however, with the power series expansion of $\cos\theta \cong 1 - \frac{1}{2}\theta^2$ to retain terms quadratic in θ in (11.36a), simplifies to $V(\theta) = \frac{1}{2}K\theta^2$, where $K \equiv mg\ell$. Hence, the power series expansion of the Lagrangian function to terms of the *second order* in $\dot{\theta}$ and θ is thus given by $L(\dot{\theta}, \theta) = \frac{1}{2}M\dot{\theta}^2 - \frac{1}{2}K\theta^2$; and, by (11.35), we find $M\ddot{\theta} + K\theta = 0$, the familiar linearized form of the exact nonlinear equation of motion (11.36c), for small oscillations of the pendulum about its equilibrium state at $\theta = 0$.

The same linearization process may be applied to any multidegree of freedom, conservative scleronomic system for which no generalized coordinates are ignorable and all remain small over time. For holonomic systems with rheonomic (time dependent) constraints and systems with ignorable coordinates, it proves best to linearize the final equations of motion, as before. For conservative scleronomic systems, however, the Lagrangian does not depend explicitly on time and the static equilibrium states in an inertial frame are readily determined by our taking all $\dot{q}_k = 0$, and hence $T = 0$, in Lagrange's equations (11.35), to obtain the static equilibrium equations $\partial V(q_r)/\partial q_k = 0$. These determine the values $q_k = q_k^*$ of the generalized coordinates in the static equilibrium state, some of which may not be stable. In the pendulum, Example 11.2, page 508, $\partial V(\theta)/\partial\theta = mg\ell\sin\theta$ vanishes at $q_1^* = \theta = 0, \pi$, the latter being an unstable equilibrium solution. Moreover, $\partial^2 V(\theta)/\partial\theta^2 = mg\ell\cos\theta$; and hence at the stable configuration $\theta = 0$, $\partial^2 V(\theta)/\partial\theta^2 = mg\ell > 0$, whereas $\partial^2 V(\theta)/\partial\theta^2 = -mg\ell < 0$ at the unstable state $\theta = \pi$. More generally, we have the following stability criterion.

Energy criterion for stability of static equilibrium: *The static equilibrium configuration $\mathscr{E}_s$ of a conservative scleronomic, holonomic system having n degrees of freedom with generalized coordinates $q_k = q_k^*$ in $\mathscr{E}_s$ is infinitesimally stable if the total potential energy of the linearized system is positive definite, that is, if*

$$\left.\frac{\partial^2 V}{\partial q_k \partial q_l}\right|_{q_k^*} u_k u_l > 0, \tag{11.80}$$

for all nonzero, n-dimensional vectors $\mathbf{u} = (u_k)$.

If the quadratic form (11.80) vanishes for some choice of u_k that are not all zero but is otherwise positive, the static equilibrium configuration at $q_k = q_k^*$ is called *neutrally stable*. If the quadratic form is negative for any choice of u_k, the equilibrium configuration is *unstable* at $q_k = q_k^*$. We shall return to this energy criterion momentarily in the presentation of the Lagrangian analysis of the theory of small vibrations about a stable equilibrium configuration of a nonlinear system.

11.13.2. The Theory of Small Vibrations

To formulate the general problem of small vibrations of a conservative and scleronomic holonomic system, let u_k denote small disturbances from a stable, static equilibrium state with corresponding specified coordinate values q_k^* so that the system has the perturbed generalized coordinates $q_k = q_k^* + u_k$, $k = 1, 2, \ldots, n$, the number of degrees of freedom. A Taylor series expansion of the total potential energy function about q_k^* to terms of the second order in u_k yields

$$V(q_r) = V^* + \frac{\partial V^*}{\partial q_k} u_k + \frac{1}{2} K_{kl} u_k u_l, \tag{11.81}$$

where repeated indices are summed over n, $V(q_r) \equiv V(q_1, q_2, \ldots, q_n)$ as usual, and quantities denoted by $*$ are evaluated in the static equilibrium state at $q_k = q_k^*$. For instance, $V^* = V(q_r^*)$, $\partial V^*/\partial q_k = \partial V(q_r)/\partial q_k|_{q_k = q_k^*}$, and, by definition, the constants

$$K_{kl} \equiv \frac{\partial^2 V^*}{\partial q_k \partial q_l} \tag{11.82}$$

are called *stiffness coefficients*. The series approximation (11.81) requires that *all* of the generalized coordinates are changed by a small perturbation, none are ignorable, and all remain small in time. Since only derivatives of V enter the equations of motion, we may omit the constant term V^* and note that in the equilibrium state $\partial V^*/\partial q_k = 0$. So far, the stiffness coefficients generally depend on the q_k^*s; however, no generality is lost in our assuming henceforward that all $q_k^* = 0$, and hence the generalized coordinates are measured from the equilibrium configuration. Then $u_r = q_r$ and the total potential energy (11.81) of the system may be written as a homogeneous quadratic function of the perturbed generalized coordinates q_k alone:

$$V(q_r) = \frac{1}{2} K_{kl} q_k q_l. \tag{11.83}$$

The matrix of stiffness coefficients (11.82) is square and symmetric: $K_{kl} = K_{lk}$, and the criterion (11.80) for stability of the equilibrium configuration requires, in matrix notation, that $K_{kl} q_k q_l \equiv K u \cdot u > 0$ for all nonzero vectors $u = (q_k)$. Therefore, for small displacements q_k from a stable, static equilibrium configuration, $V(q_r)$ is a positive definite, homogeneous quadratic form. Hence, alternatively, the stability criterion (11.80) holds if and only if $\det K$ and all of its principal minors are positive.

The total kinetic energy of a scleronomic system is a positive definite** quadratic function $T = \frac{1}{2} M_{kl}(q_r) \dot{q}_k \dot{q}_l$ of the perturbed generalized velocity components $\dot{u}_k = \dot{q}_k$. Because retention of terms linear and higher in the Taylor series expansion of $M_{kl}(q_r)$ about $q_k = q_k^* = 0$ introduces terms of order greater than the second in the total kinetic energy function, the coefficients $M_{kl}(q_r)$ may be replaced by their constant values in the equilibrium state: $M_{kl}(q_r) \cong M_{kl}^* \equiv M_{kl}$. The kinetic energy is then a homogeneous, positive definite quadratic function of the perturbed generalized velocities alone:

$$T(\dot{q}_r) = \frac{1}{2} M_{kl} \dot{q}_k \dot{q}_l. \tag{11.84}$$

The constants M_{kl} are called *inertia coefficients*. The matrix of inertia coefficients is square and symmetric: $M_{kl} = M_{lk}$.

Now, form from (11.83) and (11.84) the Lagrangian function

$$L(q_r, \dot{q}_r) = \frac{1}{2} M_{kl} \dot{q}_k \dot{q}_l - \frac{1}{2} K_{kl} q_k q_l. \tag{11.85}$$

Then Lagrange's equations (11.66), with the symmetry of K_{kl} and M_{kl} in mind, yield a system of n ordinary linear differential equations for small vibrations about

** It is easily seen that, in general, a quadratic form $F = Pu \cdot u$ that is positive definite in one reference system is positive definite in every reference system related to the first by an orthogonal transformation A. Consider the transformed quadratic form $F' = P'u' \cdot u'$, where $u' = Au$ denotes the transformed matrix vector. Then $F' = P'u' \cdot u' = P'Au \cdot Au = A^T(P'Au) \cdot u = Pu \cdot u = F$, in which the matrix $P = A^T P' A$, or $P' = APA^T$. Consequently, if the quadratic form F is positive definite in one system, it retains this property in every system related to first by an orthogonal transformation.

Now, the mass of every material object being positive, the kinetic energy, by its definition in (7.35) for a particle, (8.50) for a system of particles, and (10.90) for all bodies, is inherently positive definite. Like the potential energy function, it has the same positive definite value in every reference system. Note, however, that while the kinetic energy may be referred to any appropriate moving reference frame, it is always determined with respect to an inertial frame.

a stable, static equilibrium configuration of a conservative, scleronomic system:

$$M_{kl}\ddot{q}_l + K_{kl}q_l = 0, \qquad k, l = 1, 2, \ldots, n. \tag{11.86}$$

The stability of the equilibrium configuration must be confirmed in each application.

Example 11.11. Consider a system having $n = 2$ degrees of freedom. Then (11.86) is a coupled system of two linear equations:

$$\begin{aligned} M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + K_{11}q_1 + K_{12}q_2 = 0, \\ M_{21}\ddot{q}_1 + M_{22}\ddot{q}_2 + K_{21}q_1 + K_{22}q_2 = 0, \end{aligned} \tag{11.87a}$$

in which $M_{12} = M_{21}$ and $K_{12} = K_{21}$.

In particular, recall the conservative scleronomic system of Fig. 11.1, page 514, for which the total kinetic energy (11.40a) is a homogeneous quadratic function of the generalized velocities $\dot{q}_k = \dot{x}_k$ and the total potential energy (11.40b) is a homogeneous quadratic function of all of the generalized coordinates $q_k = x_k$, none being absent. Because both functions, without any series approximations, are homogeneous quadratic functions of $\dot{q}_k$ and q_k, respectively, the equations of motion in (11.40f) corresponding to (11.87a) hold for large amplitude oscillations of the system. The exact energy relations (11.40a) and (11.40b) are to be compared with the respective homogeneous quadratic forms (11.83) and (11.84) of the linearized theory in which the inertia and stiffness coefficient matrices of the example are identified by

$$[M_{kl}] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad [K_{kl}] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix}, \tag{11.87b}$$

evident from (11.40f). Since $Ku \cdot u = k_1(u_1^2 + u_2^2) + k_2(u_1 - u_2)^2 > 0$ for all vectors $u = (u_1, u_2) \neq 0$, the equilibrium configuration at $x_1 = x_2 = 0$ is infinitesimally stable, which also is physically evident. Alternatively, we confirm from the stiffness matrix K that $\det K = k_1^2 + 2k_1k_2 > 0$ and both of its principal minors $k_1 + k_2 > 0$; hence, K is a positive definite matrix.

To obtain the general solution of the system (11.86), we consider trial solutions of the form $q_k^T = C_k \sin(pt + \phi)$, all having the same circular frequency p and initial phase ϕ, and where C_k are n constants. Substitution of q_k^T into (11.86) yields the following homogeneous system of n algebraic equations:

$$\left(K_{kl} - p^2 M_{kl}\right)C_l = 0. \tag{11.88}$$

For nontrivial amplitudes C_l, the $n \times n$ determinant of the coefficient matrix must vanish; that is,

$$\det\left|K_{kl} - p^2 M_{kl}\right| = 0. \tag{11.89}$$

This is a polynomial of degree n in the squared circular frequency p^2 called the *characteristic equation*, and its roots are called *characteristic frequencies, eigenfrequencies*, *normal mode* or *natural frequencies*. Because both matrices K_{kl} and M_{kl} are real-valued, symmetric matrices all of the squared eigenfrequencies are positive, and only positive solutions p_m are meaningful. For each p_m obtained from (11.89), there is a corresponding trial solution of the form $q_k^T = C_{km}\sin(p_m t + \phi_m)$ for each m (no sum) and for which ratios of the amplitudes C_{km} are determined from (11.88), now cast in the form

$$\left(K_{kl} - p_m^2 M_{kl}\right) C_{lm} = 0, \tag{11.90}$$

for $k, l, m = 1, 2, \ldots, n$, sum on l, no sum on m. Hence, the general solution of the system (11.86) for each q_k is the sum of all of the trial solutions corresponding to each p_m, ϕ_m pair:

$$q_k = \sum_{m=1}^{n} C_{km}\sin(p_m t + \phi_m), \qquad k = 1, 2, \ldots, n. \tag{11.91}$$

This systematic analysis assumes that all of the characteristic frequencies are distinct and nonzero; but this is not always the case. Dynamical systems for which some roots of the characteristic equation may be repeated or may be zero, called *degenerate systems*, are handled somewhat differently. These systems are not studied here. The analysis of degenerate systems may be found in several advanced texts cited in the references.

Example 11.12. A homogeneous thin body of mass M, shown in Fig. 11.8, is suspended in the vertical plane by a thin wire of mass $m \ll M$, length a, and with its ends hinged in smooth bearings. (i) Derive the equations for small vibrations of the body about its vertical equilibrium position $\phi = \theta = 0$. Identify the inertia

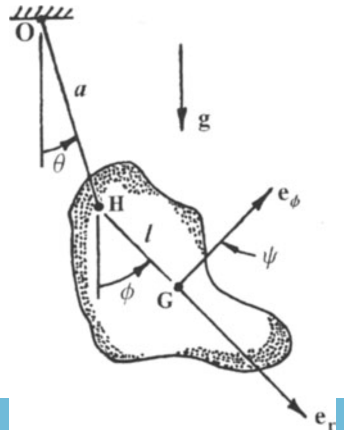


Figure 11.8. Small vibrations of a physical pendulum with two degrees of freedom.

and stiffness matrices. (ii) Sketch the general formulation of the solution. Then let $a = l$, and suppose that the body is a thin circular disk of radius $R = \sqrt{2}l$. Determine the normal mode frequencies p_m, the ratios of the amplitudes C_{lm}, and thus derive the solution of the coupled equations of motion. Identify the normal equations of motion and their normal mode solutions as functions of the generalized variables.

Solution of (i). The system is scleronomic with two degrees of freedom θ and ϕ defined in Fig. 11.8. Since the wire has negligible mass compared with M, its contribution to the total kinetic and potential energies is negligible. The total velocity of the center of mass G referred to the body frame $\psi = \{G; \mathbf{e}_r, \mathbf{e}_\phi\}$ is given by $\mathbf{v}^* = \mathbf{v}_H + \boldsymbol{\omega} \times \mathbf{l} = a\dot{\theta}\mathbf{t} + l\dot{\phi}\mathbf{e}_\phi$, where $\mathbf{t} = \sin(\phi - \theta)\mathbf{e}_r + \cos(\phi - \theta)\mathbf{e}_\phi$. The total kinetic energy of the system, by (10.101), is

$$T = \frac{1}{2}M[a^2\dot{\theta}^2 \sin^2(\phi - \theta) + (a\dot{\theta}\cos(\phi - \theta) + l\dot{\phi})^2] + \frac{1}{2}I_G\dot{\phi}^2, \quad (11.92a)$$

and the total gravitational potential energy is given by

$$V = Mg[a(1 - \cos\theta) + l(1 - \cos\phi)]. \quad (11.92b)$$

We note that no coordinates are absent. For small vibrations, only terms of second order in all of the small quantities $\phi, \dot{\phi}, \theta, \dot{\theta}$ are retained in (11.92a) and (11.92b). Therefore, the first term in (11.92a) is of higher order and may be neglected; and with $\cos(\phi - \theta) \cong 1 - (\phi - \theta)^2/2$, to terms of second order the total kinetic and potential energies of the system for small vibrations about the vertical equilibrium configuration are thus given by

$$\begin{aligned} T &= \frac{1}{2}(I_G + Ml^2)\dot{\phi}^2 + \frac{1}{2}M(2la\dot{\phi}\dot{\theta} + a^2\dot{\theta}^2), \\ V &= \frac{1}{2}Mg(l\phi^2 + a\theta^2). \end{aligned} \quad (11.92c)$$

The potential energy is a positive definite, homogeneous quadratic function of the generalized coordinates θ and ϕ. Hence, clearly, the equilibrium configuration $\phi = \theta = 0$ is infinitesimally stable. Similarly, the total kinetic energy is a homogeneous quadratic function of the generalized velocities $\dot{\theta}$ and $\dot{\phi}$. With $I_H = I_G + Ml^2$ in accordance with the parallel axis theorem in (11.92c), the symmetric inertia and stiffness matrices in (11.84) and (11.83) are thus identified as

$$[M_{kl}] = \begin{bmatrix} I_H & Mla \\ Mla & Ma^2 \end{bmatrix}, \qquad [K_{kl}] = \begin{bmatrix} Mgl & 0 \\ 0 & Mga \end{bmatrix}. \quad (11.92d)$$

We note in passing that the stiffness matrix is diagonal and all of its nontrivial components are positive; so, it is quite evident that $Ku \cdot u > 0$ holds for all $u \neq 0$.

The hinge constraints are workless, so the system is conservative with a Lagrangian function

$$L = \frac{1}{2} I_H \dot{\phi}^2 + \frac{1}{2} M(2la\dot{\phi}\dot{\theta} + a^2\dot{\theta}^2) - \frac{1}{2} Mg(l\phi^2 + a\theta^2), \qquad (11.92e)$$

and Lagrange's equations (11.66) now yield the equations of small vibration,

$$\begin{aligned} I_H\ddot{\phi} + Mal\ddot{\theta} + Mgl\phi &= 0, \\ Mla\ddot{\phi} + Ma^2\ddot{\theta} + Mga\theta &= 0. \end{aligned} \qquad (11.92f)$$

Solution of (ii). The natural frequencies for the small vibrations are determined by (11.89). With (11.92d), we have

$$\det \begin{bmatrix} Mgl - p^2 I_H & -p^2 Mla \\ -p^2 Mla & Mga - p^2 Ma^2 \end{bmatrix} = 0.$$

This reduces to a quadratic equation in p^2:

$$\left(Mgl - p^2 I_H\right)\left(Mga - p^2 Ma^2\right) - p^4 M^2 l^2 a^2 = 0, \qquad (11.92g)$$

which yields two eigenfrequencies p_m. Use of these in (11.90) provides two sets of equations for the ratios of the coefficients C_{lm}, $m = 1, 2$; namely,

$$\begin{bmatrix} Mgl - p_m^2 I_H & -p_m^2 Mla \\ -p_m^2 Mla & Mga - p_m^2 Ma^2 \end{bmatrix} \begin{bmatrix} C_{1m} \\ C_{2m} \end{bmatrix} = 0. \qquad (11.92h)$$

Finally, use of these results in (11.91) yields the solutions for $q_1 = \phi$ and $q_2 = \theta$. We omit these general details and turn to a special case for illustration.

Consider a circular disk of radius $R = \sqrt{2}l$ and let $a = l$. Then $I_G = MR^2/2 = Ml^2$, and $I_H = 2Ml^2$. The characteristic equation (11.92g) thus simplifies to $p^4 - 3p_0^2 p^2 + p_0^4 = 0$ with positive roots

$$p_1 = p_0 \sqrt{\frac{1}{2}(3 + \sqrt{5})}, \qquad p_2 = p_0 \sqrt{\frac{1}{2}(3 - \sqrt{5})}, \qquad (11.92i)$$

where $p_0 \equiv \sqrt{g/l}$, and (11.92h) becomes

$$\begin{bmatrix} p_0^2 - 2p_m^2 & -p_m^2 \\ -p_m^2 & p_0^2 - p_m^2 \end{bmatrix} \begin{bmatrix} C_{1m} \\ C_{2m} \end{bmatrix} = 0. \qquad (11.92j)$$

For $m = 1, 2$ (no sum), we thus obtain the amplitude ratios

$$\frac{C_{21}}{C_{11}} = \frac{\left(p_0^2 - 2p_1^2\right)}{p_1^2}, \qquad \frac{C_{12}}{C_{22}} = \frac{p_2^2}{\left(p_0^2 - 2p_2^2\right)}. \qquad (11.92k)$$

Substitution here of the characteristic roots (11.92i) yields

$$C_{21} = -\frac{1}{2}(1 + \sqrt{5})C_{11}, \qquad C_{12} = \frac{1}{2}(1 + \sqrt{5})C_{22}. \qquad (11.92l)$$

Introducing these in (11.91) in which $q_1 = \phi$ and $q_2 = \theta$, we obtain the general solution

$$\begin{aligned}\phi &= C_{11}\sin(p_1 t + \phi_1) + \frac{1}{2}\left(1+\sqrt{5}\right)C_{22}\sin(p_2 t + \phi_2),\\ \theta &= -\frac{1}{2}(1+\sqrt{5})C_{11}\sin(p_1 t + \phi_1) + C_{22}\sin(p_2 t + \phi_2).\end{aligned} \tag{11.92m}$$

The remaining constants $C_{11},\ C_{22},\ \phi_1$, and ϕ_2 are determined upon specification of the initial data.

The normal mode motions, readily identified from (11.92m), are defined by

$$\xi_1 = C_{11}\sin(p_1 t + \phi_1), \qquad \xi_2 = C_{22}\sin(p_2 t + \phi_2), \tag{11.92n}$$

and upon solving (11.92m) for the ξ_ks, we obtain the normal coordinates in terms of the original generalized coordinates:

$$\begin{aligned}\xi_1 &= -\frac{1}{10}\sqrt{5}\left(2\theta + \phi\left(1-\sqrt{5}\right)\right),\\ \xi_2 &= -\frac{1}{20}\left(\sqrt{5}-5\right)\left(2\theta + \phi\left(1+\sqrt{5}\right)\right).\end{aligned} \tag{11.92o}$$

These normal coordinates uncouple the original equations of motion (11.92f) applied to the circular disk so that the normal mode equations of motion are

$$\ddot{\xi}_k + p_k^2 \xi_k = 0, \qquad k = 1, 2 \text{ (no sum)}, \tag{11.92p}$$

in which the normal mode frequencies p_k are given in (11.92i). □

For linear systems having two degrees of freedom, the solution procedure illustrated above for free vibrations is straightforward. The theory of free vibrations of conservative, scleronomic systems studied above is the easiest class of problems within the theory of vibrations of systems having n degrees of freedom. For more general dynamical systems in the theory of small forced vibrations and having any number of degrees of freedom, the use of normal coordinates is especially advantageous because it eliminates the need to solve a coupled system of n simultaneous, linear nonhomogeneous differential equations of motion. Rather, in terms of normal coordinates ξ_k, a simpler system of n independent equations of the form $\ddot{\xi}_k + p_k^2 \xi_k = P_k(t)$, where $P_k(t)$ are certain generalized forcing functions corresponding to the normal coordinates, are to be solved. The methods of modal analysis and the general theory for the transformation from generalized coordinates to normal coordinates, and the analysis of degenerate systems, may be found in the works by Greenwood, Whittaker, and Yeh and Abrams, among others listed in the References.

11.14. Dissipative Dynamical Systems of the Stokes Type

In this section, Lagrange's equations are modified to account directly for the effects of linear viscous damping. We begin with a single particle and recall the Stokes drag force (6.29): $\mathbf{F}_D = -c\mathbf{v} = -c\dot{\mathbf{x}}$. Then for a scleronomic system $\mathbf{x} = \mathbf{x}(q_r)$, and the corresponding mechanical power $\mathscr{P}_D$ expended, i.e. the energy dissipated by the drag force alone, may be written as

$$\mathscr{P}_D = -c\mathbf{v}\cdot\mathbf{v} = -c\frac{\partial \mathbf{x}}{\partial q_i}\dot{q}_i \cdot \frac{\partial \mathbf{x}}{\partial q_j}\dot{q}_j = -c_{ij}\dot{q}_i\dot{q}_j < 0, \tag{11.93}$$

wherein we sum on i and j, the index range being the number of independent generalized coordinates of the particle, and, by definition,

$$c_{ij} \equiv c\frac{\partial \mathbf{x}}{\partial q_i}\cdot\frac{\partial \mathbf{x}}{\partial q_j}. \tag{11.94}$$

Of course, $c_{ij} = c_{ji}$, the *generalized damping coefficients*, are functions of the q_rs alone. For future convenience, we introduce a generalized dissipation function D defined by

$$D \equiv -\frac{\mathscr{P}_D}{2} = \frac{1}{2}c_{ij}(q_r)\dot{q}_i\dot{q}_j > 0, \tag{11.95}$$

a positive definite quadratic form in the $\dot{q}_r$s equal to the negative of half the rate at which energy is dissipated by the Stokes force. The function D is known as the *Rayleigh dissipation function.*

In the special case when the Stokes force is the only force that acts on the particle, $\mathscr{P}_D = \mathscr{P} = \dot{T}$, the total mechanical power; and then (11.93) may be written as $\dot{T} = -2\nu T = -2D$. Hence, $D = -\nu T$, where $\nu = c/m$ is a damping exponent. Let $T_0 = T(0)$ denote the initial kinetic energy of the particle. Then $T(t) = T_0 e^{-2\nu t}$, and the dissipation $D = \nu T$ decays to zero with the total kinetic energy T, as expected. In general, however, other forces that act on the particle contribute to the total power.

Now consider the virtual work done by the Stokes force in a general motion of a particle: $\delta\mathscr{W} = -c\mathbf{v}\cdot\delta\mathbf{x} = -c\mathbf{v}\cdot\partial\mathbf{x}/\partial q_r \delta q_r$; and, with the aid of (11.9), we find

$$\delta\mathscr{W} = -c\mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial \dot{q}_r}\delta q_r = \frac{\partial}{\partial \dot{q}_r}\left(-\frac{1}{2}c\mathbf{v}\cdot\mathbf{v}\right)\delta q_r = Q_r^D\delta q_r. \tag{11.96}$$

Hence, for a holonomic system, the *generalized Stokes force* Q_r^D is thus defined by $Q_r^D \equiv \partial(-\frac{1}{2}c\mathbf{v}\cdot\mathbf{v})/\partial\dot{q}_r$, and, in accordance with (11.93) for scleronomic systems, we obtain

$$Q_r^D = \frac{\partial(\mathscr{P}_D/2)}{\partial \dot{q}_r} = -\frac{\partial D}{\partial \dot{q}_r}. \tag{11.97}$$

With the total generalized force $Q_r = Q_r^D + \hat{Q}_r$, where $\hat{Q}_r$ denotes the total of all other generalized forces that are not of the Stokes type, Lagrange's equations (11.73) for scleronomic systems, accounting separately for dissipative forces of the Stokes type, may be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial D}{\partial \dot{q}_r} = \hat{Q}_r. \tag{11.98}$$

Further, with $\hat{Q}_r = Q_r^N - \partial V/\partial q_r$ in the presence of some conservative forces with total potential energy V and other nonconservative generalized forces Q_r^N, and writing $L = T - V$, as usual, we arrive at the alternative form of Lagrange's equations for holonomic systems of the scleronomic type under dissipative forces of the Stokes type:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} + \frac{\partial D}{\partial \dot{q}_r} = Q_r^N. \tag{11.99}$$

Example 11.13. For an illustration recall our earlier Example 11.4, page 510, of a particle falling from rest in a Stokes medium. In this case, there is one degree of freedom with generalized coordinate $q_1 = y$. The Lagrangian function is $L = T - V = \frac{1}{2}m\dot{y}^2 + mgy$, the Rayleigh dissipation function (11.95) is given by $D = \frac{1}{2}c_{11}\dot{y}^2$, and all other nonconservative generalized forces $Q_r^N = 0$. From (11.99) and with $c \equiv c_{11}$, we thus obtain the equation for the motion of a particle falling in a Stokes medium: $m\ddot{y} - mg + c\dot{y} = 0$. This agrees with our earlier result.

In accordance with (11.73), it follows that (11.99) holds for any general scleronomic system of particles and rigid bodies so long as the damping is characterized by a Rayleigh dissipation function of the form (11.95). In particular, for a lineal rigid body $\mathscr{L}$ of length ℓ subject to a Stokes force over its entire length and on which all other forces are workless, the total power is $\mathscr{P} = \mathscr{P}_D$ and the Rayleigh dissipation function may be read from (10.144): $D = -\mathscr{P}/2 = \beta T(\ell, t)$, in which β is a damping exponent and $T(\ell, t) = \frac{1}{2}I\dot{q}^2$ is the total kinetic energy of the body about a fixed point or about its center of mass. This example is similar to the single particle problem discussed earlier, and clearly $T(t) = T_0 e^{-2\beta t}$, as before. The system has one degree of freedom with $q = \theta$ and $\dot{q} = \omega$, the angular spin of the body. All other forces being workless, (11.98) yields the universal equation of motion $\dot{\omega} + \beta\omega = 0$, as shown previously in (10.150). Now let us consider a system having two degrees of freedom.

Example 11.14. The damped free vibration of a two degree of freedom system moving in the xy-plane has total kinetic energy $T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2$, total elastic potential energy $V = \frac{1}{2}k_1x^2 + \frac{1}{2}k_2y^2$, and a total Stokes type dissipation described by the Rayleigh function $D = \frac{1}{2}(c_1\dot{x}^2 + 2c_{12}\dot{x}\dot{y} + c_2\dot{y}^2)$. Derive the equations of motion.

Solution. The Lagrangian is $L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 - \frac{1}{2}k_1x^2 - \frac{1}{2}k_2y^2$. For the free vibrational motion $Q_r^N = 0$ and use of L and D in (11.99) yields the two equations of motion for the system:

$$\begin{aligned} m_1\ddot{x} + c_1\dot{x} + c_{12}\dot{y} + k_1x &= 0, \\ m_2\ddot{y} + c_{12}\dot{x} + c_2\dot{y} + k_2y &= 0. \end{aligned} \tag{11.100}$$

This is a coupled system of linear differential equations for which general solution methods are well known. See Whittaker in the References for further study of this topic. □

The theory of small vibrations about an equilibrium configuration of a system having n degrees of freedom is now easily extended to include dissipative forces of the Stokes type. In this case, the damping coefficients, to the lowest order, are constants: $c_{ij} = C_{ij}$, and the Rayleigh dissipation function $D = \frac{1}{2}C_{ij}\dot{q}_i\dot{q}_j$ is a homogeneous, positive definite quadratic function of only the generalized velocities. With the Lagrangian given by (11.85), the general equations (11.99) for small vibrations with Stokes damping become

$$M_{kl}\ddot{q}_l + C_{kl}\dot{q}_l + K_{kl}q_l = Q_k^N, \qquad k, l = 1, 2, \ldots, n. \tag{11.101}$$

When no additional driving forces act on the system, $Q_k^N = 0$ and these equations reduce to those for the free vibrations of a damped dynamical system having n degrees of freedom. See Problem 11.37.

Because the kinetic and potential energies and the Rayleigh dissipation function in the last example above are already quadratic functions of q_r and $\dot{q}_r$, the equations of motion (11.100) necessarily have the same form as the general equations of the theory of small vibrations in which $(q_1, q_2) = (x, y)$ and

$$[M_{kl}] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad [C_{kl}] = \begin{bmatrix} c_1 & c_{12} \\ c_{12} & c_2 \end{bmatrix}, \qquad [K_{kl}] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

The difference, however, is that the motion in the example need not be small.

11.15. Closure

We have seen that Lagrange's analytical mechanics reduces the various principles of mechanics to an invariant system of differential equations that facilitates the formulation and solution of all kinds of dynamical problems. The development of Lagrange's equations from Hamilton's principle has demonstrated that these equations may be applied to general and complex conservative and nonconservative holonomic dynamical systems having any number of degrees of freedom. It is important to remember, however, that in the applications of Lagrange's equations, the energy is determined with respect to an inertial reference frame, a concept that was never mentioned in the derivation of these equations from Hamilton's principle. Moreover, the determination of the kinetic energy for a rigid body, in particular, requires knowledge of its moment of momentum, and this entails use of a body reference frame with origin at a special point—generally a fixed point in the inertial reference frame, one having a uniform motion in the inertial frame, or the center of mass. While none of these concepts is apparent in Lagrange's formulation, the influence of Euler's ideas is evident throughout Lagrange's work. Hence, while Lagrange's equations most certainly are convenient for the derivation of the equations of motion of complex systems, in its applications we must appeal to many classical concepts and methods due to others, notably Newton and Euler, whose profound classical ideas are developed throughout this book.

In its applications to systems of particles and rigid bodies the Newton-Euler theory often proves to be particularly tedious, because generally one must deal separately with each and every particle or body in the system and introduce all of the seprate internal and external forces, including all forces of constraint. On the other hand, in applications of the Lagrangian method, though in many respects simpler than the Newton-Euler formulation, it is necessary to bear in mind certain technical details that characterize the system, the nature of its constraints, and the corresponding special technical conditions for the applicability of the equations. We recall, for example, that in the Lagrangian formulation, the work-energy principle was derived specifically for only scleronomic systems; and the simple principles of conservation of momentum and moment of momentum in the Newton-Euler theory are imbedded in Lagrange's principle of conservation of generalized momentum for an ignorable coordinate for which the corresponding nonconservative part of the generalized force vanishes. The fact that a constraint is holonomic or nonholonomic, scleronomic or rheonomic, details essential to the structure of the Lagrangian formulation, is unimportant to the mathematical structure of the Newton-Euler laws of mechanics. In the latter instance, these important technical details are brought into the analysis in different ways that usually involve determination of forces of constraint. There are, however, countless situations in engineering practice where the intensity of constraining forces that act on the system must be determined for design considerations, and these forces are not provided by the Lagrangian theory. So, we really need the full body of theory and good models to successfully analyze the motion of complex dynamical systems.

In our studies here, we have not explored nonholonomic dynamical systems, we have not investigated Lagrange's unified approach to analytical mechanics essentially based on D'Alembert's principle, and we have not studied Hamilton's form of the equations of motion. These and other topics that are outside the scope of this Introduction may be found in advanced treatises on analytical mechanics. See, for example, the works by Lanczos, Pars, Rosenberg, and Whitttaker listed in the chapter References. So, this is not the end—there is much more to be learned about dynamics. It is hoped, however, that this introductory treatment may encourage the reader to continue study of dynamical systems at the advanced and more abstract levels of theoretical mechanics and the theory of equations.

References

1. Greenwood, D. T., *Principles of Dynamics*, Prentice-Hall, Englewood Cliffs, New Jersey, 1965. Cast at an intermediate level comparable to the present text, this book provides an excellent resource for collateral study. Lagrange's equations, including a thorough discussion of constraints and the method of virtual work, are discussed in Chapter 6. The theory of vibrations, including discussion of degenerate systems, is treated in Chapter 9.
2. Housner, G. W., and Hudson, D. E., *Applied Mechanics: Dynamics*, 2nd Edition, Van-Nostrand, Princeton, New Jersey, 1959. Lagrange's equations applied to the theory of small, free, and forced vibrations are studied in Chapter 9. The authors provide good preparation throughout for the student's subsequent study of advanced topics in dynamics. Some examples and problems in the current chapter are modeled after those presented in this book.
3. Lagrange, J. L., *Analytical Mechanics*, translated from the *Mécanique analytique novelle édition* of 1811, editors A. Boissonnade and V. V. Vagliente, Kluwer, Dordrecht, The Netherlands, 1997. The original of this enduring classical work often is unavailable for study, so this English translation is a most helpful substitute. The translators' "Introduction" provides an interesting sketch of the life and times of Lagrange. Lagrange's treatise, based on the method of virtual work, begins with principles of statics in Part I; the greater emphasis is on the various principles of dynamics in Part II.
4. Lanczos, C., *The Variational Principles of Mechanics*, University of Toronto Press, Toronto, Canada, 1949. A beautifully written text highly recommended for advanced study. The focus, however, is mainly on the theoretical aspects of Lagrange's differential equations of motion and the Hamilton–Jacobi canonical theory of equations. There are very few, though well chosen examples throughout. The method of undetermined multipliers is applied in the formulation of Lagrange's equations for systems with constraints. Our discussion of nonholonomic constraints mirrors that presented in this text.
5. Long, R. R., *Engineering Science Mechanics*, Prentice-Hall, Englewood Cliffs, New Jersey, 1963. Vector and Cartesian tensor methods are integrated in this treatment of topics on engineering mechanics. Hamilton's principle and Lagrange's equations are introduced in Chapter 3. The latter, however, are not uniformly applied throughout the text, which otherwise provides a good resource for collateral reading and for additional problems and examples.
6. Marion, J. B., *Classical Dynamics of Particles and Rigid Bodies*, Academic, New York, 1965. In Chapter 9, Lagrange's equations are derived from Hamilton's principle, and the method of undetermined multipliers to characterize constraints on a system is introduced. Multidegree of freedom systems are studied in Chapter 14. Applications of Euler's equations and the energy principle for rigid bodies, including the spinning top problem, are studied in Chapter 13. See also Marion, J. B., and Thornton, S. T., *Classical Dynamics of Particles and Rigid Bodies*, Harcourt Brace, New York, 1995.
7. Pars, L. A., *A Treatise on Analytical Dynamics*, Ox Bow Press, Woodbridge, Connecticut, 1965. This text provides many examples for advanced readers.
8. Rosenberg, R. M., *Analytical Dynamics of Discrete Systems*, Plenum, New York, 1977. This is a carefully written, thorough treatment of analytical dynamics based on the geometry of the configuration space of generalized coordinates, including a precise presentation of the geometry of constraints and virtual displacements, and their relation to constrained systems. Chapter 9 is a clear analysis and description of D'Alembert's principle that leads in subsequent chapters to Lagrange's equations for arbitrary systems. There are many worked examples throughout.

9. SYNGE, J. L., and GRIFFITH, B. A., *Principles of Mechanics*, 3rd Edition, McGraw-Hill, New York, 1959. Chapter 14 treats the general problem of the spinning top and the gyroscope by Euler's equations and the energy method, and Chapter 15 deals with Lagrange's equations with several examples, including the top problem.
10. TRUESDELL, C., *Essays in the History of Mechanics*, Springer-Verlag, Berlin, Heidelberg, New York, 1968. The author argues in Chapters 2 and 5 that Euler's laws are more general than Lagrange's equations, pointing specifically to the significance of Euler's principle of moment of momentum and noting that it is never mentioned or used by Lagrange.
11. WHITTAKER, E. T., *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, 3rd Edition, Cambridge University Press, Cambridge, 1927. This classical volume, highly recommended to advanced readers, remains one of the best mathematical treatments of analytical dynamics. The treatise focuses entirely on the Lagrangian and Hamiltonian theories and includes many worked examples throughout. The reader may find the problems rather challenging, however. The theory of vibrations, including degenerate systems, is analyzed thoroughly in Chapter VII; nonholonomic systems are studied in Chapter VIII, and extension of Hamilton's principle to conservative and nonconservative, nonholonomic systems follows in Chapter IX.
12. YEH, H., and ABRAMS, J. I., *Principles of Mechanics of Solids and Fluids*, Vol. 1, *Particle and Rigid Body Mechanics* McGraw-Hill, New York, 1960. Lagrange's equations and some examples, including application to the vibrations of a structure, are discussed in Chapters 13.

Problems

11.1. Introduce $C_1 = -A/C$ and $C_2 = -B/C$, in which $A = A(q_1, q_2, q_3)$, $B = B(q_1, q_2, q_3)$, and $C = C(q_1, q_2, q_3)$, so that the differential constraint (11.5) becomes $A dq_1 + B dq_2 + C dq_3 = 0$. Show that the test condition (11.6) may be rewritten in the form

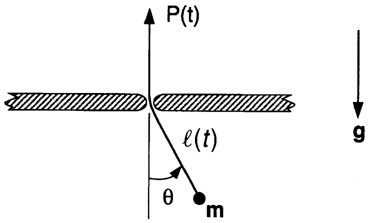
$$A\left(\frac{\partial B}{\partial q_3} - \frac{\partial C}{\partial q_2}\right) + B\left(\frac{\partial C}{\partial q_1} - \frac{\partial A}{\partial q_3}\right) + C\left(\frac{\partial A}{\partial q_2} - \frac{\partial B}{\partial q_1}\right) = 0. \tag{P11.1}$$

Notice that this is satisfied identically when the terms in parentheses vanish. In this case, the constraint is integrable and hence holonomic. If these terms do not vanish, but (P11.1) vanishes identically, the constraint is holonomic, otherwise not. In either case, however, the integral of the differential constraint is not revealed (See Rosenberg, p. 46.), and it may be quite difficult to determine. If (P11.1) yields a relation $q_3 = q_3(q_1, q_2)$ that satisfies the conditions $\partial q_3/\partial q_1 = C_1$ and $\partial q_3/\partial q_2 = C_2$, then $q_3 = q_3(q_1, q_2)$ is the holonomic constraint corresponding to (P11.1). Apply this method to decide the nature of the differential constraint relation in Exercise 11.1, page 499.

11.2. A particle P moves on a space curve with path variable $s(t)$. Apply Lagrange's method to derive the intrinsic equation of motion of P.

11.3. Introduce spherical coordinates in Example 7.15, page 260, for the spherical pendulum. Apply Lagrange's equations to derive the equations of motion, and determine their first integrals.

11.4. A small mass m is attached to a weightless, inextensible string that passes through a tiny, smooth hole in a horizontal plate. The specified time variable force $P(t)$ shown in the figure controls the cord length $\ell(t)$ as a function of time so that the mass moves in the vertical plane. (a) How many degrees of freedom does this system have? (b) Apply Lagrange's equations to derive equations to determine $\theta(t)$ and $\ell(t)$. (c) What results follow from the moment of momentum principle? (d) Derive the same equations from Newton's law.



Problem 11.4.

11.5. A particle of mass m moves on the smooth inner surface of a thin paraboloidal shell of revolution defined by $r^2 = az$ in cylindrical coordinates (r, θ, z), where a is a constant. The particle, with weight $\mathbf{W} = -mg\mathbf{k}$, encounters air resistance described by a Stokes drag force $\mathbf{F}_d = -c\mathbf{v}$. Apply Lagrange's equations to derive the equations of motion. Find the generalized forces (a) by the method of virtual work, (b) by application of (11.14), and (c) by use of (11.20).

11.6. Consider the motion of the slider block S in Problem 6.54. Identify the rheonomic constraint. Let $\mathbf{R}$ denote the force exerted on the slider by the smooth rod. Show that $Q_1 = \mathbf{R} \cdot \partial\mathbf{x}/\partial r = 0$, where $\mathbf{x}$ is the position vector of S from F at the center of the table. Use Lagrange's equations to derive the equation of motion of S for the generalized coordinate $q_1 \equiv r(t)$. Find the motion of the slider when $r(0) = 0$ and $\dot{r}\ (0) = v_o$ initially.

11.7. The slider S described in Problem 6.55 is released from rest at O, relative to the table. Apply Lagrange's method to derive the equation of motion for S, and find its relative motion $r(t)$ for all constant values of the angular speed ω. Refer all quantities to the rod frame $\varphi = \{O; \mathbf{i}_k\}$ shown in Problem 6.54.

11.8. Identify any rheonomic constraints and apply Lagrange's method to derive the equation of motion of the slider block described in Problem 6.51.

11.9. Consider the system described in Problem 6.56. (a) Identify the rheonomic constraint and derive the equation of motion for the mass m by application of Lagrange's method. (b) Relax the constraint, obtain a second equation of motion involving the constraint reaction force R exerted by the rod on the slider, and thereby determine R in the case when $\omega = \omega_0$, a constant. (c) Suppose the slider is released from rest at $x = a$ to oscillate along the smooth rod. Find R as a function of x.

11.10. Determine the generalized forces and derive the Lagrange equations of motion for the pendulum bob described in Example 6.14, page 150. Show how the bob constraining force may be found.

11.11. A particle of mass m with cylindrical coordinates (r, θ, z) moves in a gravitational field $\mathbf{g} = -\,g\mathbf{k}$ on a smooth, concave upward surface of revolution defined by $r = r(z)$ with $r(0) = 0$. Use Lagrange's equations to derive the equation of motion for $z(t)$, and outline how the angular placement $\theta(t)$ may be found.

11.12. Apply Lagrange's equations (11.15) to find the applied forces required to control the uniform motion of the particle relative to the rotating frame in Example 5.9, page 71. Identify the physical nature of the pseudoforces described by $-\partial T/\partial q_k$.

11.13. Apply Lagrange's equations to derive the equations of motion for the system described in Problem 8.16.

11.14. Use Lagrange's equations to investigate Problem 8.29.

11.15. Derive Lagrange's equations for small amplitude oscillations of the system shown in Problem 10.14.

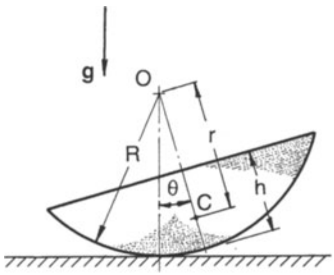
11.16. A uniform rod of mass m and length 2ℓ moves on a smooth horizontal plane with angular velocity $\boldsymbol{\omega} = \omega \mathbf{k}$. Its center C has a velocity $\mathbf{v}^* = u\mathbf{i} + v\mathbf{j}$ referred to a body frame $\varphi = \{C; \mathbf{i}_k\}$ with $\mathbf{i}$ directed along the rod. Apply Lagrange's method to find the impulsive force $\mathbf{P} = P_x\mathbf{i} + P_y\mathbf{j}$ applied at a point B distant b from C in order to bring point B instantaneously to rest. Express the result in terms of the assigned parameters.

11.17. Identify the generalized coordinates and the number of degrees of freedom of the log in Problem 10.58. Use Lagrange's method to deduce the equations of motion and thus determine the frequency of the vertical oscillations of the log.

11.18. The wire and bob assembly of the rotating simple pendulum shown in the diagram for Problem 6.47 is replaced by a thin rigid rod of length l and mass m. The rod is hinged in a smooth bearing at O and is free to slide on the smooth horizontal table. (a) Identify the rheonomic constraint and apply Lagrange's equations to derive the equation for finite amplitude oscillations of the rod relative to the table. (b) Relax the constraint, determine the generalized forces that act on the rod at its hinge bearing, and thus find the constraint reaction force as an exact function of the finite angular placement β for initial data $\beta(0) = \beta_0$ and $\dot{\beta}(0) = 0$.

11.19. Derive Lagrange's equation of motion for the rolling cylinder in Problem 10.43.

11.20. A homogeneous circular cylindrical segment of radius R, length L, height h, and mass m performs rocking oscillations without slipping on a rough horizontal surface. The center of mass is at r from the center O. The segment is released from rest at the placement $\theta(0) = \theta_0$. (a) Derive the differential equation for the finite angular motion $\theta(t)$ by (i) application of Lagrange's equations, and (ii) by use of the Newton–Euler equations. (b) Determine the first integral of the equation of motion. (c) Derive an equation for the period of the large amplitude oscillations. (d) Find the circular frequency for small oscillations.

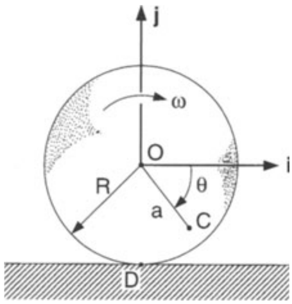


Problem 11.20.

11.21. A uniform, thin rigid rod shown in Problem 10.37 slides in the vertical plane with its ends on a smooth circle of radius r and subtending a central angle of 120°. (a) Derive the equation of motion by use of (i) Euler's laws, (ii) Lagrange's method, and (iii) the work–energy principle. (b) What is the first integral of the equation of motion? (c) Discuss briefly the exact solution for the motion $\theta(t)$ of the rod. (d) Find as functions of θ alone the contact forces acting on the rod. (e) What are the major differences among the three methods used in (a)? (f) What is the length ℓ, expressed in terms of r, of an equivalent simple pendulum having the same frequency?

11.22. Suppose the thin rod in the previous problem has its ends set in smooth bearings that slide along a circular hoop of radius r and negligible mass. The hoop rotates about its vertical central axis with a constant angular speed Ω. The rod is released from rest relative to the hoop at an angle $\theta_0 = \theta(0)$. (a) Use Lagrange's method to derive the equations of motion of the rod. Are there any surprising features of these results? (b) Derive the equations of motion by use of Euler's laws. (c) Find the bearing reaction forces exerted on the rod. (d) In what manner would the mass M of the hoop affect the results?

11.23. A nonhomogeneous circular cylinder has its center of mass C at a distance a from its geometrical center O, and its circular cross sectional plane through O is a plane of symmetry. The cylinder is released from rest when $\theta = 0$ and rolls without slipping on the horizontal surface. (a) Apply Lagrange's equations to determine the angular velocity $\boldsymbol{\omega}$ and angular acceleration $\dot{\boldsymbol{\omega}}$ of the cylinder as functions of θ. (b) Deduce the same results starting from the energy principle. (c) Find the surface reaction forces at D in terms of θ, ω, and $\dot{\omega}$. (d) Use Euler's equations to derive the equation of motion for the cylinder. (e) Discuss the principal difference between the methods of Euler and Lagrange.



Problem 11.23.

11.24. Use Lagrange's equations to formulate the equations of motion of the spring and pulley system described in Problem 7.49, about its static equilibrium state. The pulley has radius a, mass m, and rolls without slipping on its inextensible belt. How many degrees of freedom does this system have?

11.25. Use Lagrange's method to set up the equation for the finite motion of the system described in Problem 10.39 for a thin hoop whose mass m is the same as that of the thin rod. (a) Find the first integral of the equation of motion. (b) Derive an equation from which the exact period of the finite rocking oscillation is determined. (c) What is the circular frequency of small amplitude oscillations? (d) What is the length of a simple pendulum having the same small amplitude frequency as that of this system?

11.26. A smooth rigid rod shown in the figure for Problem 6.56 is attached to a table T that rotates in the horizontal plane about a smooth bearing at F. The table has mass M, radius of gyration K about F, and its variable angular speed due to an applied driving torque $\boldsymbol{\mu}_F(t) = \mu_F(t)\mathbf{k}$ about F is $\omega(t) = \dot{\theta}(t)$. The mass of the rod is negligible. A slider block of mass m, supported symmetrically by identical springs of stiffness k, is released from rest relative to the rod at a distance a from the unstretched state at O. (a) Derive the equations of motion for the system (i) by use of Lagrange's equations, and (ii) by use of the Newton–Euler laws. (b) Find the torque $\mu_F(t)$ required to sustain a stable motion of the system with a constant angular speed.

11.27. Apply Lagrange's method to derive the equations of motion for the system described in Problem 8.30. Solve these for the given initial conditions, and determine the small amplitude vertical and rotational frequencies of the motion.

11.28. Use Lagrange's equations to solve Problem 8.18.

11.29. Two uniform rigid rods, each of mass m and length 2ℓ, are connected end-to-end by a smooth hinge and placed in a straight line along the y-axis on a smooth horizontal table in the xy-plane. The end of one rod is struck suddenly by a force $\mathbf{P} = P\mathbf{i}$. Find the subsequent instantaneous generalized velocities of the system. What is the increase of the total energy of the system due to the impulse?

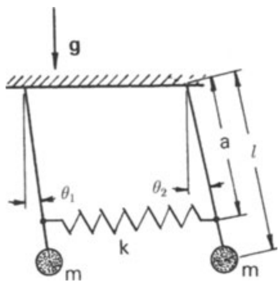
11.30. Consider the system described in Problem 6.57, but now suppose that the rigid rod is homogeneous with mass M. (i) Determine by integration the moment of inertia of the rod about the point O. (ii) Let θ denote the small angular placement from the horizontal equilibrium position. Derive the equations of motion and find the vibrational frequency of the system by use of (a) Euler's equations, (b) the energy method, and (c) Lagrange's equations. Which is the simplest, most direct method? (iii) Determine the dynamic part of the support reaction force as a function of θ for the case $b = 2a$. Does this depend on mass?

11.31. A pendulum device consists of a thin rod of mass m and length ℓ supported in the vertical plane by a smooth hinge H attached at the rim of a thin circular disk of radius R and mass M. The disk turns in the vertical plane with a steady angular speed ω about a smooth fixed axle at its center. Use Lagrange's method to derive the equations of motion.

11.32. Formulate Lagrange's equations for small vibrations of the system described in Problem 10.56.

11.33. Derive Lagrange's equations for small amplitude oscillations of the system in Problem 10.57.

11.34. Apply the theory of small vibrations to derive the equations of motion for the pendula shown in the figure, and solve these for the angular motions $\theta_1(t)$ and $\theta_2(t)$. Determine the eigenfrequencies, find the normal mode motions, and characterize these physically when the pendula are appropriately displaced and released from rest initially.



Problem 11.34.

11.35. (a) Derive Lagrange's equations of motion for the finite amplitude oscillations of the double pendulum described in Problem 8.32. Deduce from these results the equations for small amplitude oscillations. (b) Apply the theory of small vibrations to derive the latter equations of motion.

11.36. (a) Derive the equation for the finite amplitude motion of the system described in Problem 8.33. Then linearize the result to obtain the equation for small amplitude oscillations. (b) Apply the theory of small vibrations to derive the equation of motion.

11.37. Suppose that the system of pendula in Problem 11.34 moves in a Stokes medium, which might be the surrounding air for example. Find the Rayleigh dissipation function for the system of particles, and derive the equations for its small oscillations. Show that when $k = 0$, the coupled equations of motion reduce to those for the small damped oscillations of simple pendula.

Appendix C

Internal Potential Energy of a System of Particles

The internal potential energy of a particle P_j due to the mutual internal force $\mathbf{b}_{jk}$ exerted on P_j by P_k is defined by $\beta_{jk} = \psi(\mathbf{x}_j, \mathbf{x}_k)$, a scalar-valued function that depends only on the positions of both particles. By application of the principle of material frame indifference in Section 8.9.2, page 325, it is shown that for a conservative internal force, the mutual internal potential energy $\psi(\mathbf{x}_j, \mathbf{x}_k)$ for an arbitrary pair of interacting particles is a function of only the distance between the particles, namely, $\psi(\mathbf{x}_j, \mathbf{x}_k) = \psi(|\mathbf{r}_{jk}|)$, where $\mathbf{r}_{jk} \equiv \mathbf{x}_j - \mathbf{x}_k$ is the vector joining the two particles. This led to (8.75), namely, $\beta_{jk} = \psi(|\mathbf{r}_{jk}|)$. This important result, however, may be derived without use of frame indifference. Rather, it can be shown* that, *given the third law* (8.3), $\psi(\mathbf{x}_j, \mathbf{x}_k)$ *is a function of only the distance between the particles, if and only if the mutual internal forces act along their common line.*

We first show that $\beta_{jk} = \psi(\mathbf{x}_j, \mathbf{x}_k)$ is a function of only the vector $\mathbf{r}_{jk} \equiv \mathbf{x}_j - \mathbf{x}_k$ joining the two particles. To see this, fix j and $k \neq j$, and introduce the two vectors

$$\mathbf{u} = \mathbf{x}_j + \mathbf{x}_k, \qquad \mathbf{r} = \mathbf{r}_{jk} = \mathbf{x}_j - \mathbf{x}_k. \tag{C.1}$$

With (C.1) in mind, introducing $\psi(\mathbf{x}_j, \mathbf{x}_k) = \hat{\psi}(\mathbf{u}, \mathbf{r})$ and noting that for any vector $\mathbf{v}$, $\partial \mathbf{v}/\partial \mathbf{v} = \mathbf{1}$, the identity tensor, we find

$$\frac{\partial \hat{\psi}}{\partial \mathbf{x}_j} = \frac{\partial \hat{\psi}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{x}_j} + \frac{\partial \hat{\psi}}{\partial \mathbf{r}}\frac{\partial \mathbf{r}}{\partial \mathbf{x}_j} = \frac{\partial \hat{\psi}}{\partial \mathbf{u}}\mathbf{1} + \frac{\partial \hat{\psi}}{\partial \mathbf{r}}\mathbf{1} = \frac{\partial \hat{\psi}}{\partial \mathbf{u}} + \frac{\partial \hat{\psi}}{\partial \mathbf{r}}, \tag{C.2}$$

$$\frac{\partial \hat{\psi}}{\partial \mathbf{x}_k} = \frac{\partial \hat{\psi}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{x}_k} + \frac{\partial \hat{\psi}}{\partial \mathbf{r}}\frac{\partial \mathbf{r}}{\partial \mathbf{x}_k} = \frac{\partial \hat{\psi}}{\partial \mathbf{u}}\mathbf{1} - \frac{\partial \hat{\psi}}{\partial \mathbf{r}}\mathbf{1} = \frac{\partial \hat{\psi}}{\partial \mathbf{u}} - \frac{\partial \hat{\psi}}{\partial \mathbf{r}}. \tag{C.3}$$

* The development here is similar to that due to G. M. Kapoulitsas, The behavior of the internal potential energy of a system of particles in a noninertial frame, *Ingenieur-Archiv* **57,** 393–9 (1987), wherein it is also shown that the result holds in all reference frames, inertial or not.

From Newton's third law (8.3) for fixed j and $k \neq j$, and with the aid of (8.73) and $\beta_{kj} = \psi(\mathbf{x}_k, \mathbf{x}_j) = \hat{\psi}(\mathbf{u}, -\mathbf{r})$, it can be shown that

$$\frac{\partial \hat{\psi}}{\partial \mathbf{x}_j} = \frac{\partial \beta_{jk}}{\partial \mathbf{x}_j} = -\frac{\partial \beta_{kj}}{\partial \mathbf{x}_k} = -\frac{\partial \hat{\psi}}{\partial \mathbf{x}_k}. \tag{C.4}$$

Therefore, it follows from (C.2) and (C.3) that $\partial \hat{\psi}/\partial \mathbf{u} = \mathbf{0}$, that is, $\hat{\psi} = \hat{\psi}(\mathbf{r})$ is independent of $\mathbf{u}$. Consequently, by the second relation in (C.1), we have

$$\psi(\mathbf{x}_j, \mathbf{x}_k) = \hat{\psi}(\mathbf{x}_j - \mathbf{x}_k) = \hat{\psi}(\mathbf{r}_{jk}). \tag{C.5}$$

Exercise C.1. (a) Introduce the Cartesian components $u_p = x_p^j + x_p^k$, $r_p = x_p^j - x_p^k$ of the vectors $\mathbf{u}$ and $\mathbf{r}$, and show that $\nabla_j(u_p, r_p) = (\mathbf{i}_p, \mathbf{i}_p)$ and $\nabla_k(u_p, r_p) = (\mathbf{i}_p, -\mathbf{i}_p)$. Now consider $\psi(\mathbf{x}_j, \mathbf{x}_k) = \hat{\psi}(u_p, r_p)$, recall (8.73), and thus derive (C.2) and (C.3) in vector notation, wherein $\partial/\partial \mathbf{u} \equiv \sum_{p=1}^{3} (\partial/\partial u_p)\mathbf{i}_p$, for example. (b) Derive (C.4) as described above. □

We next show that the dependence in (C.5) is reflected only in the distance $|\mathbf{r}_{jk}|$ between the particles. The result follows from the *assumption* that the internal forces act along their mutual line so that the internal force exerted on P_j by P_k may be written as $\mathbf{b} \equiv \mathbf{b}_{jk} = -|\mathbf{b}_{jk}|\mathbf{e}_r$, where $\mathbf{e}_r$ is the unit vector directed from P_k toward P_j. Recall (8.73) so that $\mathbf{b} = -\nabla_j \hat{\psi}(\mathbf{r}_{jk})$ in accordance with (C.5). The reader will find that in spherical coordinates, the gradient operator (7.48) may be written as

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}, \tag{C.6}$$

in which the spherical basis vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ may be referred to a Cartesian frame $\Phi = \{F; \mathbf{I}_k\}$ in Fig. 4.21, page 277 of Volume 1. Because $\mathbf{b}$ is assumed parallel to $\mathbf{e}_r(\theta, \phi)$ in Φ, it follows from (C.6) that $\partial \hat{\psi}/\partial \theta = \partial \hat{\psi}/\partial \phi = 0$, and hence $\hat{\psi}(\mathbf{r}_{jk}) = \psi(r)$ depends on r alone, where

$$r = |\mathbf{r}_{jk}| = \sqrt{(\mathbf{x}_j - \mathbf{x}_k) \cdot (\mathbf{x}_j - \mathbf{x}_k)}. \tag{C.7}$$

Thus, in our original notation, the mutual internal potential energy for an arbitrary pair of interacting particles may be written as

$$\beta_{jk} = \psi(|\mathbf{r}_{jk}|). \tag{C.8}$$

Conversely, it follows from (C.8) that the mutual force (8.73) is directed along the line connecting the particles. See Exercise 8.8, page 325. Because the results use only the position vectors of the particles and ultimately only the vector connecting them, the result does not depend on use of any particular reference frame, inertial or not.

The result (C.8) is the same as (8.75). The important difference in comparison with the frame indifference point of view discussed in Chapter 5, however, is that here it is *assumed* that (i) the internal forces act along their mutual line and (ii) these satisfy Newton's third law. These assumptions follow as theorems in Noll's proof in Chapter 5.

Exercise C.2. Derive the gradient operator (C.6) for spherical coordinates. □

Appendix D

Properties of Homogeneous Rigid Bodies

D.1. Nomenclature

The following representative nomenclature of symbols is used in the table of properties of various homogeneous simple rigid bodies provided in Section D.3:

$\varphi = \{O; \mathbf{i}_k\}$: body reference frame at O
$\varphi^* = \{C; \mathbf{i}_k^*\}$: body reference frame at the center of mass
V : material volume of the body
A : surface area of a thin-walled body
ℓ : length measure of a lineal body
m : total mass of the body
$\mathbf{x}^*$: position vector of the center of mass C from point O in φ
$\mathbf{I}_O$: moment of inertia tensor relative to point O in φ
$\mathbf{I}_C$: moment of inertia tensor relative to C in φ^*
$\mathbf{i}_{jk} \equiv \mathbf{i}_j \otimes \mathbf{i}_k$: tensor product basis associated with φ
$\mathbf{i}_{jk}^* \equiv \mathbf{i}_j^* \otimes \mathbf{i}_k^*$: tensor product basis associated with φ^*.

It is expected, of course, that the reader will adjust to the use of diverse but similar notation for various corresponding entities encountered throughout the text, such as M for mass, $\psi\{Q; \mathbf{e}_k\}$ for a body or fixed reference frame at the point Q at which the moment of inertia tensor is $\mathbf{I}_Q = I_{rs}\mathbf{e}_{rs}$, $\mathbf{e}_{rs} \equiv \mathbf{e}_r \otimes \mathbf{e}_s$ for the tensor product basis, and many other parallel notational variations.

D.2. A Word of Caution

In a great many other dynamics texts, the products of inertia, as emphasized in (9.16), are defined as the negatives of those used in this book. Therefore, the reader must exercise caution when consulting other sources for special formulae, tables of properties, or supplementary reading. The moment of inertia tensor components defined in (9.14) and (9.15) are used throughout the text and in the table below. In every case illustrated in the table, however, both φ and φ^* are principal frames of reference with respect to which the products of inertia vanish. This generally will not be the case for an arbitrary frame orientation or for a shift of origin point of either reference frame.

D.3. Table of Properties

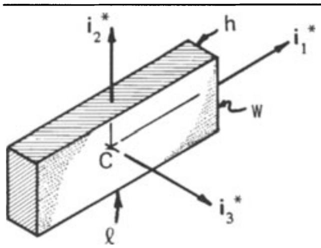
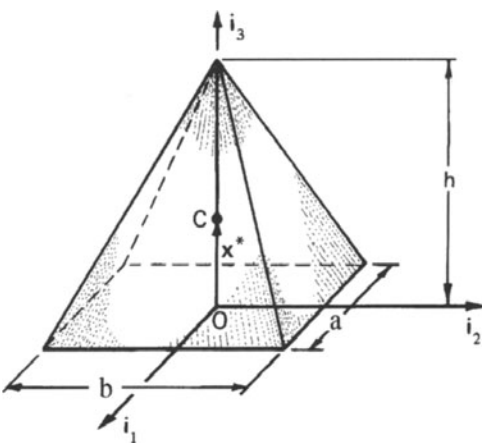
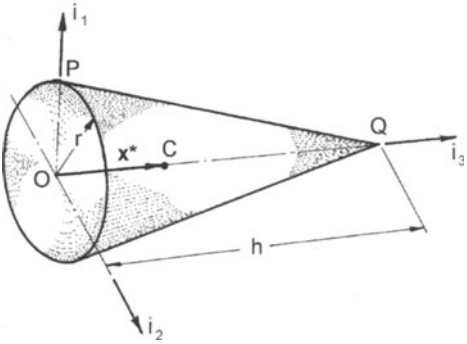
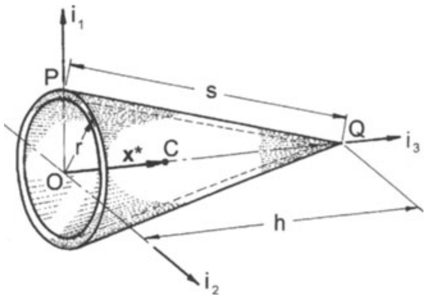
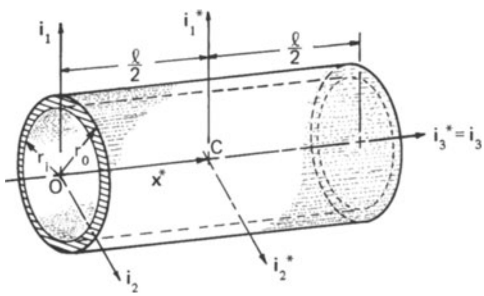
Body	Center of mass, volume/area, moment of inertia
 **Figure D.1.** Rectangular solid/cube.	Rectangular parallelepiped $V = \ell w h,$ $\mathbf{I}_C = \frac{m}{12}[(w^2+h^2)\mathbf{i}^*_{11} + (h^2+\ell^2)\mathbf{i}^*_{22} + (\ell^2+w^2)\mathbf{i}^*_{33}].$ - - - - - - - - - Cube: $\ell = w = h = a$ $V = a^3,$ $\mathbf{I}_C = \frac{m}{6}a^2\mathbf{1}.$
 **Figure D.2.** Right rectangular pyramid.	$\mathbf{x}^* = \frac{h}{4}\mathbf{i}_3, \qquad V = \frac{1}{3}abh,$ $\mathbf{I}_O = \frac{m}{20}[(b^2+2h^2)\mathbf{i}_{11} + (a^2+2h^2)\mathbf{i}_{22} + (a^2+b^2)\mathbf{i}_{33}].$

Table D.3. (*continued*)

Body	Center of mass, volume/area, moment of inertia
 **Figure D.3.** Right circular cone.	$\mathbf{x}^* = \frac{h}{4}\mathbf{i}_3, \qquad V = \frac{1}{3}\pi r^2 h,$ $\mathbf{I}_O = \frac{m}{20}[(3r^2 + 2h^2)(\mathbf{i}_{11} + \mathbf{i}_{22}) + 6r^2\mathbf{i}_{33}].$
 **Figure D.4.** Thin conical shell.	$\mathbf{x}^* = \frac{h}{3}\mathbf{i}_3, \qquad A = \pi r\sqrt{r^2 + h^2},$ $\mathbf{I}_O = \frac{m}{12}(3r^2 + 2h^2)(\mathbf{i}_{11} + \mathbf{i}_{22}) + \frac{mr^2}{2}\mathbf{i}_{33}.$
 **Figure D.5.** Circular cylindrical tube.	Thick-walled tube $\mathbf{x}^* = \frac{\ell}{2}\mathbf{i}_3, \qquad V = \pi\ell\left(r_o^2 - r_i^2\right),$ $\mathbf{I}_C = \frac{m}{4}\left(r_o^2 + r_i^2 + \frac{\ell^2}{3}\right)(\mathbf{i}^*_{11} + \mathbf{i}^*_{22}) + \frac{m}{2}\left(r_o^2 + r_i^2\right)\mathbf{i}^*_{33}.$ --- Thin-walled tube: $r_i = r_o = r$ $\mathbf{x}^* = \frac{\ell}{2}\mathbf{i}_3, \qquad A = 2\pi r\ell,$ $\mathbf{I}_C = \frac{m}{4}\left(2r^2 + \frac{\ell^2}{3}\right)(\mathbf{i}^*_{11} + \mathbf{i}^*_{22}) + mr^2\mathbf{i}^*_{33}.$

(*continued*)

Table D.3. (*continued*)

Body	Center of mass, volume/area, moment of inertia

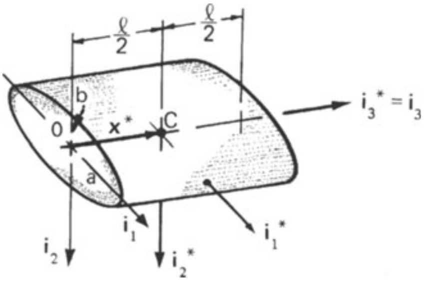


Figure D.6. Solid right cylinder.

Elliptical cylinder

$$\mathbf{x}^* = \frac{\ell}{2}\mathbf{i}_3, \qquad V = \pi a b \ell,$$

$$\mathbf{I}_C = \frac{m}{4}\left[\left(b^2 + \frac{\ell^2}{3}\right)\mathbf{i}^*_{11} + \left(a^2 + \frac{\ell^2}{3}\right)\mathbf{i}^*_{22} + (a^2 + b^2)\mathbf{i}^*_{33}\right].$$

Circular cylinder: $a = b = R$

$$\mathbf{x}^* = \frac{\ell}{2}\mathbf{i}_3, \qquad V = \pi R^2 \ell,$$

$$\mathbf{I}_C = \frac{m}{4}\left(R^2 + \frac{\ell^2}{3}\right)(\mathbf{i}^*_{11} + \mathbf{i}^*_{22}) + \frac{mR^2}{2}\mathbf{i}^*_{33}.$$

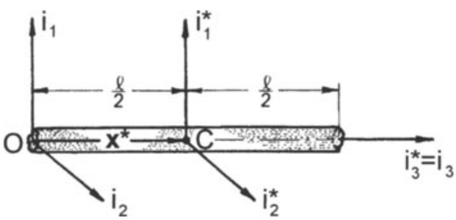


Figure D.7. Thin rod.

$$\mathbf{x}^* = \frac{\ell}{2}\mathbf{i}_3,$$

$$\mathbf{I}_C = \frac{m\ell^2}{12}(\mathbf{i}^*_{11} + \mathbf{i}^*_{22}),$$

$$\mathbf{I}_O = \frac{m\ell^2}{3}(\mathbf{i}_{11} + \mathbf{i}_{22}).$$

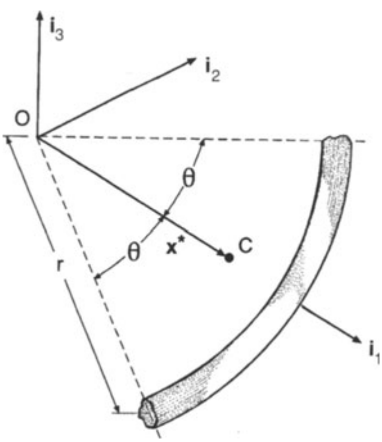


Figure D.8. Thin circular rod.

$$\mathbf{x}^* = \frac{r\sin\theta}{\theta}\mathbf{i}_1, \qquad \ell = 2r\theta,$$

$$\mathbf{I}_O = \frac{mr^2}{2}\left[\left(1 - \frac{\sin 2\theta}{2\theta}\right)\mathbf{i}_{11} + \left(1 + \frac{\sin 2\theta}{2\theta}\right)\mathbf{i}_{22} + 2\mathbf{i}_{33}\right].$$

Table D.3. (*continued*)

Body	Center of mass, volume/area, moment of inertia

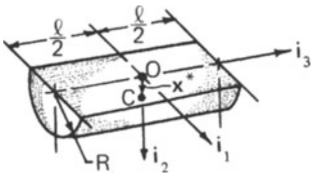


Figure D.9. Semicylinder.

$$\mathbf{x}^* = \frac{4R}{3\pi}\mathbf{i}_2, \qquad V = \frac{1}{2}\pi R^2\ell,$$

$$\mathbf{I}_O = \frac{m}{4}\left(R^2 + \frac{\ell^2}{3}\right)(\mathbf{i}_{11} + \mathbf{i}_{22}) + \frac{mR^2}{2}\mathbf{i}_{33}.$$

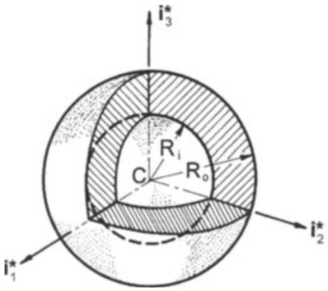


Figure D.10. Spherical shell.

Thick-walled shell	Thin-walled shell
$V = \frac{4}{3}\pi\left(R_o^3 - R_i^3\right),$	$A = 4\pi R^2,$
	$\lim R_i \to R_o = R$
$\mathbf{I}_C = \frac{2}{5}m\frac{R_o^5 - R_i^5}{R_o^3 - R_i^3}\mathbf{1}.$	$\mathbf{I}_C = \frac{2}{3}mR^2\mathbf{1}.$

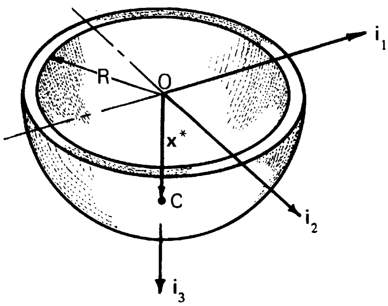


Figure D.11. Hemispherical shell.

$$\mathbf{x}^* = \frac{R}{2}\mathbf{i}_3, \qquad A = 2\pi R^2,$$

$$\mathbf{I}_O = \tfrac{2}{3}mR^2\mathbf{1}.$$

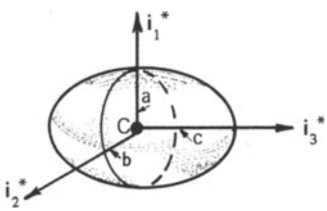


Figure D.12. Ellipsoid and sphere.

Ellipsoid

$V = \frac{4}{3}\pi abc,$

$$\mathbf{I}_C = \frac{m}{5}[(b^2 + c^2)\mathbf{i}^*_{11} + (a^2 + c^2)\mathbf{i}^*_{22} + (a^2 + b^2)\mathbf{i}^*_{33}].$$

Sphere: $a = b = c = R$

$V = \frac{4}{3}\pi R^3,$

$\mathbf{I}_C = \frac{2}{5}mR^2\mathbf{1}.$

(*continued*)

Table D.3. (*continued*)

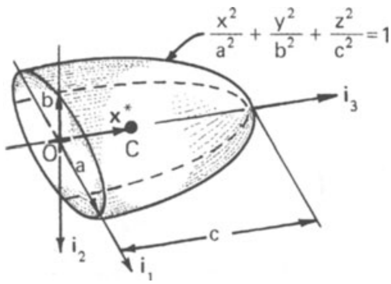
<table>
<tr><th>Body</th><th>Center of mass, volume/area, moment of inertia</th></tr>
<tr><td>


Figure D.13. Semiellipsoid/Hemisphere.
</td><td>
Semiellipsoid

$$\mathbf{x}^* = \frac{3c}{8}\mathbf{i}_3, \quad V = \frac{2}{3}\pi abc,$$

$$\mathbf{I}_O = \frac{m}{5}[(b^2+c^2)\mathbf{i}_{11} + (a^2+c^2)\mathbf{i}_{22} + (a^2+b^2)\mathbf{i}_{33}].$$

Hemisphere: $a = b = c = R$

$$\mathbf{x}^* = \frac{3R}{8}\mathbf{i}_3, \qquad V = \frac{2}{3}\pi R^3,$$

$$\mathbf{I}_O = \tfrac{2}{5}mR^2\mathbf{1}.$$
</td></tr>
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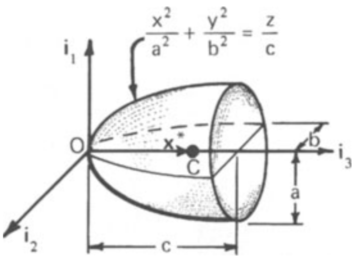

Figure D.14. Paraboloid.
</td><td>
Elliptic paraboloid

$$\mathbf{x}^* = \frac{2c}{3}\mathbf{i}_3, \qquad V = \tfrac{1}{2}\pi abc,$$

$$\mathbf{I}_O = \frac{m}{6}[(b^2+3c^2)\mathbf{i}_{11} + (a^2+3c^2)\mathbf{i}_{22} + (a^2+b^2)\mathbf{i}_{33}].$$

Paraboloid of revolution: $a = b = r$

$$\mathbf{x}^* = \frac{2c}{3}\mathbf{i}_3, \qquad V = \frac{1}{2}\pi r^2 c,$$

$$\mathbf{I}_O = \frac{m}{6}(r^2+3c^2)(\mathbf{i}_{11}+\mathbf{i}_{22}) + \frac{m}{3}r^2\mathbf{i}_{33}.$$
</td></tr>
</table>

Answers to Selected Problems

Chapter 5

5.1. (a) $10\mathbf{x}^*(\beta, 0) = 8\mathbf{i} - 6\mathbf{j} + 17\mathbf{k}$ m, (b) $\mathbf{p}(\beta, 0) = 7\mathbf{i} - 6\mathbf{j} - 16\mathbf{k}$ kg · m/sec.
5.3. $d_C = a/2$.
5.5. $\mathbf{x}^*(\mathscr{B}) = \frac{3}{4}h\mathbf{k}$ from the apex, $M = \pi\rho r^2 h/3$.
5.7. $\mathbf{p}(\mathscr{B}, 2) = 3600\mathbf{i}$ gm · cm/sec, $\mathbf{h}_A(\mathscr{B}, 2) = 43,200\mathbf{k}$ gm · cm^2/sec.
5.9. $\mathbf{h}_O = m[\omega(\ell^2 + a^2) - \ell v]\mathbf{k}$.
5.11. $\mathbf{h}_F = 30\mathbf{i} + 23\mathbf{j} - 6\mathbf{k}$ kg · m^2/sec, $\mathbf{h}_O = -2(4\mathbf{i} + 16\mathbf{j} + \mathbf{k})$ kg · m^2/sec.
5.13. $w = 2Wa/(a + g)$.
5.15. $\mathbf{F} = -2\mu GM/\pi b^2\mathbf{i}$.
5.17. $\mathbf{g}(\mathbf{X}) = -\dfrac{2GMX}{R_2^2 - R_1^2}\left(\dfrac{1}{\sqrt{X^2 + R_1^2}} - \dfrac{1}{\sqrt{X^2 + R_2^2}}\right)\mathbf{k}.$
5.21. $\mathbf{g}(Q) = \dfrac{mG}{2R^2}\mathbf{k}$ hemispherical shell, $\mathbf{g}(Q) = \mathbf{0}$ closed spherical shell.
5.23. $\mathbf{F}(R_1) = -\dfrac{mMG}{2ab}\log\left[\dfrac{(c+b)(a + \sqrt{(c-b)^2 + a^2})}{(c-b)(a + \sqrt{(c+b)^2 + a^2})}\right]\mathbf{j}.$
5.25. $g_m = 5.18$ ft/sec^2 $= 1.58$ m/sec^2; $W_m = 0.161W_e \approx W_e/6$.
5.27. (a) $P = \mu W_1\left(\dfrac{2 + \mu\tan\theta}{1 + \mu\tan\theta}\right)$, (b) $T = \dfrac{\nu W_2}{\cos\theta + \nu\sin\theta}.$
5.31. (a) $\mathbf{W}^* = 60\mathbf{j}$ lb; $\mathbf{W}^* = 260\mathbf{j}$ lb; $\mathbf{a} = 32\mathbf{j}$ ft/sec^2.
5.33. $\mathbf{F} = 0.3g\mathbf{i} - 150\sqrt{3}\omega^2\mathbf{j} + 1.5\sqrt{3}\dot{\omega}\mathbf{k}$ N.
5.35. (b) yes, (c) Any pair of the planes $x - 9y - 4 = 0$, $2y - z + 1 = 0$, $2x - 9z + 1 = 0$, (e) $\mathbf{x}_O^* = \frac{1}{86}(-3\mathbf{i} - 29\mathbf{j} + 28\mathbf{k})$ m.

Chapter 6

6.1. $\mathbf{T} = 12\mathbf{i} + 16\mathbf{j}$ N, $\mathbf{N} = -16\mathbf{j}$ N, $\mathbf{f} = -4\mathbf{i}$ N.
6.3. (b) $\mathbf{F}_A = 0.02\mathbf{I}$ N, $\mathbf{F}_B = 0.352\mathbf{J}$ N.
6.5. $v = \sqrt{2gd(\mu\cos\theta - \sin\theta)}$.
6.7. $\omega = \sqrt{\mu g/r}$.
6.9. $\mathbf{N} = 76\sqrt{2}\mathbf{n}$ lb, $\mathbf{T} = -8\mathbf{i}$ lb.

6.11. (a) $\mathbf{N} = 7\mathbf{n}$ lb, $\mathbf{T} = 2\sqrt{2}\mathbf{j}$ lb, (b) $\mathbf{N} = 10\mathbf{n}$ lb, $\mathbf{T} = 5(\mathbf{t} - \mathbf{n})$ lb, $\mathbf{f} = -3\mathbf{t}$ lb.
6.13. $N = 3.56 \times 10^{-5}$ N.
6.15. $\mathbf{F}_c = -34\mathbf{i} - 36\mathbf{j} + 47\mathbf{k}$ lb.
6.17. (b) $\dot{s}(3) = 50$ ft/sec, (c) $\mathbf{x}^*(5) = 118.5\mathbf{i} + 216\mathbf{j}$ ft.
6.19. $T = W \sec\beta$.
6.21. (a) $E_{\max} = Wd/qh$, (b) $c\mathbf{E} = 12\mathbf{i}$, (c) $y = \alpha x$, $\alpha \equiv g\cos\theta/(cE - g\sin\theta)$.
6.23. $s = v_0/k$.
6.25. $\mathbf{x}(2) = -4.54\mathbf{i} - 0.76\mathbf{j} + 49.87\mathbf{k}$ ft, $\mathbf{v}(2) = 3.46\mathbf{i} - 0.49\mathbf{j} + 44.05\mathbf{k}$ ft/sec.
6.27. (a) $v(h) = v_\infty\sqrt{1 - e^{-2\nu h}}$, (b) $v(t) = v_\infty \tanh(t/\tau)$, with ν, τ constant.
6.33. $X(P,t) \equiv x - g/p^2 = X_{\max}\cos(pt - \phi)$, $X(Q,t) \equiv x + g/p^2 = X_{\min}\cosh(pt + \psi)$,
$X_{\max} = \sqrt{X_0^2 + q^2}$, $\tan\phi = \dfrac{q}{X_0}$; $X_{\min} = \sqrt{X_0^2 - q^2}$, $\tanh\psi = \dfrac{q}{X_0}$, $q \equiv \dfrac{v_0}{p}$.
6.35. (a) $a = g/n$, (c) $x(t) = \dfrac{g}{p^2}(\cos pt - 1)$, (e) all are equivalent.
6.37. $u = (\sqrt{37}/3)\cos(pt + \phi)$ with $\tan\phi = \frac{1}{6}$.
6.39. $x_A = 1.25$ cm, $f = 14/\pi$ Hz.
6.41. (a) $k = 24$ N/cm, (b) $\tau = 2\pi/7$ sec, (c) $x(2) = 0.684$ cm.
6.43. (a) $\tau = 2\pi\sqrt{m/(k_1 + k_2)}$, (c) $x = x_A\cos(pt - \phi)$, $\tan\phi = -v_0/x_0 p$.
6.45. (b) $p = \sqrt{AE/\mu r^2}$, (c) $k_e = 2\pi AE/r$.
6.47. $\omega = 8\pi$ rad/sec, $\mathbf{T} = 0.128\pi^2\mathbf{n}$ N at β_0.
6.49. (a) $R = m\omega^2(2\sqrt{x^2 - a^2} - h)$, (b) $\mathbf{x}(S,t) = a\cosh\omega t\,\mathbf{i}$.
6.51. (a) Rod reaction force components and the motion $x(S,t)$
(b) $x(S,t) = A(\cosh(\omega + \Omega)t - \cos\omega t)$, $A \equiv a\Omega^2/(\omega^2 + (\omega + \Omega)^2)$.
6.53. (a) Stable for $\omega < \sqrt{k/m}$, (b) $k^* = k - m\omega^2$.
6.55. (a) $r_S = -g\sin\alpha/(\dfrac{k}{m} - \omega^2\cos^2\alpha)$; (b) $r(t) = v_0 t - \frac{1}{2}gt^2\sin\alpha$ for the case $k/m = \omega^2\cos^2\alpha$; (c) $r_S < 0$ stable, $r_S \geq 0$ unstable.
6.57. $f = \dfrac{a}{2\pi b}\sqrt{k/m}$.
6.59. The motion is simple harmonic with frequency $f = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{a} + \dfrac{2k}{m}\left(\dfrac{b}{a}\right)^2}$.
6.61. (b) $\theta(t) = \theta_0\cos pt + \dfrac{rq^2}{\ell(1 - q^2)}(\sin\omega t - q\sin pt)$, $q \equiv \dfrac{\omega}{p}$.
6.65. $W \ll kg/\omega^2 = 9.79$ lb.
6.67. Resonant frequency: $\Omega^* = \Omega_1/2 = 600$ rpm.
6.69. (a) $\ddot{\phi} + 2\nu\dot{\phi} + p^2\sin\phi = (F_0/mr)\cos\phi\sin\Omega t$ with $2\nu = c/m$, $p^2 = g/r$,
(b) $\tau = (2\sqrt{2}/p)\int_0^{\phi_0}(\cos\phi - \cos\phi_0)^{-1/2}d\phi$, $\tau = 2\pi\sqrt{r/g}$.
6.71. $\mathbf{R} = -2a\Omega p\sin\lambda\sin pt\mathbf{i} - (g + 2a\Omega p\cos\lambda\sin pt)\mathbf{k}$.
6.75. $\ddot{x} - \omega^2 x - 2\omega\dot{y} = -2\Omega[\dot{z}\sin\omega t\cos\lambda - (\dot{y} + \omega x)\sin\lambda]$, $\ddot{y} - \omega^2 y + 2\omega\dot{x} = -2\Omega[\dot{z}\cos\omega t\cos\lambda + (\dot{x} - \omega y)\sin\lambda]$, $\ddot{z} = -g + 2\Omega\cos\lambda[(\dot{y} + \omega x)\cos\omega t + (\dot{x} - \omega y)\sin\omega t]$.

Chapter 7

7.3. (a) $\dfrac{dv^2}{d\phi} + 2\nu v^2 = -2gr(\sin\phi + \nu\cos\phi)$, (d) $\mathscr{I} = (m + M)\sqrt{2\nu g r\phi_0}\mathbf{i}$.
7.5. (a) (i) $\mathscr{W} = 19/6$, (iii) $\mathscr{W} = 10/3$, No, (b) $\mathscr{W} = \dfrac{ab}{55}(10b^2 + 33)$.

7.7. (b) $\mathscr{W} = 2\sqrt{2}P$ and (c) $\dot{s} = 2^{5/4}\sqrt{P/m}$ are solutions for the line.
7.9. (a) $\Delta\mathscr{P} = 2(e^{4t} - 1)$, (b) $\mathbf{x} = \frac{1}{2}\mathbf{u}e^{2t}$, a straight line.
7.11. $\delta = 7.5$ cm.
7.13. (a) $\mathscr{P} = mv(g - \nu v)$, (b) $\mathscr{W} = \frac{1}{2}mv_\infty^2(1 - e^{-\nu t})^2$.
7.15. $v_B = L\sqrt{2\left(\frac{g}{L} + \frac{1}{4}\frac{k}{m}\right)}$, $v_C = L\sqrt{2\left(\frac{g}{L} + \frac{9}{32}\frac{k}{m}\right)}$.
7.17. $\boldsymbol{\beta} = \left(1 + \frac{M}{m}\right)\sqrt{2gd(\sin\alpha + \nu\cos\alpha)}\mathbf{i}$.
7.19. (a) $\mathscr{W} = 0$, (b) No.
7.21. $V(\mathbf{x}) = V_0 - \tan^{-1}\left(\frac{x}{y}\right)$ with $r \neq 0$.
7.23. $V(\mathbf{x}) = V_0 + 5z\cos x - y(x + 2yz)$, $\mathscr{W} = 12$.
7.25. $\cos\theta_2 = \frac{t_2}{t_2 - t_1}\cos\theta_1, \theta_1 > \theta_2$.
7.27. $\Omega = \omega L/(L - R)$.
7.29. $v = 8\sqrt{3}$ ft/sec.
7.31. $v_0 = 25, 145$ mph.
7.33. $\ddot{x}_m = -\frac{v_0}{\delta_1}\sqrt{v_0^2 - v_1^2}$.
7.35. (a) $d = 7.5$ in, (b) $\tau = 0.71$ sec.
7.37. (a) $V^*(y) = V_0 - mgy + m\omega^2[(1 - 4k^2A^2)\frac{y}{2k} + 2y^2]$, (b) $N(y) = \frac{m\omega^2}{2k}\sqrt{1 + 4ky}$.
7.39. (a) $\mathbf{N} = 216\mathbf{n}$ N, (b) $v_Q = 3$ m/sec, (c) $h = 25/16$ m.
7.41. (a) $v_C = 8.26$ ft/sec, (b) $\Omega/\omega = 3$.
7.43. $\delta = 40.4$ cm.
7.45. (b) $x_S = \frac{1}{4}$ ft, $p = 8$ rad/sec, (c) $x_A = \frac{5}{4}$ ft.
7.47. (a) $d = 6r$, (b) $N(\frac{\pi}{2}) = 7W$.
7.49. (a) $\ddot{z} + p^2 z = g, z_s = \frac{5mg}{4k}$, (b) $k_e = \frac{4}{5}k$, (d) one degree of freedom.
7.51. $\operatorname{curl}\mathbf{T} = -\frac{3}{\ell}W\sin\theta\mathbf{k} \neq \mathbf{0}$ everywhere.
7.53. (b) $\alpha = \beta - \frac{M}{m}\left(\frac{gD^2}{2H} + 2\nu gd\right)^{1/2}$, $(v_0)_{\max} = \sqrt{2\nu gd}$.
7.57. (a) $\nu = \sqrt{4ga}$, (b) $\frac{t}{\tau_0} = \frac{1}{2\pi\alpha}\int_0^\phi \frac{d\phi}{\sqrt{1-\alpha^{-2}\sin^2\phi}}$, $\alpha > 1$, $\frac{t}{\tau_0} = \frac{k}{2\pi}K(k)$.
7.59. (a) $\tau^* = 2.36$ sec, (b) $t = 0.281$ sec.
7.61. (i) $p = \sqrt{2ag}$, (ii) $p = \sqrt{ag}$.
7.63. $f = 0.073\sqrt{g/a}$ at $\phi_o = 1.564\pi$ rad, $f = 0.033\sqrt{g/a}$ at $\phi_o = 7.513\pi$ rad.
7.65. $s = (g/n^2)\sin\gamma$, a cycloid.
7.67. (a) $y = a$, (b) $s = 2a\sqrt{2}$.
7.69. $L = 2.622a$.

Chapter 8

8.1. (b) $\mathbf{p}^* = 16\mathbf{i} - 10\mathbf{j} + 5\mathbf{k}$ kg · m/sec, (c) $\mathbf{h}_F = 19\mathbf{i} + 8\mathbf{j} - 48\mathbf{k}$ kg · m^2/sec, (d) $\mathbf{h}_O = -6\mathbf{i} - 30\mathbf{j} - 44\mathbf{k}$ kg · m^2/sec.
8.3. $\Delta L = \ell/2$.

8.5. (a) $\dfrac{v_B}{v_A} = \dfrac{m_A}{m_B}\cos\theta(\tan\theta - \tan\phi)$.

8.7. (a) $\Omega = \omega/4$, (b) unchanged.

8.9. $\mathbf{v} = \dfrac{a^2+b^2}{2(a^2+ab+b^2)}\mathbf{v}_0$, $\boldsymbol{\omega} = \dfrac{a-b}{2(a^2+ab+b^2)}v_0\mathbf{k}$.

8.11. (a) $\mathbf{x}^*(t) = \dfrac{m_2 l}{m_1+m_2}\mathbf{i} + \left(-\tfrac{1}{2}gt^2 + \dfrac{m_2 l}{m_1+m_2}\omega_0 t\right)\mathbf{j}$, $\theta(t) = \omega_0 t$,

(b) $\mathbf{h}_C = \dfrac{m_1 m_2 l^2}{m_1+m_2}\omega_0\mathbf{k}$, $\mathbf{h}_O = m_2 l(-gt + l\omega_0)\mathbf{k}$

8.13. $\mathbf{x}^*(t) = \dfrac{1}{4m}\mathbf{F}t^2 + \mathbf{x}_0^*$, $\dot\phi = \sqrt{\dfrac{2F}{ml}\sin\phi}$.

8.15. (a) $\mathbf{A}(t) = \mathbf{B}(t) + 4m\mathbf{g} = -\dfrac{F}{2}\left(\cos\left(\dfrac{F}{8mr}t^2\right)\mathbf{i} + \sin\left(\dfrac{F}{8mr}t^2\right)\mathbf{k}\right)$ for $t \le 2$ sec,

(c) $\omega(2) = \dfrac{F}{2mr}$.

8.17. $\mathbf{A} = 805\mathbf{k}$ lb, $\mathbf{B} = -400\mathbf{j} - 773\mathbf{k}$ lb.

8.19. (a) $5m\ell^2\ddot\theta + 4k\theta - 2mg\ell\sin\theta = 0$, (b) $\theta(t) = \theta_0\cos pt$,

(c) Stability of equilibrium states θ_e requires $k - k_c\cos\theta_e > 0$. In particular, $\theta_e = 0$ is stable for $k > k_c \equiv \dfrac{mg\ell}{2}$. Other stable equilibrium states exist. Consider the case $k/k_c = 2/\pi$, for example.

8.21. $\mathbf{v}_1 = -\dfrac{m_2}{m_1+m_2}\mathbf{v}$, $\mathbf{v}_2 = \dfrac{m_1}{m_1+m_2}\mathbf{v}$, $K = \dfrac{1}{2}\dfrac{m_1 m_2}{m_1+m_2}v^2$.

8.23. $K = mg^2t^2 + m\omega^2 d^2\sin^2\phi$.

8.25. (a) $\mathbf{p}^* = -80\mathbf{i} + 2\mathbf{k}$ kg · m/sec, (b) $K_C = 229$ N · m, (c) $K = 1830$ N · m, (d) $\mathbf{h}_O = 4\mathbf{i} - 80\mathbf{j} + 175\mathbf{k}$ kg · m²/sec, (e) $\mathbf{h}_{rO} = -80\mathbf{j} + 175\mathbf{k}$ kg · m²/sec, (g) $\mathbf{h}_{rC} = 15\mathbf{k}$ kg · m²/sec.

8.27. $\delta_{\max} = 1.5$ ft, $\mathcal{I}_R^* = 201.5\mathbf{j}$ slug · ft/sec.

8.29. (a) Stability of $\theta_e = 4mg/k\ell$ requires $1 - 2\theta_e/5 > 0$, (b) $\theta(t) = \theta_0\cos pt$.

8.31. $m_1\ddot x_1 + (k_1+k_2)x_1 - k_2x_2 = m_1 g$, $m_2\ddot x_2 + k_2x_2 - k_2x_1 = m_2 g$.

8.33. (c) $\tfrac{1}{2}(M\ell^2 + 4mr^2)\dot\phi^2 + Mg\ell(\cos\phi_0 - \cos\phi) + mgr(\cos 2\phi_0 - \cos 2\phi) = 0$,

(d) $f = \dfrac{1}{2\pi}\sqrt{\dfrac{Mg\ell + 4mgr}{M\ell^2 + 4mr^2}}$.

8.35. (a) $v(\theta) = \sqrt{\dfrac{2Pb\theta}{M} + \dfrac{4gb}{\pi}(\theta + \cos\theta - 1)}$, (c) $v(\theta) = 2\sqrt{\dfrac{gb}{\pi}(\theta + \cos\theta - 1)}$.

8.37. (a) $\mathbf{v}^* = \dfrac{v}{4}(2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$, (b) $\Delta K = -\tfrac{9}{4}mv^2$.

8.39. (a) $\boldsymbol{\omega} = \dfrac{m}{M+m}\dfrac{v}{r}\mathbf{k}$, (b) $\boldsymbol{\omega} = -\dfrac{M-m}{M+m}\dfrac{v}{r}\mathbf{k}$, $\dfrac{T}{T_0} = \left(\dfrac{M-m}{M+m}\right)^2$.

Chapter 9

9.3. See Appendix D for solutions.

9.5. $\mathbf{x}^* = \dfrac{2}{3}\dfrac{\sin\theta}{\theta}\dfrac{R_o^2 + R_oR_i + R_i^2}{R_o + R_i}\mathbf{i}_1$,

$\mathbf{I}_O = \dfrac{m}{4}(R_o^2 + R_i^2)[(1 - \dfrac{\sin 2\theta}{2\theta})\mathbf{i}_{11} + (1 + \dfrac{\sin 2\theta}{2\theta})\mathbf{i}_{22} + 2\mathbf{i}_{33}] + \dfrac{mL^2}{12}(\mathbf{i}_{11} + \mathbf{i}_{22})$.

9.7. See Appendix D.

9.9. See Appendix D.
9.11. See Appendix D.
9.13. $m(\mathscr{B}) = 12.78$ slug, $R_z = 1.42$ ft.
9.15. See Appendix D.
9.17. See Appendix D.
9.19. $\mathbf{I}_Q = \dfrac{3m}{20}(r^2 + 4h^2)(\mathbf{i}_{11} + \mathbf{i}_{22}) + \dfrac{3mr^2}{10}\mathbf{i}_{33},\ I^Q_{nn} = \dfrac{3mr^2(r^2+6h^2)}{20(r^2+h^2)}$.
9.21. $\mathbf{I}_C = \dfrac{mR^2}{4}[\dfrac{5}{4}\mathbf{i}'_{11} + \mathbf{i}'_{22} + \dfrac{7}{4}\mathbf{i}'_{33} + \dfrac{\sqrt{3}}{4}(\mathbf{i}'_{13} + \mathbf{i}'_{31})]$ in ψ.
9.23. The centroid of the plane triangle at (1,1,1).
9.25. $x = \dfrac{y}{6} = \dfrac{z-12}{-20}$.
9.27. Nearest point is at (1/2, 1/2, 0); and $d_{\min} = \sqrt{6}/6$.
9.29. $V_{\max} = 9$ at $(1, -2, 2)$; $\mathscr{W} = -8$ units.
9.31. $\lambda_1 = 5 \sim \hat{\mathbf{e}}_1 = \frac{\sqrt{2}}{2}(\mathbf{e}_1 - \mathbf{e}_2); \lambda_2 = -3 \sim \hat{\mathbf{e}}_2 = \dfrac{\sqrt{2}}{2}(\mathbf{e}_1 + \mathbf{e}_2);\ \lambda_3 = 1 \sim \hat{\mathbf{e}}_3 = \mathbf{e}_3$. No, $\mathbf{T}$ cannot be a moment of inertia tensor, $\lambda_2 < 0$.
9.33. (a) $\mathbf{I}_O = \dfrac{ma^2}{6}[8\mathbf{i}_{11} + 2\mathbf{i}_{22} + 10\mathbf{i}_{33} - 3(\mathbf{i}_{12} + \mathbf{i}_{21})]$.
(c) $\mathbf{I}_O = \dfrac{ma^2}{6}[(5 + 3\sqrt{2})\hat{\mathbf{e}}_{11} + (5 - 3\sqrt{2})\hat{\mathbf{e}}_{22} + 10\hat{\mathbf{e}}_{33}]$.
9.35. (a) $7\hat{x}^2 + 5\hat{y}^2 + 2\hat{z}^2 = 1$, (b) Possible plane body ellipsoid.
9.37. (a) $\lambda_1 = A - B \sim \hat{\mathbf{e}}_1 = \dfrac{\sqrt{2}}{2}(\mathbf{l}_1 + \mathbf{l}_2);\ \lambda_2 = A + B;\ \lambda_3 = 2A \sim \hat{\mathbf{e}}_3 = \mathbf{l}_3$;
(c) $\mathbf{I}_Q = \dfrac{m}{12}[(a^2 + b^2)\hat{\mathbf{e}}_{11} + (7a^2 + b^2)\hat{\mathbf{e}}_{22} + 2(4a^2 + b^2)\hat{\mathbf{e}}_{33}]$.
9.39. (b) $\mathbf{I}_Q = \alpha[2\mathbf{e}_{11} + 10\mathbf{e}_{22} + 8\mathbf{e}_{33} - 3(\mathbf{e}_{13} + \mathbf{e}_{31})],\ \alpha \equiv \frac{2}{3}ma^2$,
(c) $\lambda_1 = \alpha(5 + 3\sqrt{2}),\ \lambda_2 = 10\alpha,\ \lambda_3 = \alpha(5 - 3\sqrt{2})$.
9.41. $\mathbf{T}^{1/2} = \dfrac{3}{2}(\mathbf{e}_{11} + \mathbf{e}_{22}) + \sqrt{3}\mathbf{e}_{33} - \dfrac{1}{2}(\mathbf{e}_{12} + \mathbf{e}_{21})$.
9.43. (a) $\mathbf{G}_Q = \sqrt{\dfrac{A-B}{m}}\hat{\mathbf{e}}_{11} + \sqrt{\dfrac{A+B}{m}}\hat{\mathbf{e}}_{22} + \sqrt{\dfrac{2A}{m}}\hat{\mathbf{e}}_{33}$.

Chapter 10

10.3. $\mathbf{F} = -\,920\mathbf{i}$ N with $\mathbf{v} = v\mathbf{i}$.
10.5. $\mathbf{h}_A = \frac{1}{2}mR^2\omega_1\mathbf{i} + m\omega_2(\frac{1}{4}R^2 + d^2)\mathbf{j}, \mathbf{M}_B = -\frac{1}{2}mR^2\omega_1\omega_2\mathbf{k}$.
10.7. (a) $\mathbf{A} = (\dfrac{W}{2} + \dfrac{P}{\pi})\mathbf{j}, \mathbf{B} = (\dfrac{W}{2} - \dfrac{P}{\pi})\mathbf{j}$, (b) $\mathbf{P} = \dfrac{\pi}{2}W\mathbf{i}, \mathbf{x}^* = \dfrac{\pi}{4}gt^2\mathbf{i}$.
10.9. $h - \dfrac{dW}{P} \le H \le h + \dfrac{dW}{P}$.
10.11. (a) $f = \dfrac{1}{\pi\ell}\sqrt{\dfrac{k}{m}}$, (b) $\theta = \theta_0 \cos pt$, (c) Show that $\mu = 2\pi mh\ell^2 f^2/a^4$.
10.13. $R_H = \sqrt{\dfrac{22}{3}}$ ft, $\tau = 2\pi\sqrt{\dfrac{11}{3g}}$ sec.

10.15. (a) $f = \dfrac{1}{2\pi}\sqrt{\dfrac{10g}{7\ell}}$, (c) $R_O = \dfrac{\ell}{3}\sqrt{\dfrac{7}{2}}$.

10.17. $\mathbf{A} = -19,839\mathbf{k}$ lb, $\mathbf{B} = -64,000\mathbf{j} + 20,161\mathbf{k}$ lb.

10.19. $M_C = \frac{1}{8}mr^2\omega^2 \sin 2\alpha,\ N = \dfrac{M_C}{2d}$.

10.21. (a) $\tau = 2\pi\sqrt{\dfrac{d^2 + R_C^2}{gd}}$, (b) $d = R_C$, (c) $b = \dfrac{R_C^2}{d}$.

10.23. $\boldsymbol{\omega} = \dfrac{3\sqrt{2}}{2ma}F^*\mathbf{k}$, $\mathbf{v}^* = \dfrac{\mathbf{F}^*}{m}$.

10.25. $\boldsymbol{\Omega} = \boldsymbol{\omega}/7$.

10.29. $\omega_A = \dfrac{b^2 I_A \omega}{b^2 I_A + a^2 I_B}$, $\omega_B = \dfrac{a\omega_A}{b}$.

10.33. $\dfrac{N_D}{N_S} = \dfrac{2}{1 + 3\sin^2\alpha}$, $\dfrac{1}{2} < \dfrac{N_D}{N_S} < 2$.

10.35. $\mathbf{T} = \dfrac{W \sin\alpha}{1 + 3\cos^2\alpha}(\cos\alpha\mathbf{i} - \sin\alpha\mathbf{j})$, $\dot{\boldsymbol{\omega}} = \dfrac{3g}{\ell}\cdot\dfrac{\sin 2\alpha}{1 + 3\cos^2\alpha}\mathbf{k}$.

10.37. $\mathbf{N}_1 = w(-\dfrac{2}{5} + \cos\theta + \dfrac{\sqrt{3}}{12}\sin\theta)(-\mathbf{i} + \sqrt{3}\mathbf{j})$,

$\mathbf{N}_2 = w(\dfrac{2}{5} - \cos\theta + \dfrac{\sqrt{3}}{12}\sin\theta)(\mathbf{i} + \sqrt{3}\mathbf{j},)$ referred to a body frame at C.

10.39. (a) $N = W + \dfrac{m\ell}{2}(\dot{\omega}\sin\theta + \omega^2\cos\theta)$, $f = \dfrac{m\ell}{2}[\dot{\omega}(2 - \cos\theta) + \omega^2\sin\theta]$,

$\dot{\omega} = -\dfrac{6}{ml}[N\sin\theta + f(2 - \cos\theta)]$, (b) $\dot{\omega}_0 = -\dfrac{3g}{8\ell} = -\dfrac{p^2}{4}$, (d) $\theta = \theta_0 \cos pt$.

10.41. $\ddot{x}_1 + p_1^2(5x_1 - x_2) = 0$, $\ddot{x}_2 + p_2^2(x_2 - x_1) = 0$, $\dfrac{p_1^2}{p_2^2} = \dfrac{2M}{3m}$.

10.43. (a) $\ddot{\theta} + p^2\sin\theta = 0,\ p^2 \equiv \dfrac{2g}{3h}$ (b) $\theta = 2\sin^{-1}\left(\sin\dfrac{\theta_0}{2}\mathrm{sn}\,pt\right)$, $\tau^* = \dfrac{2\tau}{\pi}K(k)$,

(c) $N = \frac{1}{3}mg\left(7\cos\theta - 4\cos\theta_0\right)$, $f = \frac{1}{3}mg\sin\theta$.

10.47. (a) during slipping: $v^* = \nu g t$, $\omega = -\dfrac{2\nu g}{r}t + \omega_0$,

(b) $v_D = 3\nu g t - r\omega_0$, (c) $\theta(\tau) = \dfrac{2r\omega_0^2}{9\nu g}$.

10.49. $f = \dfrac{1}{2\pi}\sqrt{\dfrac{3}{M\ell^2}\left(ka^2 + \dfrac{1}{2}W\ell\right)}$.

10.51. $\tau = 2\pi\sqrt{\dfrac{7\ell}{10g}}$.

10.53. $\beta = (1 + \dfrac{M}{m})\ell\sqrt{\dfrac{2(m + M)g\ell}{I_O}(1 - \cos\theta_0)}$.

10.55. $\dot{\theta}^2 = \dfrac{3g}{\ell}\cdot\dfrac{\cos\theta - \cos\theta_0}{4 - 3\cos\theta}$, $\tau = \dfrac{4}{p}\int_0^{\theta_0}\sqrt{\dfrac{4 - 3\cos\theta}{2(\cos\theta - \cos\theta_0)}}d\theta$, $p^2 \equiv 3g/2\ell$.

10.57. $K = \dfrac{m\ell^2}{6}(4\dot{\theta}_1^2 + 3\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)$, $V = \dfrac{m\ell^2}{4}\left[\left(\dfrac{k}{m} + \dfrac{3g}{\ell}\right)\theta_1^2 + \dfrac{g}{\ell}\theta_2^2\right]$.

10.59. $\theta(t) = A\cos pt + B\sin pt$, $p = \dfrac{1}{4}\sqrt{\dfrac{6g}{b} + \dfrac{2k}{m}}$.

10.61. (a) $\omega_{\text{tube}} = \sqrt{\dfrac{4gh}{3R^2 + r^2}}$, (b) solid wins.

10.63. (a) $\ddot{\theta} + \dfrac{3c}{m}\dot{\theta} + \dfrac{3k}{4m}\theta = 0$, (b) $\theta(t) = \dfrac{\omega_0}{\omega_d}e^{-\nu t}\sin\omega_d t$, $f = \dfrac{1}{2\pi}\sqrt{\dfrac{3k}{4m}(1 - \dfrac{3c^2}{k})}$,
(c) $\tau = 2\pi\sqrt{\dfrac{4m}{3k}}$.

10.65. (a) $\ddot{\theta} + \beta\dot{\theta} = 0$, (b) the nonviscous forces are workless, $\dot{K} + \dfrac{2c}{\sigma}K = 0$,
(c) $N = W$, $f = a\omega_0 e^{-\beta t}(2\pi ca - m\beta)$.

10.67. (a) $\omega(t) = \dfrac{\tau}{\beta} + (\omega_0 - \dfrac{\tau}{\beta})e^{-\beta t}$, $\tau \equiv \dfrac{2T}{mR^2}$, (b) $\theta = \theta_0 + \dfrac{\omega_0}{\beta}(1 - e^{-\beta t})$.

Chapter 11

11.1. Constraint is holonomic with $z = \sin^2 y + e^{2x} - e^x \sin y$.

11.3. $\ddot{\theta} - \dfrac{\gamma_\phi^2}{m^2\ell^4}\dfrac{\cos\theta}{\sin^3\theta} + \dfrac{g}{\ell}\sin\theta = 0$, $\dot{\theta}^2 + \dfrac{\gamma_\phi^2}{m^2\ell^4\sin^2\theta} - \dfrac{2g}{\ell}\cos\theta = E_0$; $\gamma_\phi =$ constant.

11.5. $\dfrac{d}{dt}\left[m\dot{r}\left(1 + 4\dfrac{r^2}{a^2}\right)\right] - 4m\dfrac{r\dot{r}}{a^2} - mr\dot{\theta}^2 + 2mg\dfrac{r}{a} = -c\dot{r}\left(1 + 4\dfrac{r^2}{a^2}\right)$,
$\dfrac{d}{dt}\left(mr^2\dot{\theta}\right) = -cr^2\dot{\theta}$.

11.7. Three familiar type solutions evolve from $\ddot{r} + \left(\dfrac{k}{m} - \omega^2\cos^2\alpha\right)r = -g\sin\alpha$.

11.9. (a) $\ddot{x} + \left(2\dfrac{k}{m} - \omega^2\right)x = 0$, (c) $R = mp^2\left(r \pm 2\dfrac{\omega_0}{p}\sqrt{a^2 - x^2}\right)$.

11.11. $(1 + (r')^2)\ddot{z} + r'r''\dot{z}^2 - \dfrac{\gamma_\theta^2}{m^2}\dfrac{r'}{r^3} + g = 0$, $\gamma_\theta =$ constant.

11.13. $(m_1 + m_2)\ddot{r} - m_1 r\dot{\phi}^2 + m_2 g = 0$, $\quad m_1 r^2\dot{\phi} = h$, constant.

11.15. $\ddot{\theta} + p^2\theta = 0$, $p^2 = 3\dfrac{k}{m}$.

11.17. $\ddot{\theta} + \dfrac{8k}{3m}\theta = \dfrac{2g}{3a}$, or $\ddot{z} + \dfrac{8k}{3m}z = \dfrac{2g}{3}$; $f = \dfrac{1}{\pi}\sqrt{\dfrac{2k}{3m}}$. One degree of freedom.

11.19. $\ddot{\theta} + p^2\sin\theta = 0$, $p = \sqrt{\dfrac{2g}{3h}}$.

11.21. (a) $\ddot{\theta} + p^2\sin\theta = 0$, $p^2 = \dfrac{g}{r}$, (b) $\dot{\theta}^2 - 2p^2\cos\theta = e$, constant,
(d) $N_1 = mg(2\cos\theta + \dfrac{\sqrt{3}}{6}\sin\theta) + \dfrac{mr}{2}e$, $N_2 = mg(2\cos\theta - \dfrac{\sqrt{3}}{6}\sin\theta) + \dfrac{mr}{2}e$.

11.23. (a) $\omega^2 = \dfrac{2mga\sin\theta}{I_C + m\left(a^2 + R^2 - 2aR\sin\theta\right)}$,
(c) $f = m[\dot{\omega}(R - a\sin\theta) - a\omega^2\cos\theta]$, $N = W + m(-\dot{\omega}a\cos\theta + a\omega^2\sin\theta)$.

11.25. (b) $\tau = 4\sqrt{\dfrac{\ell}{3g}}\displaystyle\int_0^{\theta_0}\sqrt{\dfrac{10 - 3\cos\theta_0}{\cos\theta - \cos\theta_0}}\,d\theta$, (d) $L = \dfrac{14}{3}\ell$.

11.27. $x = x_e(1 - \cos p_x t)$, $\theta = \theta_0\cos p_\theta t$, $f_x = 2f_\theta = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$.

11.29. $\dot{x} = -\dfrac{P}{m}$, $\dot{y} = 0$, $\dot{\theta}_1 = -\dfrac{3P}{4m\ell}$, $\dot{\theta}_2 = \dfrac{9P}{4m\ell}$; $\Delta T = \dfrac{7P^2}{4m}$.

11.31. $\ddot{\phi} - \dfrac{3R}{\ell}\omega^2 \sin(\omega t - \phi) + \dfrac{3g}{2\ell}\sin\phi = 0$.

11.33. $8\ddot{\theta}_1 + 3\ddot{\theta}_2 + 3\left(\dfrac{k}{m} + \dfrac{3g}{\ell}\right)\theta_1 = 0$, $3\ddot{\theta}_1 + 2\ddot{\theta}_2 + \dfrac{3g}{\ell}\theta_2 = 0$.

11.35. (a) $2m\ell^2\ddot{\phi} + m\ell^2\ddot{\theta}\cos(\theta - \phi) - m\ell^2\dot{\theta}^2\sin(\theta - \phi) + 2mg\ell\sin\phi = 0$,
$m\ell^2\ddot{\theta} + m\ell^2\ddot{\phi}\cos(\theta - \phi) + m\ell^2\dot{\phi}^2\sin(\theta - \phi) + mg\ell\sin\theta = 0$,
(b) $2\ddot{\phi} + \ddot{\theta} + \dfrac{2g}{\ell}\phi = 0$, $\ddot{\theta} + \ddot{\phi} + \dfrac{g}{\ell}\theta = 0$.

11.37. $D = \dfrac{1}{2}cl^2(\dot{\theta}_1^2 + \dot{\theta}_2^2)$.

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